The structure of the maximal development for shock-forming $3D$ compressible Euler solutions

Jared Speck
with Leo Abbrescia

Vanderbilt University

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3D compressible Euler flow

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- Two propagation phenomena: sound waves and transporting of vorticity/entropy
- Neither phenomena nor their coupling are apparent

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- Set up the shock development problem

Remarks on 1D theory

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Remarks on $1D$ theory

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- Challis (1848)
- Stokes (1850s)
- Riemann (1860)
- Oleinik (1959)
- Zabusky (1962)
- Lax (1964)
- Glimm (1965)
- Keller–Ting (1966)
- Dafermos (1970)
- Smoller (1970)
- Liu (1974)
- John (1974)
- Klainerman–Majda (1980)
- Jenssen (2000)
- Chen–Feldman (2003)
- Bianchini–Bressan (2005)

Riemann invariants

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- $\mathcal{R}_\pm = v^1 \pm F(\varrho)$ are Riemann invariants
- F determined by equation of state $p = p(\varrho, s)$
- $L = \partial_t + (v^1 + c)\partial_1$
- $\underline{L} = \partial_t + (v^1 - c)\partial_1$
- $c = \sqrt{\frac{\partial p}{\partial \varrho}} =$ speed of sound > 0

Shocks for 1D isentropic compressible Euler

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- Picture is qualitatively different compared to Burgers' equation: Cauchy horizons.
- Cauchy horizons can rescue uniqueness of classical solutions. So far, this is understood only locally in the regime with transversal convexity.

Maximal globally hyperbolic development for 1D isentropic compressible Euler solutions

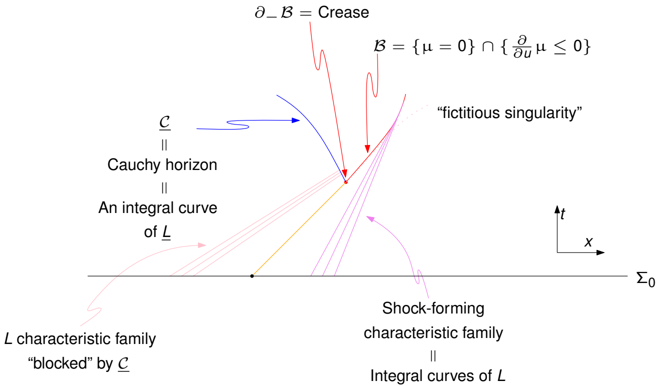

Figure: Local structure of MGHD for $\mathcal{R}_{+}$-dominated 1D isentropic compressible Euler solutions

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Hence, the modern (starting in late 1990s) approach in multi-dimensions:

- Detailed study of all the structures that can arise in singular flows
- Geometry plays a key role
- Relies on energy estimates, which are very difficult near singularities

Multi-D shocks and singularities

- Majda (1980s)
- Alinhac (late 1990s)
- Christodoulou (2007, 2019)
- Christodoulou–Miao (2014)
- Miao–Yu (2016)
- Holzegel–Luk–Speck–Wong (2016)
- Luk–Speck (2016, 2020s)
- Merle–Raphael–Rodnianski–Szeftel (2020s)
- Abbrescia–Speck (2020s)
- Buckmaster–Iyer (2020s)
- Buckmaster–Drivas–Shkoller–Vicol (2020s)
- Ginsburg–Rodnianski (pre-print)
- (Luo–Yu) (irrotational rarefaction waves in 2D)
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With Abbrescia, for open sets of data in 3D, we have given the first complete description of the full structure of the singular set and the Cauchy horizon

Infinite density of the characteristics

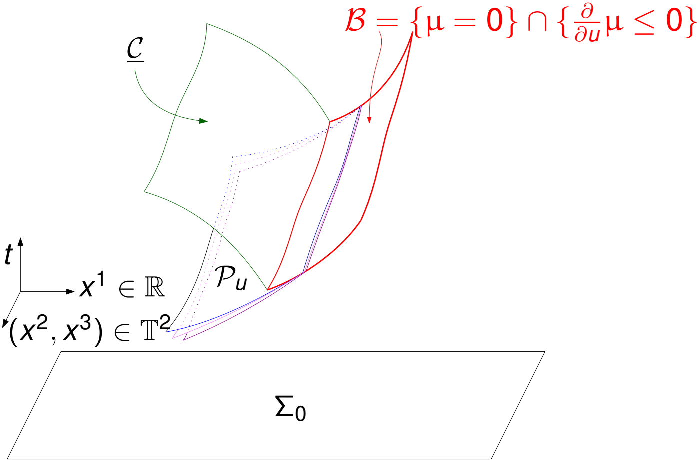

Figure: Infinite density of the characteristics $\mathcal{P}_u$ on $\mathcal{B}$

New results with L. Abbrescia

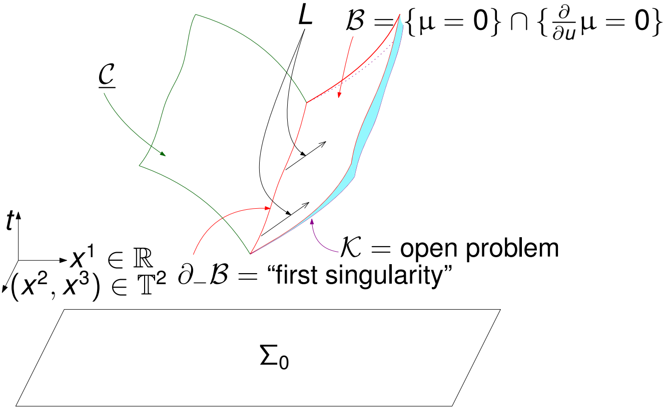

Figure: A localized subset of the maximal classical development and the shock hypersurface in Cartesian space

Acoustical metric

The acoustical metric is tied to sound wave propagation.

Definition (The acoustical metric and its inverse)

$$\mathbf{g} := -dt \otimes dt + c^{-2} \sum_{a=1}^{3} (dx^a - v^a dt) \otimes (dx^a - v^a dt),$$

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Material derivative vectorfield **B** is **g**-timelike and thus transverse to acoustically null hypersurfaces:

$$\boxed{\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1}$$

Acoustic eikonal function

Definition (The acoustic eikonal function)

The acoustic eikonal function u solves:

$$\boxed{\begin{aligned}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u &= 0, \\ u|_{t=0} &= -x^1, \qquad\qquad \partial_t u > 0\end{aligned}}$$

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Definition (Geometric coordinates)

We refer to (t, u, x^2, x^3) as the geometric coordinates.

Inverse foliation density

Definition

$$\boxed{\mu := -\frac{1}{(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha t \partial_\beta u} = \frac{1}{\mathbf{B}u}}$$

Can show:

$$\boxed{\mu|_{t=0} \approx 1}$$

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$\mu = 0$ signifies a shock (infinite density of characteristics and blowup of ∂u)

Proof philosophy

Big idea (Alinhac and Christodoulou): Solution remains rather smooth in geometric coordinates

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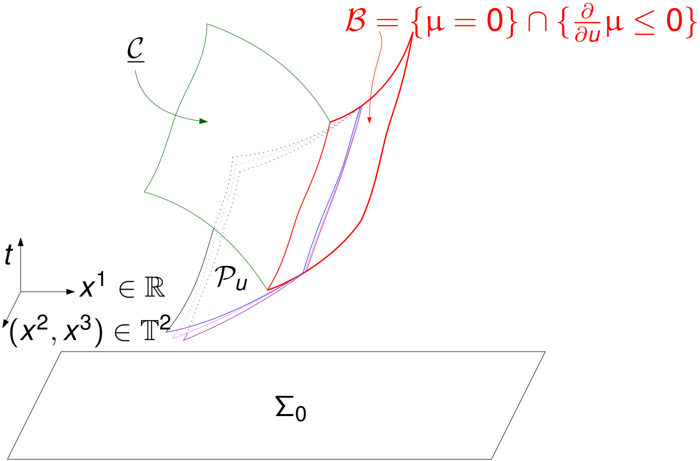
- $\partial_\alpha \sim \frac{1}{\mu}\frac{\partial}{\partial u} + \frac{\partial}{\partial t} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}$
- Hence, $\mu = 0$ represents a degeneracy between Cartesian and geometric partial derivatives

Infinite density of the characteristics



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Strictly convex sub-regime

The strictly convex sub-regime is easier to study:

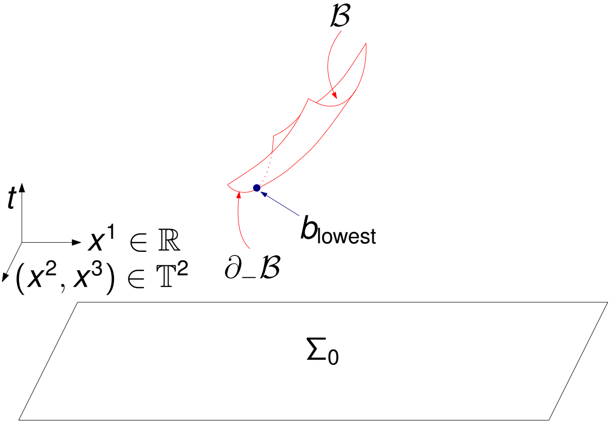

Figure: Strictly convex crease and singular boundary in Cartesian coordinate space

Null vectorfields

Definition

Null vectorfields

$$\boxed{\begin{aligned} L^\alpha_{(geo)} &:= -(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta u, \\ L^\alpha &:= \mu L^\alpha_{(geo)} \end{aligned}}$$

Easy to see:

$$\boxed{Lt = 1}$$

In plane symmetry, L agrees with the vectorfield defined explicitly in terms of Riemann invariants

Vectorfield frames constructed from u

Definition (Frame vectorfields)

- X is Σ_t-tangent, left-pointing, satisfies $\mathbf{g}(X, X) = 1$, and $\mathbf{g}$-orthogonal to $\ell_{t,u} := \Sigma_t \cap \mathcal{P}_u$
- $\breve{X} := \mu X$ (satisfies $\breve{X} u = 1$)
- For $A = 2, 3$, $Y_{(A)} :=$ $\mathbf{g}$-orthogonal projection of (rectangular) ∂_A onto $\ell_{t,u}$

Definition (Frame adapted to the characteristics)

The rescaled frame is:

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- Derive regular estimates relative to the rescaled frame
- Shows that the solution and its $\{L, \breve{X}, Y_{(2)}, Y_{(3)}\}$-derivatives remain rather smooth (equivalently, smooth w.r.t. (t, u, x^2, x^3) and smooth in directions tangent to $\mathcal{P}_u$)

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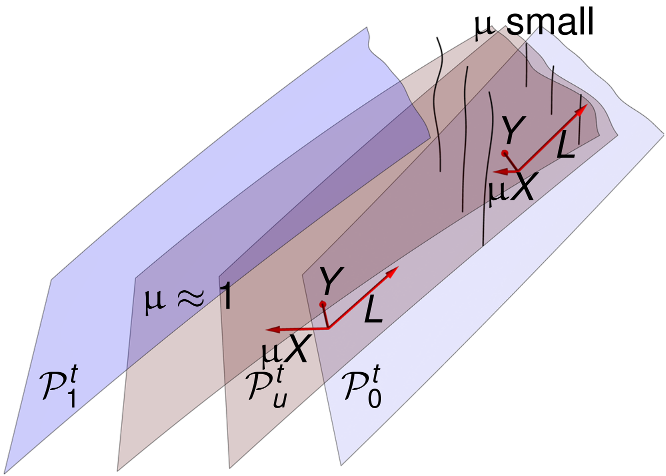
- Big technical difficulty: High order geometric energies can blow up as $\upmu \downarrow 0$: $\mathbb{E}_{\text{Top}} \lesssim \upmu^{-10}$, $\mathbb{E}_{\text{Top}-1} \lesssim \upmu^{-8}$, $\cdots$, $\mathbb{E}_{\text{Mid}} \lesssim 1$

A picture of the dynamics



Statement of main results

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- *Writing-in-progress: we give a complete description of a neighborhood of the Cauchy horizon $\underline{\mathcal{C}}$ that includes the entire crease.*
- *In total, we reveal a portion of the maximal (classical) globally hyperbolic development, including a neighborhood of the boundary.*

New results with L. Abbrescia

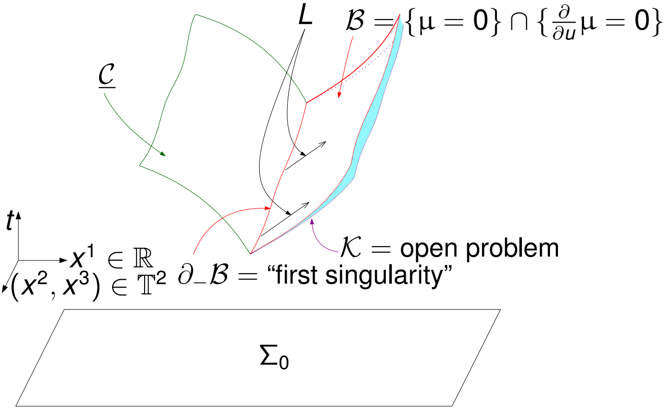
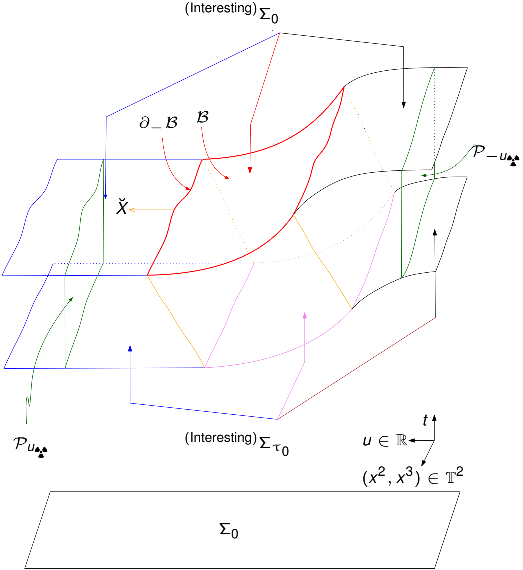


Figure: A localized subset of the maximal classical development and the shock hypersurface in Cartesian space

The crease and the singular boundary



Construction of $\underline{u}$ for the Cauchy horizon

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Construction of $\underline{u}$ for the Cauchy horizon

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Then $\underline{u}$ is propagated via the eikonal equation:

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The Cauchy horizon region

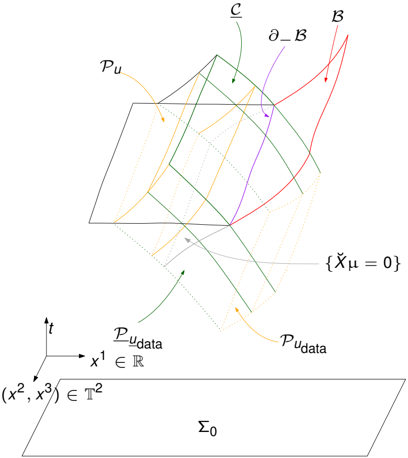

Figure: The Cauchy horizon region in Cartesian space

Connection to wave equations

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- Also useful for low-regularity well-posedness

New formulation of 3D compressible Euler

Theorem (J. Luk–JS; M. Disconzi–JS in relativistic case)

Consider smooth compressible Euler solutions in 3D. *For* $\Psi \in \vec{\Psi} := (\varrho, v^1, v^2, v^3, s)$, *we have*, *schematically*:

$$\begin{aligned} \square_{\mathbf{g}(\vec{\Psi})} \Psi &= \nabla \times \left(\frac{\nabla \times v}{\varrho} \right) + \text{div}\, \nabla s \\ &\quad + \mathbf{g}-\textit{null forms}, \\ \mathbf{B} \left(\frac{\nabla \times v}{\varrho} \right) &= \nabla \vec{\Psi} \cdot \left(\frac{\nabla \times v}{\varrho} \right) + \nabla \vec{\Psi} \cdot \nabla s, \\ \mathbf{B} \nabla s &= \nabla \vec{\Psi} \cdot \nabla s \end{aligned}$$

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  → With L. Abbrescia, we derived suitable "elliptic-hyperbolic" identities for $\frac{\nabla \times v}{\varrho}$ and ∇s on arbitrary globally hyperbolic domains for 3D compressible Euler solutions

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Would require the development of new geometry.

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Burgers' equation for $\Psi(t,x)$:

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- Second boxed equation is a Riccati-type ODE in $\partial_x\Psi$; typically, $\partial_x\Psi$ blows up in finite time.

Singular curve

The set of singular points can be parameterized by $(0, z) \in \Sigma_0$:

$$\mathcal{S}(z) = \left(-\frac{1}{\frac{d}{dz}\mathring{\Psi}(z)}, z - \frac{(1 + \mathring{\Psi}(z))}{\frac{d}{dz}\mathring{\Psi}(z)} \right) = (\text{Blowup-time}, \text{Blowup } x\text{-coordinate})$$

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$$= \frac{\frac{d^2}{dz^2}\mathring{\Psi}(z)}{(\frac{d}{dz}\mathring{\Psi}(z))^2} L|_{\mathring{\Psi}(z)}$$

The Cartesian coordinates picture

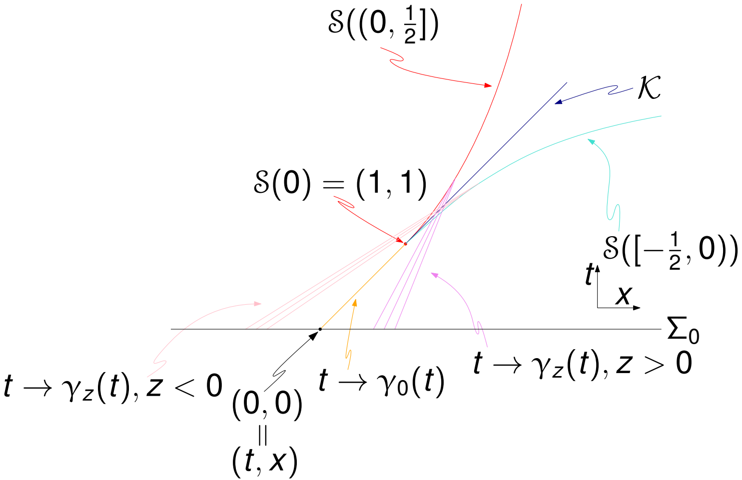

Figure: The singular curve and shock hypersurface in Cartesian coordinates for $\mathring{\Psi}(x) = -x + \frac{1}{3}x^3$

Crucial and under-appreciated fact

The Burgers' equation singularity is renormalizable.
That is, the solution is globally smooth relative to well-constructed geometric coordinates.

"Hiding" the singularity

By solving

$$\boxed{Lu = 0, \qquad u|_{t=0} = -x,}$$

we can construct geometric coordinates (t, u) such that:

$$\frac{\partial}{\partial t} := \frac{\partial}{\partial t}|_u = \partial_t + (1 + \Psi)\partial_x = L = \partial_t + (1 + \Psi)\partial_x$$

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Thus,

$$\boxed{\Psi(t, u) = \mathring{\Psi}(u) := \Psi(0, u)}$$

The inverse foliation density μ

Set:

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Can easily solve:

$$\mu(t,u) = 1 - t\frac{d}{du}\mathring{\Psi}(u)$$

How to think about the singularity

CHOV relation $\implies \partial_x \Psi$ blows up when $\mu \to 0$:

$$\boxed{\partial_x \Psi = -\frac{1}{\mu} \frac{\partial}{\partial u} \Psi = -\frac{1}{\mu} \frac{d}{du} \mathring{\Psi}(u)}$$

The geometric coordinates picture

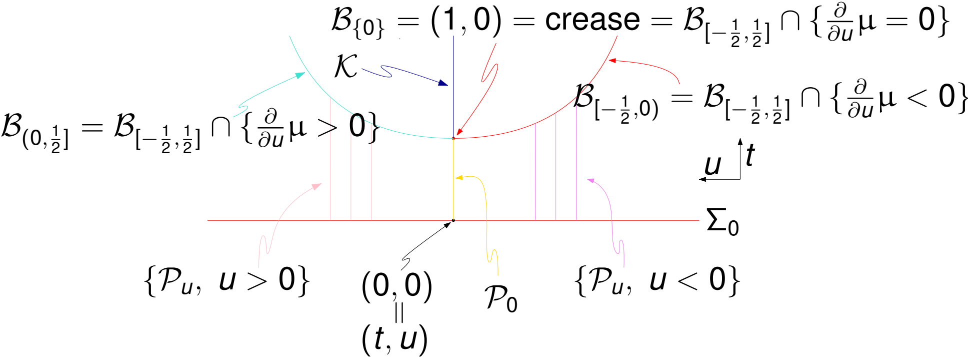

Figure: Portions $\mathcal{B}_J = \{\mu = 0\} \cap \{u \in J\}$ for Burgers' equation in geometric coordinates with $\mathring{\Psi}(u) = u - \frac{1}{3}u^3$