Cauchy problem	Goals	Kasner	Incompleteness	Oscillatory vs Monotonic	Results	The Gauge	Proof Hints	Future
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Cosmological Singularities in GR: The Complete Sub-Critical Regime

Jared Speck

Vanderbilt University

April 27, 2021

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Incompleteness

Cauchy problem

Goals

Kasner

$$\begin{split} \textbf{Ric}_{\mu\nu} &- \frac{1}{2}\textbf{Rg}_{\mu\nu} = \textbf{T}_{\mu\nu} := \textbf{D}_{\mu}\phi\textbf{D}_{\nu}\phi - \frac{1}{2}\textbf{g}_{\mu\nu}\textbf{D}\phi\cdot\textbf{D}\phi, \\ & \Box_{\textbf{g}}\phi = \textbf{0} \end{split}$$

Oscillatory vs Monotonic

The Gauge

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Results

- ϕ . Some results I will describe hold when $\phi=0$.
- Data on $\Sigma_1 := \mathbb{T}^n$ are tensors $(\hat{g}; k, \phi_0, \phi_1)$ verifying the Gauss and Codazzi constraints
- Our data will be Sobolev-close to Kasner data
- Choquet-Bruhat and Geroch: data verifying constraints launch a unique maximal globally hyperbolic development (*M*1, g, *i*)

Cauchy problem

Goals

Kasner

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Oscillatory vs Monotonic

Results

The Gauge

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• Some results I will describe hold when $\phi \equiv 0$

Cauchy problem

Goals

Kasner

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Oscillatory vs Monotonic

Results

The Gauge

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Future

- Some results I will describe hold when $\phi \equiv 0$
- Data on Σ₁ = T^D are tensors (ġ, k̇, φ̇₀, φ̇₁) verifying the Gauss and Codazzi constraints

Choquet-Bruhat and Geroch: data verifying constraints launch a unique maximal globally hyperbolic development (\mathcal{M}_{0} , q, ϕ)

Incompleteness

Cauchy problem

Goals

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Oscillatory vs Monotonic

Results

The Gauge

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Cauchy problem

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Oscillatory vs Monotonic

Results

The Gauge

Proof Hints

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Future

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- Choquet-Bruhat and Geroch: data verifying constraints launch a unique maximal globally hyperbolic development (*M*, g, φ)

Cauchy problem o	Goals ●○	Kasner o	Incompleteness	Oscillatory vs Monotonic	Results	The Gauge	Proof Hints	Future ○
Goal								

Goal: Understand the formation of stable spacelike singularities in $(\mathcal{M}, \mathbf{g}, \phi)$.

Dynamic stability of the Big Bang



Cauchy problem	Goals ●○	Kasner o	Incompleteness	Oscillatory vs Monotonic	Results	The Gauge	Proof Hints	Future ○
Goal								

Goal: Understand the formation of stable spacelike singularities in $(\mathcal{M}, \mathbf{g}, \phi)$.

Math problem: For which open sets of data does **Riem_{\alpha\beta\gamma\delta} Riem^{\alpha\beta\gamma\delta} blow up on a spacelike hypersurface?**

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Cauchy problem	Goals ●○	Kasner o	Incompleteness	Oscillatory vs Monotonic	Results	The Gauge	Proof Hints	Future ○
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Math problem: For which open sets of data does $\operatorname{Riem}_{\alpha\beta\gamma\delta}\operatorname{Riem}^{\alpha\beta\gamma\delta}$ blow up on a spacelike hypersurface? "Dynamic stability of the Big Bang"

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Some sources of inspiration

Incompleteness

Cauchy problem

Goals

Kasner

- Hawking–Penrose "singularity" theorems.
- Explicit solutions, especially FLRW and Kasner.

Oscillatory vs Monotonic

Results

The Gauge

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Future

- Heuristics from the physics literature.
- Numerical work on singularities.
- Rigorous results in symmetry and analytic class.
- Dafermos–Luk.

Incompleteness

Kasner

$$\mathbf{g}_{\mathcal{KAS}} = -dt \otimes dt + \sum_{l=1}^{D} t^{2q_l} dx^l \otimes dx^l, \quad \phi_{\mathcal{KAS}} = B \ln t$$

Oscillatory vs Monotonic

Results

The Gauge

The $q \in (-1, 1]$ and $B \ge 0$ verify the Kasner constraints:



 $\mathsf{Riem}_{ab\gamma b}\mathsf{Riem}^{ab\gamma b}=\mathsf{Cl}^{-b}$

where C > 0 (unless one q equals 1 and the rest vanish

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Goals

Kasner

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Oscillatory vs Monotonic

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The $q_l \in (-1, 1]$ and $B \ge 0$ verify the Kasner constraints:

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Incompleteness

Cauchy problem

Goals

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Oscillatory vs Monotonic

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$$\operatorname{Riem}_{lphaeta\gamma\delta}\operatorname{Riem}^{lphaeta\gamma\delta}=Ct^{-4}$$

where C > 0 (unless one q_l equals 1 and the rest vanish)

Future

Cauchy problem

Goals

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Oscillatory vs Monotonic

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where C > 0 (unless one q_i equals 1 and the rest vanish)

"Big Bang" singularity at t = 0

Future

Hawking's incompleteness theorem

Theorem (Hawking)

Assume

(*M*, g, φ) is the maximal globally hyperbolic development of data (g, k, φ₀, φ₁) on Σ₁ ≃ T^D
 trk < -C < 0

Hawking's theorem applies to perturbations of Kasnerc:

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Hawking's incompleteness theorem

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Then no past-directed timelike geodesic emanating from Σ_1 is longer than $C'<\infty.$

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Then no past-directed timelike geodesic emanating from Σ_1 is longer than $C' < \infty$.

• Hawking's theorem applies to perturbations of Kasner: $tr \mathring{k}_{KAS} = -1$.

Cauchy problem	Goals	Kasner o	Incompleteness	Oscillatory vs Monotonic	Results	The Gauge	Proof Hints	Future o
Why?								

Glaring question:

• Why are the timelike geodesics incomplete?





Glaring question:

- Why are the timelike geodesics incomplete?
- For Kasner, incompleteness ↔ Big Bang, but what about perturbations?

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Cauchy problem

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Curvature blowup/crushing singularities à la Kasner

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Cauchy problem

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• Curvature blowup/crushing singularities à la Kasner

Oscillatory vs Monotonic

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 Cauchy horizon formation à la Kerr black hole interiors

Kasner

Incompleteness

New result with Rodnianski and Fournodavlos: Kasner Big Bang is dynamically stable assuming a sub-criticality condition:

Oscillatory vs Monotonic

Results

The Gauge

$$\max_{\substack{I,J,B=1,\cdots,D\\I < J}} \{q_I + q_J - q_B\} < 1$$

Key takeways:

- In GR, distinct kinds of incompleteness occurs in different solution regimes
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Incompleteness

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Oscillatory vs Monotonic

Results

The Gauge

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• \exists sub-critical vacuum Kasner solutions $\iff D \ge 10$ (Demaret–Henneaux–Spindel)

Key takeways:

Goals

Kasner

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Incompleteness

Cauchy problem

Goals

Kasner

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Oscillatory vs Monotonic

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Oscillatory vs Monotonic

Results

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Oscillatory vs Monotonic

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Key takeways:

Cauchy problem

Goals

Kasner

- In GR, distinct kinds of incompleteness occurs in different solution regimes
- In principle, other stable pathologies could dynamically develop in other (not-yet-understood) regimes

Kasner

Goals

Belinskii-Khalatnikov-Lifshitz considered tensorfields:

Oscillatory vs Monotonic

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Results

The Gauge

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$$\begin{split} \mathbf{g}_{BKL} &= -dt \otimes dt + \sum_{l=1}^{D} t^{2q_l(x)} dx^l \otimes dx^l, \ \phi_{BKL} = B(x) \ln t, \\ &\sum_{l=1}^{D} q_l(x) = 1, \qquad \sum_{l=1}^{D} (q_l(x))^2 = 1 - (B(x))^2 \end{split}$$

(g_{akts} ϕ_{akt}) are typically <u>not</u> solutions.

3D vacuum Kasner: Sub-criticality condition fails.

Kasner

Goals

Belinskii-Khalatnikov-Lifshitz considered tensorfields:

Oscillatory vs Monotonic

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Results

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Kasner

Goals

Belinskii–Khalatnikov–Lifshitz considered tensorfields:

Incompleteness

Oscillatory vs Monotonic

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Kasner

Cauchy problem

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Belinskii-Khalatnikov-Lifshitz considered tensorfields:

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3D vacuum Kasner: Sub-criticality condition fails.
Part of BKL saga: In 3D vacuum, near spacelike singularities, "most solutions" "should" oscillate violently in time;

Kasner

Cauchy problem

Goals

Belinskii-Khalatnikov-Lifshitz considered tensorfields:

Incompleteness

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 g_{BKL} metrics are typically at best "short-time approximations" (Kasner epochs)

Kasner

Cauchy problem

Goals

Belinskii-Khalatnikov-Lifshitz considered tensorfields:

Incompleteness

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Future

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 Fournodavlos–Luk: ∃ large family of non-oscillatory, Sobolev-class 3D Einstein-vacuum solutions that are asymptotic to g_{BKL}-type metrics;

Kasner

Cauchy problem

Goals

Belinskii-Khalatnikov-Lifshitz considered tensorfields:

Incompleteness

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 Fournodavlos–Luk: ∃ large family of non-oscillatory, Sobolev-class 3D Einstein-vacuum solutions that are asymptotic to g_{BKL}-type metrics; 3 functional degrees of freedom (compared to 4 for the Cauchy problem)

"Monotonic" regimes

Works by BK, Barrow, Demaret–Henneaux–Spindel, Andersson–Rendall,

Damour–Henneaux–Rendall–Weaver suggest that a D–dimensional Kasner Big Bang might be dynamically stable under the sub-criticality condition:

$$\max_{\substack{I,J,B=1,\cdots,D\\I < J}} \{q_I + q_J - q_B\} < 1$$

 Significance: Houristics suggest that time derivative terms will dominate; "Asymptotically Velocity Term Dominated"
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"Monotonic" regimes

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- Significance: Heuristics suggest that time derivative terms will dominate; "Asymptotically Velocity Term Dominated"
- With symmetry, stability might hold for "even more q's"
The singularity industry: A sampler

Incompleteness

Cauchy problem

Goals

Kasner

• Numerical works: e.g. Berger, Garfinkle, Isenberg, Lim, Moncrief, Weaver, ···

Oscillatory vs Monotonic

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- Symmetry: e.g. Alexakis–Fournodavlos, Chruściel–Isenberg–Moncrief, Ellis, Isenberg–Kichenassamy, Isenberg–Moncrief, Liebscher, Ringström, Wainwright, ···
- Linear: e.g. Alho-Franzen-Fournodavlos, Ringström
- **Construction of singular solutions**: e.g. Ames, Andersson, Anguige, Beyer, Choquet-Bruhat, Damour, Demaret, Fournodavlos, Henneaux, Isenberg, LeFloch, Luk, Kichenassamy, Rendall, Spindel, Ståhl, Todd, Weaver, · · ·
- Oscillatory investigations: e.g. BKL, Damour, van Elst, Heinzle, Hsu, Lecian, Liebscher, Misner, Nicolai, Uggla, Reiterer, Ringström, Tchapnda, Trubowitz, · · ·

Cauchy problem	Goals	Kasner o	Incompleteness	Oscillatory vs Monotonic	Results ●0	The Gauge	Proof Hints	Future o

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Main theorem

Theorem (JS, G. Fournodavlos, and I. Rodnianski)

If the sub-criticality condition

$$\max_{\substack{I,J,B=1,\cdots,D\\I < J}} \{q_I + q_J - q_B\} < 1$$

holds, then near its Big Bang, $\mathbf{g}_{\text{KAS}} := -dt \otimes dt + \sum_{l=1}^{D} t^{2q_l} dx^l \otimes dx^l, \ \phi_{\text{KAS}} = B \ln t \text{ is a}$ dynamically stable solution to the Einstein-scalar field system under Sobolev-class perturbations of the data on $\{t = 1\}.$

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• Relative to CMC time t (i.e., $trk|_{\Sigma_t} = -t^{-1}$):

Cauchy problem	Goals 00	Kasner o	Incompleteness	Oscillatory vs Monotonic	Results ●○	The Gauge	Proof Hints	Future o

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• Relative to CMC time t (i.e., $trk|_{\Sigma_t} = -t^{-1}$): $|k| \sim t^{-1}$, Riem_{$\alpha\beta\gamma\delta$}Riem^{$\alpha\beta\gamma\delta$} $\sim t^{-4}$, $\sqrt{|detg|} \sim t$

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Cauchy problem Goals Kasner Incompleteness Oscillatory vs Monotonic e Oscil

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 $\mathbf{g}_{KAS} := -dt \otimes dt + \sum_{l=1}^{D} t^{2q_l} dx^l \otimes dx^l$, $\phi_{KAS} = B \ln t$ is a dynamically stable solution to the Einstein-scalar field system under Sobolev-class perturbations of the data on $\{t = 1\}$.

- Relative to CMC time t (i.e., $\operatorname{tr} k|_{\Sigma_t} = -t^{-1}$): $|k| \sim t^{-1}$, Riem $_{\alpha\beta\gamma\delta}$ Riem $^{\alpha\beta\gamma\delta} \sim t^{-4}$, $\sqrt{|\det g|} \sim t$
- Lapse n := |g(Dt, Dt)|^{-1/2} solves an elliptic PDE; synchronizes the singularity. 0 shift.

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Main theorem

Theorem (JS, G. Fournodavlos, and I. Rodnianski)

If the sub-criticality condition

$$\max_{\substack{I,J,B=1,\cdots,D\\I < J}} \{q_I + q_J - q_B\} < 1$$

holds, then near its Big Bang,

 $\mathbf{g}_{KAS} := -dt \otimes dt + \sum_{l=1}^{D} t^{2q_l} dx^l \otimes dx^l$, $\phi_{KAS} = B \ln t$ is a dynamically stable solution to the Einstein-scalar field system under Sobolev-class perturbations of the data on $\{t = 1\}$.

- Relative to CMC time t (i.e., $\operatorname{tr} k|_{\Sigma_t} = -t^{-1}$): $|k| \sim t^{-1}$, Riem_{$\alpha\beta\gamma\delta$}Riem^{$\alpha\beta\gamma\delta$} $\sim t^{-4}$, $\sqrt{|\det g|} \sim t$
- Lapse n := |g(Dt, Dt)|^{-1/2} solves an elliptic PDE; synchronizes the singularity. 0 shift.

Moreover, when D = 3 and B = 0, under polarized U(1)-symmetric perturbations (i.e., $g_{13} = g_{23} \equiv 0$ and no x^3 -dependence), all Kasner Big Bangs are dynamically stable.

Cauchy problem Goals Kasner Incompleteness Oscillatory vs Monotonic e Oscil

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 Effectively covers the entire (asymmetric) regime where BK-type heuristics suggest stable blowup. Cauchy problem Goals Cauchy problem Goals Cauchy problem Goals Cauchy problem Cau

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Main theorem

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If the sub-criticality condition

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Moreover, when D = 3 and B = 0, under polarized U(1)-symmetric perturbations (i.e., $g_{13} = g_{23} \equiv 0$ and no x^3 -dependence), **all** Kasner Big Bangs are dynamically stable.

- Effectively covers the entire (asymmetric) regime where BK-type heuristics suggest stable blowup.
- Previously with Rodnianski, we had treated i) D = 3 with $q_1 = q_2 = q_3 = 1/3$. i.e. stability for FLRW; and ii) $D \ge 38$ with $\max_{l=1,\dots,D} |q_l| < 1/6$ and $\phi \equiv 0$

Cauchy problem Goals Kasner Incompleteness Oscillatory vs Monotonic Results The Gauge Proof Hints Future

The singularities in our main results are crushing:



due to blowup of $|k|^2 \sim t^{-2}$, $k := 2^{nd}$ F.F. of $\{t = \text{const}\}$

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The singularities in our main results are crushing:

$$\int_{Spacetime} |Christoffel|^2 \underbrace{dvol}_{O(t)dtdx} = |\ln(0)| = \infty$$

due to blowup of $|k|^2 \sim t^{-2}$, $k := 2^{nd}$ F.F. of $\{t = \text{const}\}$

This shows that in the chosen gauge, the solution cannot be continued weakly.

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• 0 shift decomposition: $\mathbf{g} = -n^2 dt \otimes dt + g_{ab} dx^a \otimes dx^b$

• $k_i := -g(D_{ij}, e_{ij}, \theta_j) := -\frac{1}{2}e_0g_j$ • CMC slices: $k_i^a := -t^{-1} \implies$ Elliptic PDE for n

Key new ingredient:



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0 shift decomposition: g = -n²dt ⊗ dt + g_{ab}dx^a ⊗ dx^b
e₀ := n⁻¹∂_t = unit normal to Σ_t

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• $e_0 := n^{-1} \partial_t =$ unit normal to Σ_t

•
$$k_{ij} := -\mathbf{g}(\mathbf{D}_{\partial_i} \mathbf{e}_0, \partial_j) = -\frac{1}{2} \mathbf{e}_0 g_{ij}$$

Kasner

Cauchy problem

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Future

• $e_0 := n^{-1} \partial_t =$ unit normal to Σ_t

- $k_{ij} := -\mathbf{g}(\mathbf{D}_{\partial_i} \mathbf{e}_0, \partial_j) = -\frac{1}{2} \mathbf{e}_0 g_{ij}$
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Future

• $e_0 := n^{-1} \partial_t =$ unit normal to Σ_t

- $k_{ij} := -\mathbf{g}(\mathbf{D}_{\partial_i} \mathbf{e}_0, \partial_j) = -\frac{1}{2} \mathbf{e}_0 \mathbf{g}_{ij}$
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• 0 shift decomposition: $\mathbf{g} = -n^2 dt \otimes dt + g_{ab} dx^a \otimes dx^b$

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Incompleteness

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- CMC slices: $k_a^a = -t^{-1} \implies$ Elliptic PDE for *n*

Key new ingredient:

Fermi-Walker-propagated Σ_t -tangent orthonormal spatial frame $\{e_l\}_{l=1,\dots,D}$; with $e_l = e_l^c \partial_c$:

$$e_0 e_l^i = k_{lC} e_C^i$$

Recast Einstein's equations as an elliptic-hyperbolic PDE system for scalar frame-component functions

The unknowns are:

- The lapse n
- Spatial connection coefficients γ_{UB} := g(∇_n, e_J, e_B)
 b₁ := k₁, e^c e^f
- The coordinate components $\{g_i^{\prime}\}_{i,i=1,\cdots,N}$ where $a_i=a_i^{\prime}\partial_i$

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 $e_{0}\phi$ and $e_{\ell}\phi$ if scalar field is present

Proof philosophy

Recast Einstein's equations as an elliptic-hyperbolic PDE system for scalar frame-component functions

The unknowns are:

- The lapse *n*
- Spatial connection coefficients $\gamma_{IJB} := g(\nabla_{e_I} e_J, e_B)$
- $k_{IJ} := k_{cd} e_I^c e_J^d$
- The coordinate components $\{e_l^i\}_{l,i=1,\dots,D}$, where $e_l = e_l^c \partial_c$

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Proof philosophy

Recast Einstein's equations as an elliptic-hyperbolic PDE system for scalar frame-component functions

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- Spatial connection coefficients γ_{IJB} := g(∇_{e_I}e_J, e_B)
- $k_{IJ} := k_{cd} e_I^c e_J^d$
- The coordinate components $\{e_l^i\}_{l,i=1,\dots,D}$, where $e_l = e_l^c \partial_c$
- $e_0\phi$ and $e_I\phi$ if scalar field is present

Einstein-vacuum equations in our gauge

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Evolution equations

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$$\partial_t k_{IJ} = -\frac{n}{t} k_{IJ} - e_I e_J n + n e_C \gamma_{IJC} - n e_I \gamma_{CJC}$$

$$+ \gamma_{IJC} e_C n - n \gamma_{DIC} \gamma_{CJD} - n \gamma_{DDC} \gamma_{IJC},$$

$$\partial_t \gamma_{IJB} = n e_B k_{IJ} - n e_J k_{BI}$$

$$- n k_{IC} \gamma_{BJC} + n k_{IC} \gamma_{JBC} + n k_{IC} \gamma_{CJB}$$

$$- n k_{CJ} \gamma_{BIC} + n k_{BC} \gamma_{JIC}$$

$$+ (e_B n) k_{IJ} - (e_J n) k_{BI}$$

Elliptic lapse PDE

$$\begin{split} e_{C}e_{C}(n-1) - t^{-2}(n-1) &= \gamma_{CCD}e_{D}(n-1) + 2ne_{C}\gamma_{DDC} \\ &- n\left\{\gamma_{CDE}\gamma_{EDC} + \gamma_{CCD}\gamma_{EED}\right\} \end{split}$$

Constraint equations

$$\begin{split} k_{CD}k_{CD} - t^{-2} &= 2e_C\gamma_{DDC} - \gamma_{CDE}\gamma_{EDC} - \gamma_{CCD}\gamma_{EED}, \\ e_Ck_{CI} &= \gamma_{CCD}k_{ID} + \gamma_{CID}k_{CD} \end{split}$$

Elliptic PDE $\Delta_q n = \cdots$ synchronizes singularity

• $S_{i,a} := g([a_i, a_i], a_i) = \gamma_{i,a} = \gamma_{i,a}$ • Diagonal structure: $\partial_i S_{i,a} + \frac{1}{2} (\underline{q}_i + \underline{q}_i - \underline{q}_a) S_{i,a} = PDE Error Terms$ a_i • $\longrightarrow mp_i |S_{i,a}| \lesssim 1^{-4}, q := c + mp_i (\underline{q}_i + \underline{q}_i - \underline{q}_a).$ • Integrability: T^{-4} is integrable in time near t = 0.

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Kasner

- PDE $e_0 e'_i = k_0 e'_0$ suggests e_i is as regular as k_0 • However: special structure of Einstein's equations $\longrightarrow \gamma_{LR} := g(\nabla_{e_i} e_i, e_i)$ is as regular as k_0 .
 - \implies Gain of one derivative for eq.

Goals

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Elliptic PDE $\Delta_g n = \cdots$ synchronizes singularity

Results

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Singularity strength via structure coefficients:

•
$$S_{IJB} := \mathbf{g}([e_I, e_J], e_B) = \gamma_{IJB} - \gamma_{JIB}$$

 $\circ \longrightarrow \max_{n,n} |S_{un}| \lesssim t^{-q}, q := c + \max_{n,n} (q) + q_t = q_n$ \circ Integrability: t^{-q} is integrable in time near t = 0.

• PDE $e_0 e'_1 = k_0 e'_0$ suggests e_1 is as regular as k_0 • However: special structure of Einstein's equations $\implies \gamma_{20} = g(\nabla_{e_1}e_1, e_0)$ is as regular as k_0 .

 \implies Gain of one derivative for e_l

Elliptic PDE $\Delta_q n = \cdots$ synchronizes singularity

Singularity strength via structure coefficients:

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$$S_{IJB} := \mathbf{g}([e_I, e_J], e_B) = \gamma_{IJB} - \gamma_{JIB}$$

Diagonal structure:

Goals

Kasner

$$\partial_t S_{IJB} + \frac{1}{t} \underbrace{(q_I + q_J - q_B)}_{<1} S_{IJB} = \mathsf{PDE} \; \mathsf{Error} \; \mathsf{Terms}$$

Oscillatory vs Monotonic

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The Gauge

• PDE $e_0 e'_0 = h_0 e'_0$ suggests e_0 is as regular as h_0 • However: special structure of Einstein's equations $\Rightarrow \gamma_{00} = g(\nabla_{e_1} e_0)$ is as regular as h_0 .

Elliptic PDE $\Delta_g n = \cdots$ synchronizes singularity

Singularity strength via structure coefficients:

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$$S_{IJB} := \mathbf{g}([e_I, e_J], e_B) = \gamma_{IJB} - \gamma_{JIB}$$

Diagonal structure:

Goals

Kasner

$$\partial_t S_{IJB} + \frac{1}{t} \underbrace{(q_I + q_J - q_B)}_{\leq lJB} S_{IJB} = \mathsf{PDE} \text{ Error Terms}$$

$$\bullet \implies \max_{I,J,B} |S_{IJB}| \lesssim t^{-q}, \ q := \epsilon + \max_{I,J,B} (q_I + q_J - q_B).$$

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 PDE e₀e'₀ = k₁₀e'₀ suggests e₀ is as regular as k₀.
 However: special structure of Einstein's equations → γ₁₀e = g(∇_{i0}e₃, e₀) is as regular as k₀.

Incompleteness

Elliptic PDE $\Delta_g n = \cdots$ synchronizes singularity

Singularity strength via structure coefficients:

•
$$S_{IJB} := \mathbf{g}([e_I, e_J], e_B) = \gamma_{IJB} - \gamma_{JIB}$$

Diagonal structure:

Cauchy problem

Goals

Kasner

$$\partial_t S_{IJB} + \frac{1}{t} \underbrace{(q_I + q_J - q_B)}_{<1} S_{IJB} = \mathsf{PDE} \; \mathsf{Error} \; \mathsf{Terms}$$

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$$\implies \max_{I,J,B} |S_{IJB}| \lesssim t^{-q}, \ \boldsymbol{q} := \epsilon + \max_{I,J,B} (\boldsymbol{q}_I + \boldsymbol{q}_J - \boldsymbol{q}_B).$$

• Integrability: t^{-q} is integrable in time near t = 0.

• PDE $e_0 e'_1 = h_0 e'_0$ suggests e_1 is as regular as $k_0 = 0$. • However: special structure of Einstein's equations $\Rightarrow \gamma_{00} = g(\nabla_{e_1} e_2, e_3)$ is as regular as k_0 .

Elliptic PDE $\Delta_g n = \cdots$ synchronizes singularity

Singularity strength via structure coefficients:

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$$S_{IJB} := \mathbf{g}([e_I, e_J], e_B) = \gamma_{IJB} - \gamma_{JIB}$$

Diagonal structure:

$$\partial_t S_{IJB} + \frac{1}{t} \underbrace{(q_I + q_J - q_B)}_{\leq 1} S_{IJB} = \mathsf{PDE} \; \mathsf{Error} \; \mathsf{Terms}$$

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$$\bullet \implies \max_{I,J,B} |S_{IJB}| \lesssim t^{-q}, \ \boldsymbol{q} := \epsilon + \max_{I,J,B} (\boldsymbol{q}_I + \boldsymbol{q}_J - \boldsymbol{q}_B).$$

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Regularity

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Cauchy problem

Goals

Kasner

• PDE $e_0 e_l^i = k_{lC} e_C^i$ suggests e_l is as regular as k_{lJ}

Elliptic PDE $\Delta_g n = \cdots$ synchronizes singularity

Singularity strength via structure coefficients:

•
$$S_{IJB} := \mathbf{g}([e_I, e_J], e_B) = \gamma_{IJB} - \gamma_{JIB}$$

Diagonal structure:

$$\partial_t S_{IJB} + \frac{1}{t} \underbrace{(q_I + q_J - q_B)}_{<1} S_{IJB} = \mathsf{PDE} \; \mathsf{Error} \; \mathsf{Terms}$$

Oscillatory vs Monotonic

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Future

•
$$\implies \max_{I,J,B} |S_{IJB}| \lesssim t^{-q}, \ q := \epsilon + \max_{I,J,B} (q_I + q_J - q_B).$$

• Integrability: t^{-q} is integrable in time near t = 0.

Regularity

Cauchy problem

Goals

Kasner

- PDE $e_0 e_l^i = k_{lC} e_C^i$ suggests e_l is as regular as k_{lJ}
- However: special structure of Einstein's equations $\implies \gamma_{IJB} := g(\nabla_{e_I} e_J, e_B)$ is as regular as k_{IJ} .

Incompleteness

Elliptic PDE $\Delta_g n = \cdots$ synchronizes singularity

Singularity strength via structure coefficients:

- $S_{IJB} := \mathbf{g}([e_I, e_J], e_B) = \gamma_{IJB} \gamma_{JIB}$
- Diagonal structure:

$$\partial_t S_{IJB} + \frac{1}{t} \underbrace{(q_I + q_J - q_B)}_{<1} S_{IJB} = \mathsf{PDE} \; \mathsf{Error} \; \mathsf{Terms}$$

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The Gauge

Future

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$$\implies \max_{I,J,B} |S_{IJB}| \lesssim t^{-q}, \ q := \epsilon + \max_{I,J,B} (q_I + q_J - q_B).$$

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Regularity

Cauchy problem

Goals

Kasner

- PDE $e_0 e_l^i = k_{lC} e_C^i$ suggests e_l is as regular as k_{lJ}
- However: special structure of Einstein's equations
 ⇒ γ_{IJB} := g(∇_{e_l}e_J, e_B) is as regular as k_{IJ}.
 ⇒ Gain of one derivative for e_l

Goals

Kasner

The hard part is showing that the solution exists all the way to t = 0. The key is to prove: $|tk_{IJ}(t, x)|$ is bounded.

Oscillatory vs Monotonic

Results

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Proof Hints

- Low-norm bootstrap assumptions (slightly worse than Kasner): $\|G_{1,2}^{(1)}\| \ge \|G_{1,2}^{(2)}\| \le \|G_$
- N and A are parameters, with A large and N chosen large relative to A
- chosen small relative to N and A
- \sim Interpolation: $||\sigma_{ij}||_{L^{\infty}(\Omega_{ij})} \lesssim c^{1-(2n+\delta)}$, where $c^{1-(2n+\delta)}$, $||\sigma_{ij}||_{L^{\infty}(\Omega_{ij})}$, $||\sigma_{ij}||_{L^{\infty}(\Omega_{ij})}$.
- $\partial_t(k_0) = ie_{tY} + i_{Y'} \cdot Y + \cdots \leq e^{1-(2q+2\delta)}$

Goals

Kasner

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Proof Hints

•
$$\sigma > 0$$
 small, $q := \max_{I,J,B} (q_I + q_J - q_B) + \sigma < 1$

 $\label{eq:constraint} \begin{array}{l} \|A\|_{\mathcal{M}(\mathbf{p}_0)} \leq d^{-(A+1)} \|A\|_{\mathcal{M}(\mathbf{p}_0)} \leq d^{-(A+1)} \\ & \mbox{ and } A \mbox{ are parameters, with } A \mbox{ large and } N \mbox{ chosen large relative to } A \end{array}$

- chosen small relative to N and A
- * Interpolation: $||e_{\ell'}||_{\ell^{\infty}(\mathbb{R}^3)} \lesssim c\ell^{-(2q+\delta)}$, where
- $=\partial_i(b_{ij}) = ie_i\gamma + i\gamma \cdot \gamma + \dots \leq d^{1-(e_0+\infty)}$

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Cauchy problem

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- $\sigma > 0$ small, $q := \max_{I,J,B} (q_I + q_J q_B) + \sigma < 1$
- Low-norm bootstrap assumptions (slightly worse than Kasner): ||eⁱ_l||_{L[∞](Σt)} ≤ t^{-q}, ||γ||_{L[∞](Σt)} ≤ εt^{-q}
- M and A are parameters, with A large and N chosen large relative to A
- chosen small relative to M and A.
- * Interpolation: $\| \varphi_{\mathcal{H}} \|_{L^{\infty}(\Sigma_{\ell})} \lesssim e^{-i2\phi(\theta)}$, where
- $\delta = \delta(N, A) = 0$
- $O_{i}(R_{0,i}) = IO_{i}(i+1) \cdots O_{i}(i+1) \cdots O_{i}(i+1)$

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Cauchy problem

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- Low-norm bootstrap assumptions (slightly worse than Kasner): ||e_lⁱ||_{L[∞](Σ_t)} ≤ t^{-q}, ||γ||_{L[∞](Σ_t)} ≤ εt^{-q}
- High-norm bootstrap assumptions: $\|e_l^i\|_{\dot{H}^N(\Sigma_t)} \leq t^{-(A+q)}$, $\|k\|_{\dot{H}^N(\Sigma_t)} \leq \epsilon t^{-(A+1)}$, $\|\gamma\|_{\dot{H}^N(\Sigma_t)} \leq \epsilon t^{-(A+1)}$

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Cauchy problem

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Proof Hints

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- High-norm bootstrap assumptions: $\|e_l^i\|_{\dot{H}^N(\Sigma_t)} \leq t^{-(A+q)}$, $\|k\|_{\dot{H}^N(\Sigma_t)} \leq \epsilon t^{-(A+1)}$, $\|\gamma\|_{\dot{H}^N(\Sigma_t)} \leq \epsilon t^{-(A+1)}$
- N and Å are parameters, with A large and N chosen large relative to A
- ϵ chosen small relative to N and A

- Interpolation: $\|e_l\gamma\|_{L^{\infty}(\Sigma_t)} \lesssim \epsilon t^{-(2q+\delta)}$, where
 - $\delta = \delta(N, A) \rightarrow 0$ as $N \rightarrow \infty$ with A fixed

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Cauchy problem

The hard part is showing that the solution exists all the way to t = 0. The key is to prove: $|tk_{IJ}(t, x)|$ is bounded.

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 small, $q := \max_{I,J,B} (q_I + q_J - q_B) + \sigma < 1$

- Low-norm bootstrap assumptions (slightly worse than Kasner): ||e_lⁱ||_{L[∞](Σ_t)} ≤ t^{-q}, ||γ||_{L[∞](Σ_t)} ≤ εt^{-q}
- High-norm bootstrap assumptions: $\|e_l^i\|_{\dot{H}^N(\Sigma_t)} \leq t^{-(A+q)}$, $\|k\|_{\dot{H}^N(\Sigma_t)} \leq \epsilon t^{-(A+1)}$, $\|\gamma\|_{\dot{H}^N(\Sigma_t)} \leq \epsilon t^{-(A+1)}$
- N and Å are parameters, with A large and N chosen large relative to A
- ϵ chosen small relative to N and A

- Interpolation: $\|e_{i\gamma}\|_{L^{\infty}(\Sigma_{t})} \lesssim \epsilon t^{-(2q+\delta)}$, where $\delta = \delta(N, A) \to 0$ as $N \to \infty$ with A fixed
- $\partial_t(tk_{IJ}) = te_I\gamma + t\gamma \cdot \gamma + \cdots \lesssim \epsilon t^{1-(2q+2\delta)}$

Goals

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The hard part is showing that the solution exists all the way to t = 0. The key is to prove: $|tk_{IJ}(t, x)|$ is bounded.

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$$\sigma > 0$$
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- $\partial_t(tk_{IJ}) = te_I\gamma + t\gamma \cdot \gamma + \cdots \lesssim \epsilon t^{1-(2q+2\delta)}$
- Thus, integrability of $t^{1-(2q+2\delta)}$ (for large *N*) implies that for $t \in (0, 1]$: $|tk_{IJ}(t, x) - k_{IJ}(1, x)| \leq \epsilon$
Kasner

Goals

• Similar argument $\implies \exists \kappa_{IJ}^{(\infty)}(x)$ such that $\left| t k_{IJ}(t,x) - \kappa_{IJ}^{(\infty)}(x) \right| \rightarrow 0$ as $t \downarrow 0$.

- The $\{q_i^{(m)}(x_i)\}_{i=1,...,d}$ are the "asymptotic Kasner exponents" of the perturbed solution.
- The set of "limiting end states" is infinite-dimensional.

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- Our proof does not suggest that t-rescaled versions of the component functions e(t, x) should have finite, non-trivial limits as t 1.0.
- $k_{ij} = t_{k_{ij}} e_{j}^{*} e_{j}^{*}^{*}$ converges, but $t_{k_{j}}$ might not.

Kasner

Incompleteness

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- Similar argument $\implies \exists \kappa_{IJ}^{(\infty)}(x)$ such that $\left| tk_{IJ}(t,x) \kappa_{IJ}^{(\infty)}(x) \right| \rightarrow 0$ as $t \downarrow 0$.
- Eigenvalues of the symmetric matrix (κ_l(x))_{l,J=1,...,D} are functions {q_l^(∞)(x)}_{l=1,...,D} on T^D.

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Kasner

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Cauchy problem

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Kasner

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Incompleteness

We prove that for $t \in (0, 1]$, we have:

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$$\begin{split} \|t^{A+1}k\|_{\dot{H}^{N}(\Sigma_{t})}^{2} + \|t^{A+1}\gamma\|_{\dot{H}^{N}(\Sigma_{t})}^{2} \\ &\leq \mathsf{Data} \\ &+ \{C_{\star} - A\} \int_{t}^{1} s^{-1} \left\{ \|s^{A+1}\gamma\|_{\dot{H}^{N}(\Sigma_{s})}^{2} + \|s^{A+1}k\|_{\dot{H}^{N}(\Sigma_{s})}^{2} \right\} ds \\ &+ \cdots, \end{split}$$

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Proof Hints

where

- C_{*} can be large but is independent of N and A
- ··· denotes time-integrable error terms

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where

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- Large $A \implies$ very singular top-order energy estimates

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Problems to think about

• What happens in the presence of "timelike" matter (e.g. fluid)?

Problems to think about

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• What happens in the presence of "timelike" matter (e.g. fluid)?

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Future

 What can be proved outside of the "monotonic" regime?