A New Formulation of Relativistic Euler Flow: Miraculous Geo-Analytic Structures and Applications

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Context

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- New formulation $\implies$ can exploit geometric vectorfield method.

Long-term goal: Understand global structure of piecewise smooth solutions with shocks

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  Shocks can form: $|\partial\Psi| \to \infty$, $|\Psi| < \infty$

Interesting results for model problem

- **Stable shock formation**
  • Christodoulou: small data for irrotational compressible fluids
  • Speck: small data for wave equations
  • Miao–Yu and Speck–Holzegel–Luk–Wong: new solution regimes

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- **Local well-posedness below** $H^{(5/2)^+}$
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  - Wang: geometric physical space proof of Smith–Tataru result
- Unifying theme: **nonlinear geometric optics** via eikonal functions; $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha U \partial_\beta U = 0$

Big Idea:

For some applications (especially ones involving eikonal functions), one can treat the relativistic Euler equations as perturbations of the model problem $\square_{\mathbf{g}(\Psi)}\Psi = 0$

Schematic depiction of the "wave part"

The "wave part" Ψ of relativistic Euler satisfies:

$$\boxed{\begin{aligned}\square_{\mathbf{g}(\Psi)}\Psi = \text{4-curl (vorticity)} + \text{div } (\nabla \text{ entropy}) \\ + \, \mathbf{g}-\text{null forms}\end{aligned}}$$

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  ❗ This can be achieved via div-curl-transport systems that enjoy good null structure.

Christodoulou's sharp picture of relativistic Euler shock formation (irrotational case)

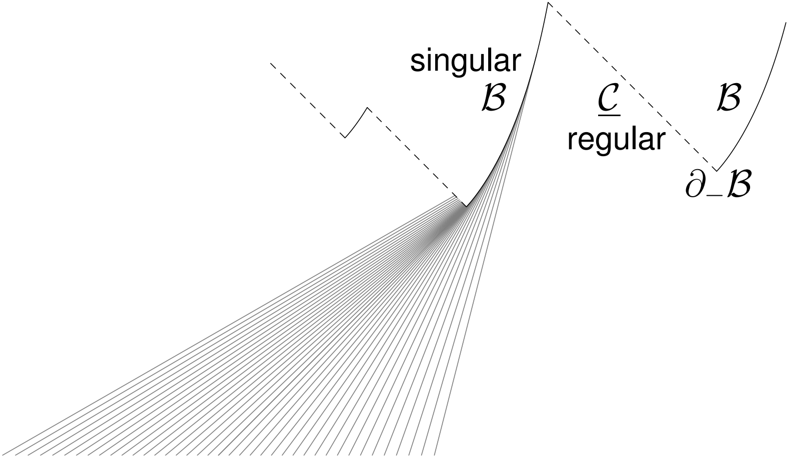


Figure: The maximal development

Boundary of the Maximal Development and Shock Hypersurface

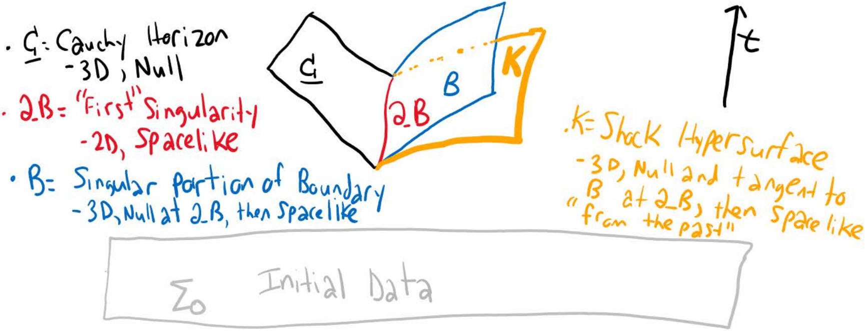
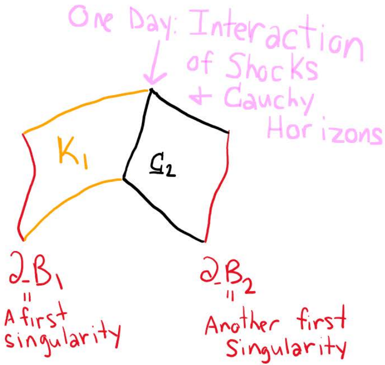


Figure: Boundary of the maximal development and the shock hypersurface

Interaction of shock hypersurfaces and Cauchy horizons



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- s is crucial for the theory of solutions with shocks

Geometric tensors associated to the flow

The four-velocity transports vorticity and entropy.

Definition (The four-velocity vectorfield)

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Definition (The acoustical metric and its inverse)

$$\boxed{\begin{aligned} \mathbf{g}_{\alpha\beta}(\vec{\Psi}) &:= c^{-2}\eta_{\alpha\beta} + (c^{-2} - 1)u_{\alpha}u_{\beta}, \\ (\mathbf{g}^{-1})^{\alpha\beta}(\vec{\Psi}) &= c^{2}(\eta^{-1})^{\alpha\beta} + (c^{2} - 1)u^{\alpha}u^{\beta} \end{aligned}}$$

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u is $\mathbf{g}$-timelike and thus transverse to acoustically null hypersurfaces:

$$\boxed{\mathbf{g}(u, u) = -1}$$

Additional fluid variables

Definition (The *u*-orthogonal vorticity of a one-form)

$$\boxed{\mathrm{vort}^{\alpha}(V) := -\epsilon^{\alpha\beta\gamma\delta} u_{\beta} \partial_{\gamma} V_{\delta}}$$

Definition (Vorticity vectorfield)

$$\boxed{\varpi^{\alpha} := \mathrm{vort}^{\alpha}(Hu) = -\epsilon^{\alpha\beta\gamma\delta} u_{\beta} \partial_{\gamma} (H u_{\delta})}$$

Definition (Entropy gradient one-form)

$$\boxed{S_{\alpha} := \partial_{\alpha} s}$$

Modified fluid variables

- Exhibit improved regularity
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$$
\begin{aligned}
\mathcal{C}^\alpha := & \operatorname{vort}^\alpha(\varpi) + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi_\delta \\
& + (\theta - \theta_{;h})S^\alpha(\partial_\kappa u^\kappa) + (\theta - \theta_{;h})u^\alpha(S^\kappa\partial_\kappa h) \\
& + (\theta_{;h} - \theta)S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\kappa), \\
\mathcal{D} := & \frac{1}{n}(\partial_\kappa S^\kappa) + \frac{1}{n}(S^\kappa\partial_\kappa h) - \frac{1}{n}c^{-2}(S^\kappa\partial_\kappa h)
\end{aligned}
$$

- Temperature $\theta(h, s)$ and number density $n(h, s)$ determined by equation of state
- $\theta_{;h} := \frac{\partial}{\partial h}\theta$

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$$\mathcal{Q}^{(\mathbf{g})}(\partial\phi, \partial\widetilde{\phi}) := (\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\widetilde{\phi},$$
$$\mathcal{Q}_{(\alpha\beta)}(\partial\phi, \partial\widetilde{\phi}) := \partial_\alpha\phi\partial_\beta\widetilde{\phi} - \partial_\alpha\widetilde{\phi}\partial_\beta\phi$$

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The new formulation allows for the application of geometric techniques from mathematical GR and nonlinear wave equations.

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  "multiple characteristic speeds"

A new formulation of relativistic Euler

Theorem (JS with M. Disconzi)

For $\Psi \in \vec{\Psi} := (h, u^0, u^1, u^2, u^3, s)$, $\mathcal{Q} :=$ *combinations of null forms, regular solutions satisfy, up to lower-order terms:*

$$\begin{aligned} \square_{\mathbf{g}(\vec{\Psi})} \Psi &= \mathcal{C} + \mathcal{D} + \mathcal{Q}(\boldsymbol{\partial} \vec{\Psi}, \boldsymbol{\partial} \vec{\Psi}), \\ u^{\kappa} \partial_{\kappa} \varpi^{\alpha} &= \boldsymbol{\partial} \vec{\Psi}, \\ u^{\kappa} \partial_{\kappa} S^{\alpha} &= \boldsymbol{\partial} \vec{\Psi} \end{aligned}$$

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- *Formally,* $\mathcal{C}, \mathcal{D} \sim \partial\partial\vec{\Psi}$, *but they are actually better from various points of view. In fact,* $\partial\varpi, \partial S$ *are better:*

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- In practice, one needs L^2 regularity for $\boldsymbol{\partial}\varpi$ along Σ_t.
- To achieve this, one also considers the PDEs $u^\kappa \partial_\kappa \varpi^\alpha = RHS$ and $u_\alpha \varpi^\alpha = 0$ (and thus $u_\alpha \boldsymbol{\partial} \varpi^\alpha = -(\boldsymbol{\partial} u_\alpha) \varpi^\alpha$).

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- Thesis work in progress by Sifan Wu: low regularity sound waves (à la my work with Disconzi, Luo, Mazzone and Wang's work in the non-relativistic case).

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- Level sets $\mathcal{C}_U$ of U are **g**-null hypersurfaces.
- Play a critical role in many delicate local and global results for wave equations.
- The regularity theory of U is difficult, tensorial, influenced by the Euler solution, especially the vorticity and entropy.

Acoustic null frame

An acoustic null frame $\{L, \underline{L}, e_1, e_2\}$:

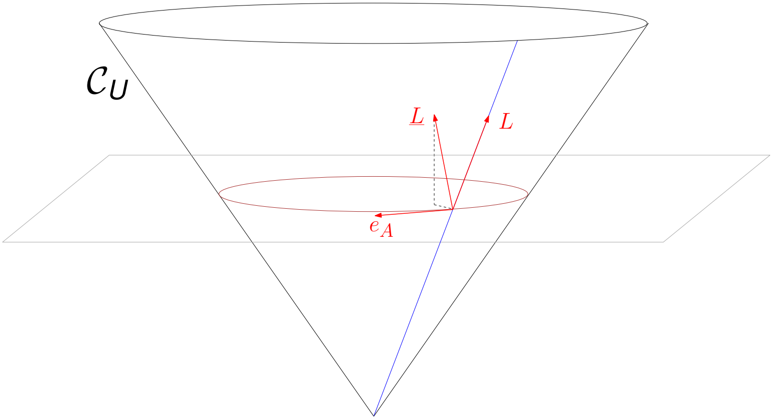

Figure: Null (with respect to **g**) frame

A crucial geometric quantity

- $S_{t,U} :=$ intersection of level sets of t and U, equipped with $\mathbf{g}$-orthonormal frame $\{e_A\}_{A=1,2}$
- $L :=$ rescaled version of $-\mathbf{D}U$; it is $\mathbf{g}$-null (often normalized by $L = \frac{\partial}{\partial t}$)

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The null mean curvature plays a key role in the analysis and regularity theory of U:

Definition (Null mean curvature of $S_{t,U}$)

With $\mathbf{D}$ the connection of $\mathbf{g}$, we define:

$$\boxed{\mathrm{tr}\chi := \sum_{A=1,2} \mathbf{g}(\mathbf{D}_{e_A} L, e_A)}$$

- $\mathrm{tr}\chi = 2/r$ along standard Euclidean spheres in Minkowski light cones

Renormalized Raychaudhuri equation

To control $\mathrm{tr}\chi$, one starts with the Renormalized Raychaudhuri equation, where $\Gamma \sim (\mathbf{g}^{-1})^2 \partial \mathbf{g} \sim \partial \vec{\Psi} :=$ contracted rectangular Christoffel symbol:

$$L(\mathrm{tr}\chi + \Gamma_L) = (1/2) L^\alpha L^\beta \square_g \mathbf{g}_{\alpha\beta}(\vec{\Psi}) + \partial \vec{\Psi} \cdot \partial \vec{\Psi} + \cdots$$

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The main contribution of the vorticity and entropy to the evolution of $\mathrm{tr}\chi$ is through the special combinations $\mathcal{C}$ and $\mathcal{D}$, which enjoy improved regularity properties compared to $\partial \varpi$ and ∂S

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- Long-time behavior of solutions with shocks (at least in a perturbative regime)
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Would require the development of new geometry.