Cosmological Singularities in GR: The Complete Sub-Critical Regime

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Cauchy Problem for Einstein's equations

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$$\square_{\mathbf{g}}\phi = 0$$

- Some results I will describe hold when $\phi \equiv 0$
- Data on $\Sigma_1 = \mathbb{T}^3$ are tensors $(\mathring{g}, \mathring{k}, \mathring{\phi}_0, \mathring{\phi}_1)$ verifying the Gauss and Codazzi constraints
- Our data will be Sobolev-close to Kasner data
- Choquet-Bruhat and Geroch: data verifying constraints launch a unique maximal globally hyperbolic development $(\mathcal{M}, \mathbf{g}, \phi)$

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"Dynamic stability of the Big Bang"

Some sources of inspiration

- Hawking–Penrose “singularity” theorems.
- Explicit solutions, especially FLRW and Kasner.
- Heuristics from the physics literature.
- Numerical work on singularities.
- Rigorous results in symmetry and analytic class.
- Dafermos–Luk.

"Generalized" Kasner solutions

$$\boxed{\mathbf{g}_{KAS} = -dt \otimes dt + \sum_{I=1}^{D} t^{2q_I} dx^I \otimes dx^I, \qquad \phi_{KAS} = B \ln t}$$

The $q_I \in (-1, 1)$ and $B \geq 0$ verify the Kasner constraints:

$$\sum_{I=1}^{D} q_I = 1, \qquad \sum_{I=1}^{D} (q_I)^2 = 1 - B^2$$

$$\mathbf{Riem}_{\alpha\beta\gamma\delta}\mathbf{Riem}^{\alpha\beta\gamma\delta} = Ct^{-4}$$

where $C > 0$ (unless one q_I equals 1 and the rest vanish)

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"Big Bang" singularity at $t = 0$

Hawking's incompleteness theorem

Theorem (Hawking)

Assume

- $(\mathcal{M}, \mathbf{g}, \phi)$ *is the maximal globally hyperbolic development of data* $(\mathring{g}, \mathring{k}, \mathring{\phi}_0, \mathring{\phi}_1)$ *on* $\Sigma_1 \simeq \mathbb{T}^D$
- $\mathrm{tr}\mathring{k} < -C < 0$

Then no past-directed timelike geodesic emanating from Σ_1 is longer than $C' < \infty$.

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- For Kasner, incompleteness $\leftrightarrow$ Big Bang, but what about perturbations?

Potential sources of incompleteness

- Curvature blowup/crushing singularities à la Kasner
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Near-Kasner incompleteness

New result with Rodnianski and Fournodavlos: Kasner Big Bang is dynamically stable assuming a sub-criticality condition:

$$\boxed{\max_{\substack{I,J,B=1,\cdots,D \\ I<J}} \{q_I + q_J - q_B\} < 1}$$

$\exists$ sub-critical vacuum Kasner solutions $\iff D \geq 10$ (Demaret–Henneaux–Spindel)

Key takeaways:

- In GR, distinct kinds of incompleteness occurs in different solution regimes
- In principle, other stable pathologies could dynamically develop in other (not-yet-understood) regimes

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Belinskiĭ–Khalatnikov–Lifshitz considered tensorfields:

$$\mathbf{g}_{BKL} = -dt \otimes dt + \sum_{I=1}^{D} t^{2q_I(x)} dx^I \otimes dx^I, \; \phi_{BKL} = B(x) \ln t,$$

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- Part of BKL saga: In $3D$ vacuum, near spacelike singularities, "most solutions" "should" oscillate violently in time;
- $\mathbf{g}_{BKL}$ metrics are typically at best "short-time approximations" (Kasner epochs)
- Fournodavlos–Luk: ∃ large family of non-oscillatory Sobolev-class $3D$ Einstein-vacuum solutions that are asymptotic to $\mathbf{g}_{BKL}$-type metrics; 3 functional degrees of freedom (compared to 4 for the Cauchy problem)

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“Monotonic” regimes

Works by BK, Barrow, Demaret–Henneaux–Spindel, Andersson–Rendall, Damour–Henneaux–Rendall–Weaver suggest that a $D-$dimensional Kasner Big Bang might be dynamically stable under the sub-criticality condition:

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- Significance: Heuristics suggest that time derivative terms will dominate: “Asymptotically Velocity Term Dominated”
- With symmetry, stability might hold for even more q’s

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The singularity industry: A sampler

- **Numerical works**: e.g. Berger, Garfinkle, Isenberg, Lim, Moncrief, Weaver, $\cdots$
- **Symmetry**: e.g. Alexakis–Fournodavlos, Chruściel–Isenberg–Moncrief, Ellis, Isenberg–Kichenassamy, Isenberg–Moncrief, Liebscher, Ringström, Wainwright, $\cdots$
- **Linear**: e.g. Alho–Franzen–Fournodavlos, Ringström
- **Construction of singular solutions**: e.g. Ames, Andersson, Anguige, Beyer, Choquet-Bruhat, Damour, Demaret, Fournodavlos, Henneaux, Isenberg, LeFloch, Luk, Kichenassamy, Rendall, Spindel, Ståhl, Todd, Weaver, $\cdots$
- **Oscillatory investigations**: e.g. BKL, Damour, van Elst, Heinzle, Hsu, Lecian, Liebscher, Misner, Nicolai, Uggla, Reiterer, Ringström, Tchapnda, Trubowitz, $\cdots$

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If the *sub-criticality condition*

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holds, then near its Big Bang,
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- Effectively covers the entire (asymmetric) regime where BK-type heuristics suggest stable blowup.

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Moreover, when $D = 3$ *and* $B = 0$*, under polarized* $U(1)$*-symmetric perturbations (i.e.,* $g_{13} = g_{23} \equiv 0$ *and no* x^3*-dependence),* ***all*** *Kasner Big Bangs are dynamically stable.*

- Effectively covers the entire (asymmetric) regime where BK-type heuristics suggest stable blowup.
- Previously with Rodnianski, we had treated **i)** $D = 3$ with $q_1 = q_2 = q_3 = 1/3$. i.e. stability for FLRW; and **ii)** $D \geq 38$ with $\max_{I=1,\cdots,D} |q_I| < 1/6$ and $\phi \equiv 0$

Crushing singularities

The singularities in our main results are crushing:

$$\boxed{\int_{Spacetime} |Christoffel|^2 \underbrace{dvol}_{O(t)dtdx} = |\ln(0)| = \infty}$$

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This shows that in the chosen gauge, the solution cannot be continued weakly.

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Fermi-Walker-propagated Σ_t-tangent orthonormal spatial frame $\{e_I\}_{I=1,\cdots,D}$; with $e_I = e_I^c \partial_c$:

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- $e_0\phi$ and $e_I\phi$ if scalar field is present

Einstein-vacuum equations in our gauge

Evolution equations

$$
\begin{aligned}
\partial_t k_{IJ} &= -\frac{n}{t} k_{IJ} - e_I e_J n + n e_C \gamma_{IJC} - n e_I \gamma_{CJC} \\
&\quad + \gamma_{IJC} e_C n - n \gamma_{DIC} \gamma_{CJD} - n \gamma_{DDC} \gamma_{IJC}, \\
\partial_t \gamma_{IJB} &= n e_B k_{IJ} - n e_J k_{BI} \\
&\quad - n k_{IC} \gamma_{BJC} + n k_{IC} \gamma_{JBC} + n k_{IC} \gamma_{CJB} \\
&\quad - n k_{CJ} \gamma_{BIC} + n k_{BC} \gamma_{JIC} \\
&\quad + (e_B n) k_{IJ} - (e_J n) k_{BI}
\end{aligned}
$$

Elliptic lapse PDE

$$
\begin{aligned}
e_C e_C (n-1) - t^{-2}(n-1) &= \gamma_{CCD} e_D (n-1) + 2 n e_C \gamma_{DDC} \\
&\quad - n \left\{ \gamma_{CDE} \gamma_{EDC} + \gamma_{CCD} \gamma_{EED} \right\}
\end{aligned}
$$

Constraint equations

$$
\begin{aligned}
k_{CD} k_{CD} - t^{-2} &= 2 e_C \gamma_{DDC} - \gamma_{CDE} \gamma_{EDC} - \gamma_{CCD} \gamma_{EED}, \\
e_C k_{CI} &= \gamma_{CCD} k_{ID} + \gamma_{CID} k_{CD}
\end{aligned}
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Elliptic PDE $\Delta_g n = \cdots$ synchronizes singularity

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- $\partial_t(tk_{IJ}) = te_I\gamma + t\gamma\cdot\gamma + \cdots \lesssim \epsilon t^{1-(2q+2\delta)}$
- Thus, integrability of $t^{1-(2q+2\delta)}$ (for large N) implies that for $t \in (0,1]$: $|tk_{IJ}(t,x) - k_{IJ}(1,x)| \lesssim \epsilon$

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Top-order energy estimates

We prove that for $t \in (0, 1]$, we have:

$$\|t^{A+1}k\|^2_{\dot{H}^N(\Sigma_t)} + \|t^{A+1}\gamma\|^2_{\dot{H}^N(\Sigma_t)}$$
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$$+ \{C_\star - A\} \int_t^1 s^{-1} \left\{ \|s^{A+1}\gamma\|^2_{\dot{H}^N(\Sigma_s)} + \|s^{A+1}k\|^2_{\dot{H}^N(\Sigma_s)} \right\} ds$$
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