Advances in the theory of multi-dimensional shock waves for the compressible Euler equations

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1 Compressible fluids & shocks

In 1757, Euler formulated a now-famous set of evolution PDEs, presently known as the *compressible Euler equations*, modeling the motion of a compressible fluid without viscosity (with viscosity = the compressible Navier–Stokes equations). The equations remain a fascinating source of challenging mathematical problems connecting many branches of mathematics, including analysis, PDE, geometry, and topology. There are also related fluid PDEs accounting for relativistic effects, the *relativistic Euler equations*, and they are a fundamental model in cosmology and astrophysics. While much has been rigorously understood about fluids in the idealized setting of $1D$ (one spatial dimension), much less is known about multi-dimensional fluids, the main subject of this article.

The compressible Euler equations are quasilinear hyperbolic conservation laws, i.e., they can be written in divergence form, leading (upon integrating over space and using the divergence theorem) to conserved quantities including total mass, momentum, and energy. They have a well-posed initial value problem formulation, which means that given suitably regular initial data (say at time 0), one can uniquely solve the equations for a regular solution, at least for short times. Importantly, the equations exhibit *finite speed of propagation*, which means that the fluid's future state is locally determined only by nearby points in the present.

"Compressible" means that the fluid can be squished, and squishy-ness lies behind a fascinating phenomenon: *shock singularities*. This means that when there is sufficient compression, then even starting from very smooth initial conditions, the fluid variables can develop infinite gradient singularities while remaining bounded. In particular, finite-time blowup can occur, despite the conserved quantities. The mathematical problem of describing this kind of blowup is known as the *shock formation problem*. Starting from a suitable set of blowup-points known as a "first singularity," one can reformulate the fluid flow in a *weak form*, essentially by demanding that an integrated version of the conservation laws should hold, even in the presence of singularities (integrals of functions with gradient singularities can still make sense, even when the PDEs don't!). In weak form, one finds that from the first singularity, a super-sonic (faster than sound waves) "shock hypersurface" can emerge, across which the fluid experiences a jump discontinuity. Such discontinuous super-sonic solutions are known as *shock waves*, and they are associated with natural phenomena such as sonic booms and the crack of a whip; see Sect. 4 for further discussion.

In total, the picture described above is one of a fluid that starts out smooth, then develops a gradient singularity, which in turn engenders a discontinuous shock wave solution. In $1D$, the dynamics are relatively well understood because they fall under the scope of one-dimensional hyperbolic conservation law theory, for which a robust set of mathematical tools has been developed [Daf10]. In multiple spatial di-

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mensions, the theory is much less understood, and parts of the picture described above have not been rigorously proved, even for short amounts of time. Nevertheless, as we will describe, there has been dramatic progress on multi-dimensional shocks.

Fascinatingly, another kind of fluid singularity can occur, one known as an *implosion*, in which the density blows up in finite time. These were first discovered in the breakthrough work [MRRS22], which showed that implosions can develop from C^∞ initial data. Shocks are of special interest because they tend to be stable, while implosions are conjectured to be unstable. This article is focused on shocks.

1.1 A first-order formulation

For $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, the $3D$ compressible Euler equations can be formulated as a first-order PDE system in the velocity $v : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, the density $\varrho : \mathbb{R} \times \mathbb{R}^3 \to [0,\infty)$, and the entropy $s : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, which accounts for thermodynamics. Euler's original formulation of the equations was under-determined, but the theory of thermodynamics developed in the 1800's supplied the key missing ingredient: the notion of an *equation of state*, which postulates the fluid pressure p as a function of ϱ and s, i.e., $p = p(\varrho, s)$. The *speed* of sound, defined by $c := \sqrt{\frac{\partial p}{\partial \varrho}}$, is a fundamental quantity in the analysis. We will consider only the non-degenerate regime in which $c > 0$. Relative to standard "Cartesian coordinates" $x^0 := t, x^1, x^2, x^3$ on "Cartesian space" $\mathbb{R}^{1+3}$, the equations can be written as follows,[1] where $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$:

$$\partial_t \varrho + \partial_a(\varrho v^a) = 0, \tag{1a}$$

$$\varrho \mathbf{B} v^i = -\partial_i p, \qquad (i = 1,2,3) \tag{1b}$$

$$\mathbf{B} s = 0. \tag{1c}$$

Above,

$$\mathbf{B} := \partial_t + v^a \partial_a, \tag{2}$$

is the *material derivative vectorfield*, and throughout $\mathbf{X} f := \mathbf{X}^\alpha \partial_\alpha$ denotes the derivative of the function f in the direction of the vectorfield $\mathbf{X}$. Importantly, as we mentioned earlier, there is a divergence-form version of the equations, which is equivalent to (1a)–(1c) for C^1 solutions:

$$\partial_t \varrho + \partial_a(\varrho v^a) = 0, \tag{3a}$$

$$\partial_t(\varrho v^i) + \partial_a(\varrho v^a v^i) + \partial_i p = 0, \quad (i = 1,2,3), \tag{3b}$$

$$\partial_t E + \partial_a \{(E + p) v^a\} = 0, \tag{3c}$$

where $E = \varrho e + \frac{1}{2}\varrho |v|^2$ is the total energy, e is the specific internal energy per particle, related to entropy by the 2^{nd} law of thermodynamics: $de = T ds - p d(\frac{1}{\varrho})$, where the temperature T is determined by the equation of state.

1.2 Sound waves, transporting, and the acoustical metric

The compressible Euler equations exhibit two kids of phenomena: *sound waves*, and transporting, as is indicated e.g. by the "transport equation" (1c). Solutions such that $\nabla \times v = 0$ and $s \equiv$ constant are known as *irrotational and isentropic* and are much easier to study because the transport phenomena are absent, and there are only sound waves. In this case, one can introduce a potential function Φ, and it can be shown that v and ϱ are completely determined by its spacetime gradient $\boldsymbol{\partial}\Phi$ (the condition $s \equiv$ constant is preserved by equation (1c)). One can show that the dynamics reduces to a single equation:

$$(\mathbf{g}^{-1})^{\alpha\beta}(\boldsymbol{\partial}\Phi)\partial_\alpha \partial_\beta \Phi = 0, \tag{4}$$

where $\mathbf{g}$ is the *acoustical metric*, a solution-dependent Lorentzian metric (a bilinear form of signature $(-,+,+,+)$) determined by the equation of state. Equation (4) is a quasilinear (i.e., nonlinear but linear in the second derivatives of Φ) PDE of wave type, i.e., sound waves are modeled by (4). The acoustical metric is fundamental even outside of the irrotational and isentropic class of solutions, and generally takes the following form:

$$\mathbf{g} := -\mathrm{d}t \otimes \mathrm{d}t + c^{-2} \sum_{a=1}^{3} (\mathrm{d}x^a - v^a \mathrm{d}t) \otimes (\mathrm{d}x^a - v^a \mathrm{d}t). \tag{5}$$

[1] We use Einstein summation convention in that repeated adjacent indices are summed over their respective ranges, which is $1-3$ for Latin indices and $0-3$ for Greek indices.

2 Shock formation & geometry

2.1 Model problems

Shock formation is intimately connected to the blowup that occurs in solutions to Riccati's ODE:

$$\dot{y} = -y^2, \qquad y(0) = y_0, \tag{6}$$

which has the explicit solution $y = \frac{y_0}{1-ty_0}$ that develops a singularity at $t = \frac{1}{y_0}$. The connection between shocks and (6) can be readily seen in Burgers' equation[2], a toy model on $\mathbb{R}^{1+1}$ (equipped with standard coordinates (t, x)):

$$\partial_t \Psi(t,x) + (1 + \Psi(t,x))\partial_x \Psi = 0, \quad \Psi(0,x) = \mathring{\Psi}(x), \tag{7}$$

where $\mathring{\Psi}(x)$ is given initial data. Taking a ∂_x derivative of (7), we deduce that

$$L\partial_x \Psi = -(\partial_x \Psi)^2, \tag{8}$$

where

$$L := \partial_t + (1 + \Psi)\partial_x \tag{9}$$

is the (solution-dependent) *characteristic vectorfield*, whose integral curves are called *characteristics*. Equation (8) is of Riccati-type in $\partial_x \Psi$ and admits many solutions that blow up along the integral curves of L. On the other hand, (7) is equivalent to $L\Psi = 0$, so Ψ is constant along the integral curves of L and never blows up; this is the crudest picture of the formation of a shock. Burgers' equation also allows for an unusually easy illustration of finite speed of propagation, as each characteristic – as well as Ψ along it – is completely determined by the initial condition of Ψ at the point where the curve hits the x axis.

The blowup of $\partial_x \Psi$ is tied to the infinite density of the characteristics. In Fig. 1, we have exhibited two families of characteristics forming infinite density on the *singular boundary* $\mathcal{B}_1 \cup \mathcal{B}_2$ (the red curve), which has two branches ($\mathcal{B}_1$ and $\mathcal{B}_2$) separated by a cusp. Since Ψ is constant along the integral curves of L, Fig. 1 also exhibits *non-uniqueness of classical solutions* in regions where the characteristics cross.

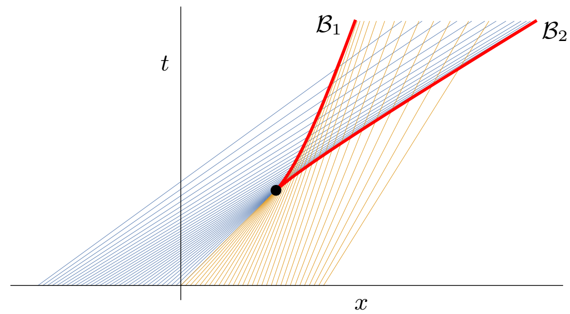

Figure 1: A singular curve $\mathcal{B}_1 \cup \mathcal{B}_2$ for Burgers' equation in Cartesian coordinates for data $\mathring{\Psi}(x) = -x + \frac{1}{3}x^3$

2.2 Nonlinear geometric optics for Burgers' equation

We now provide a second description of the Burgers' equation blowup, one that is much closer in spirit to techniques that have proven successful in multi-dimensions. We first implement *nonlinear geometric optics* via an eikonal function u, where L is as in (9):

$$Lu = 0, \qquad u(0,x) = -x. \tag{10}$$

We refer to (t, u) as "geometric coordinates" and denote the corresponding partial derivatives by $\frac{\partial}{\partial t}, \frac{\partial}{\partial u}$. Since $Lt = 1$ and $Lu = 0$, we must have $L = \frac{\partial}{\partial t}$. Hence, in geometric coordinates, equation (7) becomes a *linear PDE*:

$$\frac{\partial}{\partial t}\Psi(t,u) = 0, \quad \Psi(0,u) = \Psi_0(u) := \mathring{\Psi}(-u). \tag{11}$$

The solution is $\Psi(t,u) = \Psi_0(u)$. In particular, in geometric coordinates, *smooth initial data gives rise to eternally smooth solutions.* A similar phenomenon occurs in multi-D, which is what makes the PDE analysis tractable in suitable geometric coordinates.

[2] Burgers' equation is usually written as $\partial_t \Psi + \Psi \partial_x \Psi = 0$ instead of (7). The difference is harmless for the purposes of this article, and we included the extra 1 factor only to tilt Fig.1 for easier comparisons with the compressible Euler equations.

To see the singularity in the original (t,x) coordinates, we introduce the *inverse foliation density*:

$$\mu := -\frac{1}{\partial_x u}. \qquad (12)$$

When $\mu = 0$, the density of the characteristics is *infinite* and a shock singularity has formed. In particular, $\mu = 0$ along the singular curve $\mathcal{B}_1 \cup \mathcal{B}_2$ in Fig. 1. From (12), one easily deduces the change of variables relation:

$$\partial_x = -\frac{1}{\mu}\frac{\partial}{\partial u}. \qquad (13)$$

By differentiating (10) with ∂_x and using (7) and (10), one can show that in geometric coordinates, μ satisfies a transport equation:

$$\frac{\partial}{\partial t}\mu(t,u) = -\frac{\partial}{\partial u}\Psi(t,u) = -\frac{d}{du}\Psi_0(u), \quad \mu(0,u) = 1. \qquad (14)$$

From (14), one readily deduces that $\mu(t,u)$ vanishes in finite time at values of u such that $\frac{d}{du}\Psi_0(u) > 0$, and then (13) implies that $\partial_x \Psi$ must blow up "like $1/\mu$" at such points. Note that the ∂_x derivative is transverse to the characteristics, while L is tangent. Thus, the derivatives of Ψ in directions transverse to the characteristics blow up, while equation (7) is equivalent to $L\Psi = 0$ and thus Ψ remains smooth in directions tangent to the characteristics. Crucially, these phenomena are also present in the multi-dimensional fluid flows.

2.3 Riemann invariants for $1D$ isentropic compressible Euler flow

In the foundational paper [Rie60], Riemann studied $1D$ solutions to (1a)–(1c) with $s \equiv 0$, where "$1D$ means that $v^2 = v^3 \equiv 0$ and ρ, v^1 are functions of $(t,x^1) \in \mathbb{R}\times\mathbb{R}$. He showed that for C^1 solutions, the equations are equivalent to:

$$L\mathcal{R}_{(+)} = 0, \qquad \underline{L}\mathcal{R}_{(-)} = 0, \qquad (15)$$

where $\mathcal{R}_{(\pm)} := v^1 \pm F(\varrho)$ are the *Riemann invariants*, F is an invertible function determined by the equation of state, and

$$L := \partial_t + (v^1 + c)\partial_1, \quad \underline{L} := \partial_t + (v^1 - c)\partial_1 \qquad (16)$$

are explicitly determined *characteristic* vectorfields. L and $\underline{L}$ are intimately connected to the $1D$ acoustical metric, i.e., (5) with the x^2 and x^3 directions suppressed. One can compute that:

$$(\mathbf{g}^{-1})^{\alpha\beta} = -\frac{1}{2}L^\alpha \underline{L}^\beta - \frac{1}{2}\underline{L}^\alpha L^\beta, \quad (\alpha,\beta = 0,1). \qquad (17)$$

(15) can be thought of as a system of Burgers'-like equations. Even though there are two directions of propagation, the methods of Sect. 2.2 can be used to prove shock formation for data that are dominated by one Riemann invariant, e.g., data for which the gradient of $\mathcal{R}_{(+)}$ is large while that of $\mathcal{R}_{(-)}$ is small; see Fig. 5. In particular, one can carry out the analysis with the help of an L-adapted eikonal function, i.e., a solution to $Lu = 0$, which in view of (17), is a solution to the (acoustic) *eikonal equation*, which has been used extensively in mathematical general relativity (GR):

$$(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0. \qquad (18)$$

Equation (18) plays a fundamental role in the study of multi-dimensional shocks. Importantly, because $\mathcal{R}_{(-)}$ propagates in a different direction than $\mathcal{R}_{(+)}$, one can show that for $\mathcal{R}_{(+)}$-dominated initial data, $\partial_1\mathcal{R}_{(-)}$ remains bounded at the points where $\partial_1\mathcal{R}_{(+)}$ blows up, i.e., $\mathcal{R}_{(-)}$ is more regular than $\mathcal{R}_{(+)}$.

2.4 Multi-dimensional irrotational and isentropic shock formation

The biggest difference between proving shock formation in $1D$ versus multi-dimensions is that all known multi-dimensional well-posedness results (i.e., existence, uniqueness, and continuous dependence on initial conditions) rely on *energy estimates*, leading to well-posedness in L^2-type Sobolev spaces. In particular, Rauch showed [Rau86] that Bounded Variation function spaces, which form the crux of many aspects of $1D$ theory [Daf10], cannot be used in multi-dimensions. Energies are analogs of the quantity $\int_{\mathbb{R}^3}\left\{(\partial_t\Psi(t,x))^2 + |\nabla\Psi(t,x)|^2\right\} \mathrm{d}^3x$, which, upon time-differentiating under the integral, integrating by parts, can be shown to be constant in time for solutions Ψ to the $3D$ linear wave equation

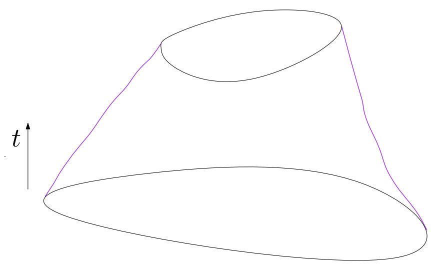

Figure 2: A domain of dependence on which energy identities hold

$-\partial_t^2\Psi + \Delta\Psi = 0$. The key point is that deriving energy estimates up to a singularity can be very hard, and this difficulty is absent in $1D$ because energy estimates can be avoided. Importantly, "energy identities" can be localized to "domains of dependence," a manifestation of finite speed of propagation that is schematically depicted in Fig. 2.

Alinhac was the first to prove stable shock multidimensional formation [Ali99]. He studied equations of type (4) and a used an eikonal function, i.e., a solution u to (18), with $\mathbf{g}(\boldsymbol{\partial}\Phi)$ the Lorentzian metric in (4), as well as a related geometric coordinate system in the spirit of the (t, u)-one from Sect. 2.2. It turns out that it is very difficult to derive energy estimates in geometric coordinates due to "change of variables" terms that appear to "lose derivatives," i.e., have insufficient regularity. The difficulty is that the regularity of u is limited by the regularity of the principal coefficients $\mathbf{g}(\boldsymbol{\partial}\Psi)$ in equation (4), and the change of variables between geometric and standard Cartesian coordinates already depends on one derivative of u. Nevertheless, Alinhac was able to avoid losing derivatives by invoking a Nash–Moser iteration scheme. In total, he was able to follow the solution up to the constant-time hypersurface of first blowup, but not further, and his proof worked only in "non-degenerate" situations such that there is only a single blowup-point at the time of first blowup.

The next big breakthrough was by Christodoulou, whose monumental monograph [Chr07] concerned irrotational and isentropic perturbations of the non-vacuum constant-state solutions[3] to the $3D$ relativistic Euler equations. Due to the irrotationality and isentropicity, the flow is described by a wave equation of type (4). Like Alinhac, Christodoulou used an eikonal function, i.e., a solution u to (18), with $\mathbf{g}$ a relativistic analog of the acoustical metric (5). This allowed him, in particular, to construct a multi-dimensional analog of the $1D$ characteristic vectorfield L featured in (16), and it played a fundamental role in his PDE analysis. Crucially, in multi-dimensions, L is no longer explicitly given, but rather is proportional to the gradient vectorfield $-(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta u$ (normalized such that $Lt = 1$) and thus is determined by the acoustical metric (which depends on the fluid) and u. As a consequence of the eikonal equation (18), L is null, i.e., $\mathbf{g}(L, L) := \mathbf{g}_{\alpha\beta}L^\alpha L^\beta = 0$, and thus the level sets of u are *null hypersurfaces.* Moreover, owing in part to his experience using eikonal functions in mathematical GR, notably in the proof of the global stability of Minkowski space [CK90], Christodoulou found a sharper, more geometric way to derive energy estimates up to the singularity without relying on Nash–Moser estimates. This allowed him to prove that the non-vacuum constant-state solutions are unstable to shock formation under small and smooth perturbations of the initial data, and it allowed him to study the perturbed solutions in a large region, going beyond the first singular point, as is described in Sect. 3. A particularly compelling aspect of the proof is that to close his energy estimates, Christodoulou had to control the mean curvature of the null hypersurfaces, which satisfies *Raychaudhuri's equation*, famous among the general relativists. Raychaudhuri's equation involves the acoustical Ricci curvature $\mathbf{Ric}(\mathbf{g})$ and thus Christodoulou's approach to studying shock formation brought into play geometric-analytic techniques from Lorentzian geometry.

[3]Non-vacuum constant solutions have a constant non-zero density and vanishing velocity.

2.5 Multi-dimensional shock formation beyond the irrotational and isentropic class

The multi-dimensional shock formation results of Christodoulou were extended to allow for solutions with vorticity in $2D$ [LS18], and then later also with entropy in $3D$ [LS24]. The proofs relied on a discovery [Spe19] (extending earlier work by Luk–Speck) that the compressible Euler equations (1a)–(1c) imply a geometric system of wave-transport-div-curl type, which can be schematically depicted as follows, where $\Psi \in \{\varrho, v^1, v^2, v^3, s\}$, $\square_{\mathbf{g}}\Psi := \frac{1}{\sqrt{|\det \mathbf{g}|}}\partial_\alpha \left\{ \sqrt{|\det \mathbf{g}|}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta \Psi \right\}$ is the covariant wave operator of the acoustical metric $\mathbf{g} = \mathbf{g}(\Psi)$, $\boldsymbol{\partial}$ denotes spacetime gradient, and ∇ denotes spatial gradient:

$$\square_{\mathbf{g}(\Psi)}\Psi = \mathrm{curl}(\mathrm{curl} v) + \mathrm{div}\nabla s, \tag{19a}$$

$$\mathbf{B}\mathrm{curl} v = \boldsymbol{\partial}\Psi \cdot \mathrm{curl} v, \tag{19b}$$

$$\mathbf{B}\nabla s = \boldsymbol{\partial}\Psi \cdot \nabla s. \tag{19c}$$

The system (19a)–(19c) allows for a sharp implementation of nonlinear geometric options and exhibits many remarkable properties, as is explained in [Spe19]. Equation (19a) in particular gives a sharp formulation of the coupling between sound waves and vorticity/entropy. It is much more difficult to derive energy estimates for the system (19a)–(19c) compared to the irrotational case, i.e., compared to equation (4), because difficult div-curl-type estimates are needed. It is even harder to localize the estimates to compact domains of dependence (see Fig. 2). For these reasons, [LS18, LS24] followed the solution only up to the constant-time hypersurface of first blowup.

An alternative method of proving shock formation in $3D$ with vorticity and entropy, based on understanding perturbations of self-similar solutions, was developed in [BSV23]. The authors constructed an open set of fully non-degenerate data, analogous to the one studied by Alinhac [Ali99], such that a self-similar Burgers'-type shock forms at an *isolated* point. This is distinct from [LS24], where the blowup points need not be isolated.

3 Maximal globally hyperbolic developments

Due to finite speed of propagation, a singularity at one point does not preclude one from continuing the solution classically to other regions. This suggests the following compelling question:

> What is the *largest* spacetime region for which the PDE admits a *classical* solution that is uniquely determined by initial data?

The answer to this question is extraordinarily subtle and depends on the geometry of the solution, which is Lorentzian in the case of the compressible Euler equations, in view of the formulation (19a)–(19c), which features the acoustical metric $\mathbf{g}$. Given the Lorentzian geometry, the natural condition guaranteeing uniqueness of solutions on a spacetime region $\mathcal{M} \subset \mathbb{R}^{1+3}$ arose in the study of mathematical general relativity: *global hyperbolicity*. This means that there exists a hypersurface $\Sigma \subset \mathcal{M}$ such that every smooth inextendible timelike[4] curve in $\mathcal{M}$ intersects Σ precisely once. Such a Σ is called a Cauchy hypersurface. Such $\mathcal{M}$ arise naturally when Σ is spacelike[5] and one solves the initial value problem with initial data given on Σ. We call such an $\mathcal{M}$ a *globally hyperbolic development* (GHD) of the data on Σ.

Definition 3.1 (MGHD) *A GHD $\mathcal{M}$ is said to be a* ***maximal globally hyperbolic development*** *(MGHD) if it is inextendible as a GHD of the data on Σ.*

The crucial word in Def. 3.1 that answers the question posed above is the word *inextendible*, i.e., it is not possible to find a larger GHD containing the initial data surface Σ. MGHDs are the holy grail of classical solutions. The importance MGHDs was first revealed in mathematical general relativity in the seminal work of Choquet-Bruhat–Geroch, who

[4] A curve γ is timelike (with respect to $\mathbf{g}$) if $\mathbf{g}(\dot\gamma, \dot\gamma) < 0$.

[5] A hypersurface Σ is $\mathbf{g}$-spacelike if its $\mathbf{g}$-normal satisfies $\mathbf{g}(N, N) < 0$. One can intuitively think that $\mathbf{g}$-spacelike hypersurfaces are like the constant time slices $\{t\} \times \mathbb{R}^3$.

proved [CBG69] the existence of a *unique* MGHD for sufficiently regular initial data for Einstein's equations, a hyperbolic PDE system that in the vacuum case take the form $\mathbf{Ric} = 0$, where $\mathbf{Ric}$ is the Ricci curvature of the Lorentzian spacetime metric.

It is now easy to see why in the context of fluid mechanics, MGHDs and shock singularities should be analyzed in unison: points at which a shock has started to form, i.e., at which the fluid gradient has blown up, constitute *boundary* points of MGHDs. From this perspective, the first point of blowup identified by Alinhac [Ali99] is the "lowest point" on the boundary of the MGHD for the quasilinear wave equation (4), i.e., a multi-dimensional analog of the cusp in Fig. 1.

For a large class of solutions of the $3D$ compressible Euler equations, specifically for general (asymmetric) perturbations of the $1D$ solutions from Sect. 2.3, the boundary of the MGHD in a future neighborhood of a shock singularity locally consists of three parts (see Figs.3–4 below):

- A co-dimension 2 submanifold known as the *crease* and denoted by '$\partial_-\mathcal{B}$', along which the fluid's first-order Cartesian coordinate partial derivatives blow up. The crease can be shown to be spacelike with respect to the acoustical metric $\mathbf{g}$, and it plays the role of the "true initial singularity" in the theory of shock formation.

- A hypersurface $\mathcal{B}$ that emanates from $\partial_-\mathcal{B}$, termed the *singular boundary*, along which $\mu = 0$ and the fluid's gradient continues to blowup.

- A *Cauchy horizon* $\underline{\mathcal{C}}$, which is a hypersurface that is null with respect to $\mathbf{g}$ (i.e., its normal has vanishing $\mathbf{g}$-length) and also emanates from the crease, but along which the solution extends smoothly (except at the crease).

We clarify that $\mathcal{B} \cup \underline{\mathcal{C}}$ makes up only a *localized* portion of the boundary of *an* MGHD for the compressible Euler equations. Moreover, to date, it is the only component of $\partial\mathcal{M}$ that has been understood. By "generic shock-forming solutions," we mean those whose inverse foliation density satisfies the following bound[6] relative to geometric coordinates (t, u, x^2, x^3):

$$\frac{\partial^2 \mu}{\partial u^2}\Big|_{\partial_-\mathcal{B}} \geq C > 0, \tag{20}$$

where the word "generic" stems from the fact that μ is an everywhere positive function that vanishes everywhere prior to the crease. We call this property *transversal convexity*; see Fig. 3.

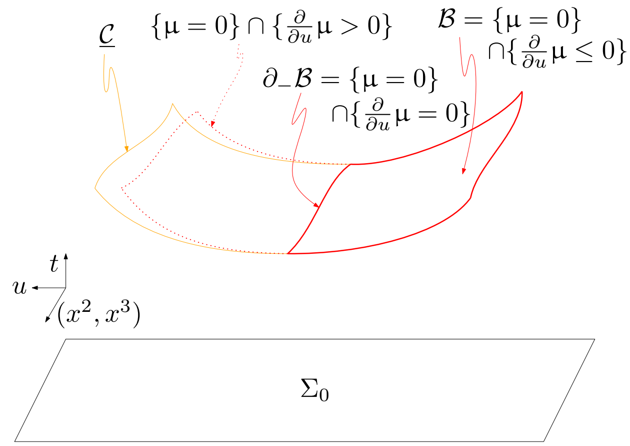

Figure 3: $\mathcal{B}$ and $\underline{\mathcal{C}}$ in geometric coordinates

The first breakthrough for compressible fluid MGHDs was Christodoulou's work [Chr07] on the 3D irrotational isentropic relativistic Euler equations, described in Sect. 2. Christodoulou assumed an inequality that is equivalent to (20), and as is explained on [Chr07, Pages 929, 968–969], by varying the "angle of tilt" of families of flat spacelike (with respect to $\mathbf{g}$) hypersurfaces Σ_{flat}, he used them to foliate spacetime and reveal an implicit portion of the boundary of the MGHD. If one strengthens (20) by assuming "strict convexity of the crease in (u, x^2, x^3)," then his approach allows one to access the entire crease. While Christodoulou's framework yields a description of some part of an MGHD, the precise portion that it reveals is not made explicit through the construction, in particular when strict convexity fails. Moreover,

[6] There are alternative, more geometric ways to formulate (20), but we have omitted them to keep the presentation short.

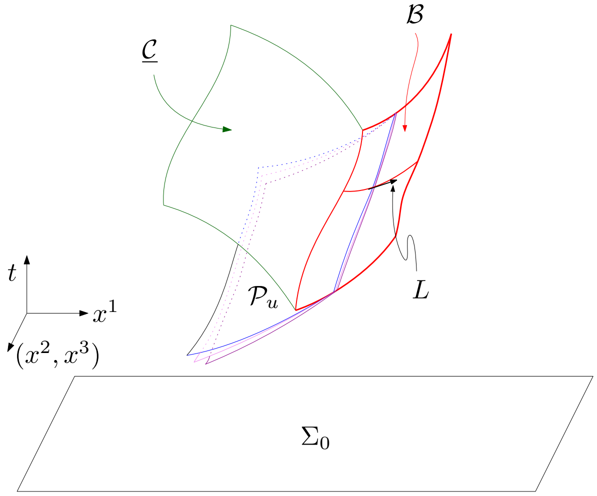

Figure 4: $\mathcal{B}$ and $\underline{\mathcal{C}}$ in Cartesian coordinates

strict convexity does not hold for all shock-forming solutions. For example, it does not hold for $1D$ solutions (viewed as $3D$ solutions with symmetry), nor for their general perturbations.

The first constructive proof of a localized portion of the MGHD for the $3D$ compressible Euler solutions was our work [AS22]. The two big advancements were that the paper went beyond the irrotational case, i.e., it treated open sets of solutions with vorticity and dynamic entropy, and it did not assume strict convexity of the crease. The proof also relied on new, solution-dependent foliations of spacetime, precisely adapted to the shape of the boundary. We briefly summarize the results of [AS22]:

Theorem 3.1 *Fix any $1D$ isentropic initial data such that the corresponding shock-forming solution satisfies* (20)*. Then for sufficiently small* ***general*** *$3D$* ***perturbations*** *of the $1D$ initial data without any irrotationality, isentropicity, or strict convexity assumptions, there holds:*

- *A sharp description of the formation and stability of a full connected component of $\partial_-\mathcal{B}$ and $\mathcal{B}$, along which the fluid's first-order Cartesian coordinate partial derivatives blow up. In particular, relative to geometric coordinates (t, u, x^2, x^3), the crease is characterized by $\partial_-\mathcal{B} = \{\upmu = 0\} \cap \{\frac{\partial}{\partial u}\upmu = 0\}$ and the singular boundary is characterized by $\mathcal{B} = \{\upmu = 0\} \cap \{\frac{\partial}{\partial u}\upmu \leq 0\}$.*
- *A causal[7] intrinsic description of $\partial_-\mathcal{B}$ and $\mathcal{B}$.*
- *A causal extrinsic description of $\partial_-\mathcal{B}$ and $\mathcal{B}$, including sharp quantitative control of its embedding into Cartesian space $\mathbb{R}^{1+3}$.*

[AS22] shows that the boundary of the MGHD is extremely intricate. As is depicted in Fig. 4, the characteristics pile up along all of $\mathcal{B}$ with infinite density in the Cartesian differential structure. This implies that relative to the Cartesian coordinates (t, x^1, x^2, x^3), the extrinsic geometry of $\mathcal{B}$ is that of a $\mathbf{g}$-spacelike hypersurface, as past-directed $\mathbf{g}$-null geodesics[8] emanating from $\mathcal{B}$ do not remain in $\mathcal{B}$. Nevertheless, it was shown in [AS22] that $\mathcal{B}$ is ruled by integral curves of L in the Cartesian differential structure and hence its intrinsic geometry is that of a null hypersurface; see Fig. 4, where one such integral curve is displayed. We stress that the acoustical metric $\mathbf{g}$ depends only on the *undifferentiated* fluid variables in Cartesian coordinates and can be shown to extend continuously up to $\mathcal{B}$, even though it is <u>not</u> C^1 up to it. The lack differentiability of $\mathbf{g}$ in Cartesian coordinates leads to lack of uniqueness of the integral curves of L, which in turn leads to some fascinating geometric and analytic degeneracies along $\mathcal{B}$, derived in [AS22]. Nevertheless, the continuity of $\mathbf{g}$ up to $\mathcal{B}$ allows one to "measure" the aforementioned integral curves of L as being $\mathbf{g}$-null along $\mathcal{B}$.

The remaining portion of the boundary of the MGHD in Figs. 3–4 is the Cauchy horizon $\underline{\mathcal{C}}$, which for some $2D$ solutions exhibiting a strictly convex boundary, was constructed in the recent important work [SV24].

Theorem 3.2 *There exists an open set of initial data for the $2D$ isentropic compressible Euler equations obeying a strict convexity condition and with*

[7]The word "causal" refers to geometric properties as measured by $\mathbf{g}$. The name is inspired because questions of causality and determinism in special relativity are determined by the Minkowski metric.

[8]A curve γ is $\mathbf{g}$-null if it satisfies $\mathbf{g}(\dot{\gamma}, \dot{\gamma}) = 0$. One can think that $\mathbf{g}$-null curves propagate at the speed of sound.

gradients of size $1/\varepsilon$, such that when ε is small and positive, within a rectangular region of spacetime sandwiched by $\mathcal{O}(\varepsilon)$-separated Cartesian-time hypersurfaces, there holds:

- *A sharp description of the formation, stability, and gradient blowup of the fluid on an $\mathcal{O}(\varepsilon)$-size strictly convex portion of $\partial_-\mathcal{B}$ and $\mathcal{B}$.*
- *A sharp description of an $\mathcal{O}(\varepsilon)$-size strictly convex portion of the Cauchy horizon $\underline{\mathcal{C}}$, where no fluid blowup occurs, except along $\partial_-\mathcal{B}$.*

The paper [SV24] also introduced two new methods: a way to generalize the Riemann invariants to multi-dimensions, and novel energy identities that allow one to close the proof assuming less regularity for the eikonal function and vorticity compared to prior works.

For shock-forming compressible fluids, the Cauchy horizon plays a crucial role as the savior of classical determinism and uniqueness in that it prevents the "classical-uniqueness-destroying part" of the set $\{\mu = 0\}$ from having a chance to form. This is easy to see for the $1D$ isentropic solutions described in Sect. 2.3, which we depict in Fig. 5. In the figure, one can see that the Cauchy horizon $\underline{\mathcal{C}}$, which in that setting is an integral curve of $\underline{L}$ emanating from the crease (which looks like a cusp in the figure), delineates the boundary of the region of global hyperbolicity and prevents the "fictitious portion of the singularity boundary," denoted by a dotted curve, from forming. This is in stark contrast to the non-uniqueness we saw for Burgers' equation in Fig. 1, where there is no analogous notion of global hyperbolicity, the singular boundary has two branches separated by a cusp, and there is no mechanism to prevent the characteristics from strictly crossing.

Note that the portion $\{\mu = 0\} \cap \{\frac{\partial}{\partial u}\mu > 0\}$ is omitted in Thms. 3.1–3.2. This is the multi-dimensional analog of the "fictitious portion of $\mathcal{B}$" featured in Fig. 5. This fictitious portion can be described from a geometric/causal perspective. That is, by doing *formal* Taylor expansions in geometric coordinates starting from $\partial_-\mathcal{B}$, one finds that the fictitious portion – if it existed – would lie in the $\mathbf{g}$-timelike future of the Cauchy horizon $\underline{\mathcal{C}}$; see the dotted por-

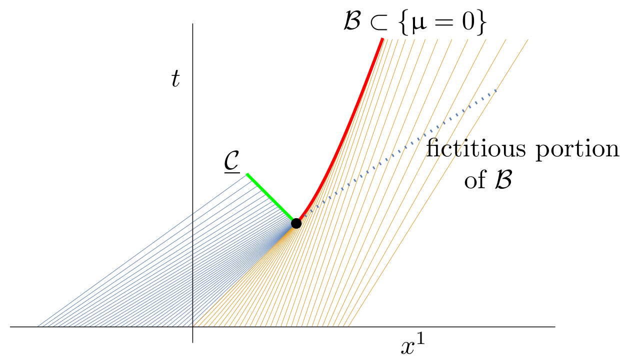

Figure 5: Localized MGHD in Cartesian coordinates for shock-forming $\mathcal{R}_+$-dominated $1D$ isentropic compressible Euler solutions

tion in Fig. 3. Importantly, the map from geometric coordinates, where all of the analysis is done in [AS22, SV24], to Cartesian coordinates[9] would have *failed to be injective* on the region in Fig. 3 that would have included the fictitious portion of the singular boundary, where the multi-valuedness corresponds to the multi-valuedness (non-uniqueness) of classical solutions that we saw for Burgers' equation when the characteristics cross. The works [AS22, SV24] on the multi-dimensional compressible Euler equations show that the change of variable map is in fact injective in the region trapped between Σ_0 and $\mathcal{B} \cup \underline{\mathcal{C}}$, and hence there is a GHD of the data there, without the problem of multi-valuedness.

4 Shock developments vs. shock fronts

Having established that MGHDs define the limits of the classical theory for the initial value problem, we can now entertain the question of whether it is possible to meaningfully solve the compressible Euler equations *beyond* a shock singularity. A *weak solu-*

[9] More precisely, [SV24] works with the *inverse* map from Cartesian coordinates to geometric coordinates, which they call Arbitrary Lagrangian Eulerian (ALE) coordinates.

tion solution is one that satisfies (3a)–(3c) upon multiplying by any smooth, compactly supported "test function," integrating over a spacetime domain, and integrating by parts to put the derivatives on the test function. In this way, one may consider the equations as solved and still allow singularities, as functions with sufficiently mild singularities (such as jump discontinuities) can be integrated. The divergence theorem can be used to show that classical solutions are weak solutions and also that sufficiently regular weak solutions are classical solutions.

4.1 The shock development problem

A compelling way[10] to study weak solutions with shocks is to look for solutions (see Fig. 6) satisfying:

> Emanating from the crease, a new *shock hypersurface* $\mathcal{K}$ emerges, across which the solution exhibits a jump discontinuity, in accordance with "jump conditions" described below. The solution should be weak in a neighborhood of $\mathcal{K}$ but pointwise smooth – and hence classical – on either side of $\mathcal{K}$.

It turns out that having the solution be weak across $\mathcal{K}$ forces a coupling of the spacetime normal of $\mathcal{K}$ with the average state of the fluid on either side of it. That is, the "speed and direction of the shock surface" is determined by the "jump" in the fluid across $\mathcal{K}$. The precise nonlinear relationships between the two are called the *Rankine–Hugoniot (RH) jump conditions.* The "jump" is what we see in shock waves from mother nature such as volcanic eruptions or supernova explosions. Mathematically, $\mathcal{K}$ is a *new unknown* that must be solved for in conjunction with the fluid using the RH jump conditions, making these weak solutions a fully nonlinear evolutionary free boundary problem.

Describing the *transition* from a classical solution, to forming a shock singularity, to a weak solution with a shock hypersurface $\mathcal{K}$ is called the *shock development problem.* The first solution to a shock development problem was by Lebaud [Leb94], in which she treated the $1D$ isentropic compressible Euler equations.

To date, the only shock development result in multiple space dimensions without symmetry is Christodoulou's breakthrough monograph on the *restricted* shock development problem [Chr19]. "Restricted" means that he studied only irrotational and isentropic solutions and ignored the jump in entropy and vorticity across the shock hypersurface; the RH jump conditions plus the laws of thermodynamics imply that in a true compressible fluid, the vorticity and entropy should jump across $\mathcal{K}$. In [Chr19], he constructed a local, piecewise smooth solution to a hyperbolic PDE that approximates the compressible Euler equations, and the two smooth regions were separated by a shock hypersurface $\mathcal{K}$, which he also constructed. The "initial conditions" were a portion of an MGHD all the way up to $\mathcal{B} \cup \underline{\mathcal{C}}$. In other words, a sharp (local) understanding of the MGHD is necessary to properly set up the PDE analysis of the shock development problem. Christodoulou studied solutions satisfying a convexity condition of type (20), and without such a condition, it is not clear whether the shock development problem is well-posed.

Importantly, *$\mathcal{K}$ protrudes into the MGHD of the classical solution*; see Fig. 6. Accounting for the jump across $\mathcal{K}$, this means that *the weak solution and the classical solution disagree on the shaded depicted in Fig. 6.* The cause for this is that the RH jump conditions force $\mathcal{K}$ to be *super-sonic*, while the singular boundary $\mathcal{B}$ is characteristic and hence propagates at the speed of sound; see Sect. 3. This does not diminish the importance of the MGHD, since a sharp understanding of its boundary has played a role in setting up and solving the shock development problem in the restricted case [Chr19], and moreover, $\mathcal{B}$ and $\mathcal{K}$ are asymptotically tangent at the crease. We clarify that only the speed of $\mathcal{K}$ defined by the fluid state *to its past* is super-sonic; the speed of $\mathcal{K}$ defined by the fluid state to its future is *sub-sonic.* This agrees with experimental data taken from shock waves observed in nature, and, at least in some solution regimes, it is equivalent to having the entropy

[10] One can consider weak solutions with drastically different geometric configurations to those with a single shock hypersurface as in Fig. 6. The ones discussed here are the most heavily studied, in part due to there being a well-developed uniqueness theory for them in $1D$ [Daf10].

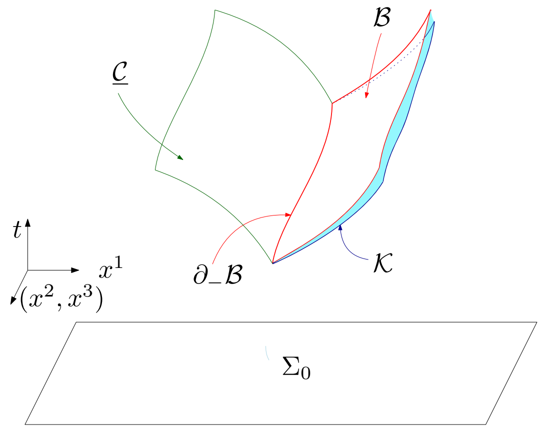

Figure 6: The shock hypersurface $\mathcal{K}$

increase across $\mathcal{K}$. Solutions satisfying this condition are sometimes called *entropy solutions*, and the key takeaway is that *uniqueness of solutions to the shock development problem is only expected when $\mathcal{K}$ is super-sonic to its past and sub-sonic to its future.* Even in $1D$, weak solutions can be non-unique in $1D$ without the entropy condition; see [Daf10].

4.2 The shock front problem

An important problem related to – but distinct from – shock development is the *shock front problem.* Instead of studying the *transition* from classical solutions to those with jumps, one studies the flow with initial data that already have a discontinuity satisfying the RH jump conditions across a hypersurface. The goal is to propagate the jump discontinuity with a piece-wise smooth fluid solution divided by a shock hypersurface (which also must be constructed). The shock front problem was shown to be well-posed for small times in Majda's celebrated work [Maj83].

5 Outlook

Christodoulou's book [Chr07] on shock formation initiated an explosion of outstanding progress in both the classical and weak theory of shocks that has been sustained for nearly two decades so far; see e.g. [LS18, Chr19, AS22, BSV23, SV24, GR24]. This progress shows no signs of slowing down. We conclude with a *far-from-exhaustive* list of future directions that we believe to be compelling.

Existence & Uniqueness of MGHDs with shocks: The outstanding result [ERS19] shows that one cannot ensure uniqueness of an MGHD (the paper gave examples of hyperbolic PDEs where uniqueness fails!) unless one constructs it in its entirety and then proves that it enjoys some crucial *global* structural properties. Roughly, the paper shows that a sufficient condition for uniqueness is that the "MGHD lies on one side of its boundary," i.e., a global version of Fig. 6. The works [AS22, SV24] construct *not the full MGHD of the data*, but rather only a compact portion of one and its boundary. Presently, we do not have any examples of unique, entire MGHDs for compressible Euler solutions with shocks.

Shock development: The laws of thermodynamics and RH jump conditions force entropy and vorticity to jump across a shock hypersurface. Although Christodoulou's resolution [Chr19] of the restricted shock development shed considerable light, there is a great leap in difficulty to allow for dynamic entropy and vorticity, and the general shock development problem therefore remains open.

General relativity: [Chr07] highlighted the tremendous synergy between shock waves and techniques developed to study general relativity. It remains an outstanding open problem to prove shock formation for the Einstein–Euler system, a general-relativistic PDE system modeling a self-gravitating fluid. The main new difficulty is that fluid characteristics propagate at a strictly slower speed than gravitational waves and so the techniques from [Chr07] do not apply directly. There is exciting work-in-progress on this problem for irrotational isentropic solutions by John Anderson and Jonathan Luk.

Weak interactions: Shock development, complicated as it may be, is only one type of fluid phenomenon that can occur. We are far from fully understanding the possible behavior of weak solutions. What is likely more tractable is to use our current understanding of shock waves as "building blocks" for more complex fluid configurations. That is, one could study the interaction of e.g. two shock hypersurfaces $\mathcal{K}_1$ and $\mathcal{K}_2$ colliding, or the interactions of two Cauchy horizons $\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$, or the interaction of a shock hypersurface $\mathcal{K}_1$ from one shock and a Cauchy horizon $\underline{\mathcal{C}}_2$ from another.

Global weak solutions: Majda produced solutions to the shock front problem for short times in [Maj83]. There has been very recent progress on the global-in-time well-posedness of weak irrotational and isentropic solutions, starting with two hypersurfaces of jump discontinuities [GR24]. It is not at all clear – but would be fascinating to understand – if any global result can be proved in the presence of vorticity.

Inviscid limits: A physically and mathematically important question is whether *shock-forming* compressible Euler solutions can be rigorously be shown to be limits of compressible Navier–Stokes solutions as the viscosity (which tends to have a regularizing effect) in the Navier–Stokes equations goes to zero. This research program goes by the name of *inviscid limits* and has recently been well understood in the context of Burgers' equation (7) for shock-forming initial data with transversal convexity by Chaturvedi–Graham [CG23]. An important open problem, even in $1D$, is to extend the results of [CG23] to the compressible Euler equations.

Other systems: There are many other physical phenomena, including nonlinear electromagnetism and elasticity, such that multi-dimensional shocks are expected to form. In the corresponding PDE systems, the geometry is captured by a tensor that is more complicated than a Lorentzian metric and thus brand new ideas are needed to study shocks.

The theory has come a long way since Riemann's foundational gradient blowup-result [Rie60] for $1D$ compressible fluids. Although making progress on the multi-dimensional front-lines is very challenging, the field has never before been more prepared to handle the difficulties. We are truly at the heart of a golden age in the multi-dimensional theory of shocks.

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