LECTURE NOTES ON: THE EMERGENCE OF THE SINGULAR BOUNDARY FROM THE CREASE IN $3D$ COMPRESSIBLE EULER FLOW

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Abstract. These lecture notes are an extended version of our paper [1]. Here, we provide some additional details that, while standard, might be helpful to those who are learning the field. Moreover, we eliminate the assumption of compact support on the initial data, which we assumed in [1] to simplify some technical but non-essential aspects of the analysis.

We study the Cauchy problem for the $3D$ compressible Euler equations under an arbitrary equation of state with positive speed of sound, aside from that of a Chaplygin gas. For open sets of smooth initial data with non-trivial vorticity and entropy, our main results yield a constructive proof of the formation, structure, and stability of the singular boundary, which is the set of points where the solution forms a shock singularity, i.e., where some first-order Cartesian coordinate partial derivatives of the velocity and density blow up. We prove that in the solution regime under study, the singular boundary has the structure of a degenerate $3D$ sub-manifold-with-boundary that is ruled by acoustically null curves. Our approach yields the full structure of a neighborhood of a connected component of the crease, which is a $2D$ acoustically spacelike sub-manifold equal to the past boundary of the singular boundary. In the study of shocks, the crease plays the role of the "true initial singularity" from which the singular boundary emerges, and it is a crucial ingredient for setting up the shock development problem. These are the first results revealing the totality of these structures without symmetry, irrotationality, or isentropicity assumptions. Moreover, even within the sub-class of irrotational and isentropic solutions, these are the first constructive results revealing these structures without a strict convexity assumption on the shape of the singular boundary. Our proof relies on a new method: the construction of rough foliations of spacetime, dynamically adapted to the exact shape of the singular boundary and crease, where the latter is provably two degrees less differentiable than the fluid. Our results also set the stage for our forthcoming paper, in which we will prove the emergence and stability of a Cauchy horizon, which emanates from the crease and "evolves" in a direction that is "opposite" the singular boundary in a sense determined by the intrinsic acoustic geometry of the flow.

Keywords: Cauchy horizon; characteristics; characteristic current; compressible Euler equations; eikonal function; maximal development; null condition; null hypersurface; null structure; shock development problem; shock formation; singular boundary; stable singularity formation; wave breaking; vectorfield method

Mathematics Subject Classification (2010) Primary: 35L67 - Secondary: 35L05, 35Q31, 74J40, 76N10

Contents

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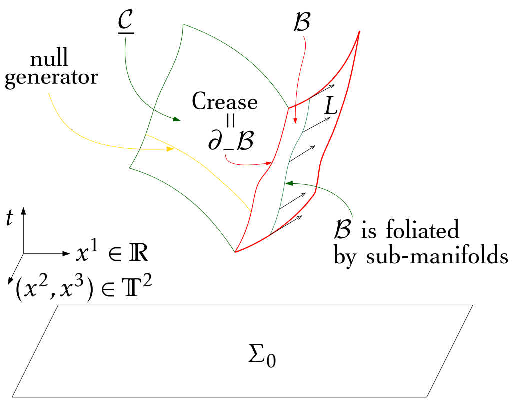
† LA gratefully acknowledges support from an NSF Postdoctoral Fellowship.
††JS gratefully acknowledges support from NSF grant # DMS-2054184 and NSF CAREER grant # DMS-1914537.

1. Introduction

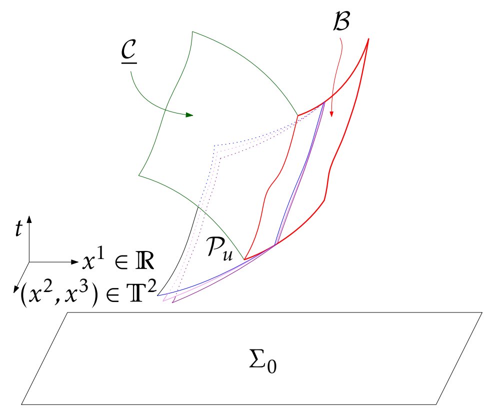
These lecture notes are an extended version of our paper [1]. Here, we provide some additional details that, while standard, might be helpful to those who are learning the field. Moreover, we eliminate the assumption of compact support on the initial data, which we assumed in [1] to simplify some technical but non-essential aspects of the analysis.

We study the Cauchy problem for the $3D$ compressible Euler equations with vorticity and dynamic entropy, and without symmetry assumptions. This is the first of two papers in which we construct a large (though bounded) portion of the maximal classical globally hyperbolic development (which we refer to as the "maximal development" for short from now on) – up to the boundary – of the initial data for open sets of initially smooth, *shock-forming* solutions. When a shock forms, the gradients of the density ϱ and the velocity v blow up, though ϱ and v remain bounded. This phenomenon is also known in the literature as *wave breaking*. Roughly speaking, the maximal development is the largest possible classical solution + region that is determined by given, regular initial data. Our main results are the constructions of the (gradient-singularity-forming) solution + localized region depicted in Fig. 1 (see Remark 1.3 for comments on our notation in the figure) and a proof of their stability under Sobolev-class perturbations of the initial data on the Cauchy hypersurface $\Sigma_0 \stackrel{\text{def}}{=} \{t = 0\}$.

> Our papers provide the first results that construct and fully justify Fig. 1 for open sets of initial data without symmetry, irrotationality, or isentropicity assumptions.



(A) A localized subset of the maximal classical development in Cartesian coordinate space



(B) Infinite density of the characteristics $\mathcal{P}_u$ on $\mathcal{B}$

Figure 1. Cartesian coordinate space illustrations of the main results

The present paper concerns the analysis up to the *singular boundary*, which we denote by[1] "$\mathcal{B}$," while our companion work [3] concerns the analysis up to the *Cauchy horizon*, which we denote by "$\underline{\mathcal{C}}$." Roughly, $\mathcal{B}$ is the sub-manifold-with-boundary[2] of points where the fluid's first-order partial derivatives blow up, while $\underline{\mathcal{C}}$ is a future boundary that "feels the influence" of the singularity, even though the solution can be uniquely smoothly extended to $\underline{\mathcal{C}} \backslash \partial_-\mathcal{B}$. Our work [3] will show that even though $\underline{\mathcal{C}}$ is a boundary of the classical solution region, the solution canonically induces a limit geometry on it so that in the standard differential structure associated to the Cartesian coordinates, $\underline{\mathcal{C}}$ has the structure of an acoustically[3] null submanifold-with-boundary. Readers can jump to Theorem 1.4 for an abbreviated version of the main results of this paper, and Theorems 31.1 and 34.1 for precise, extended statements. Readers can also consult Appendix C, where, in the drastically simplified contexts of the $1D$ Burgers' equation and the $1D$ compressible Euler equations, we provide a gentle introduction to various subtleties and degeneracies that we encounter in our analysis (some of which have not been described in prior literature, even in $1D$!) as well as an introduction to the various geometric points of view that we use to study the flow. As we describe in Sect. 1.4, our papers resolve several open problems and allow one to

[1] More precisely, we follow the solution up to a compact portion of the singular boundary that we denote by $\mathcal{B}^{[0,\mathfrak{n}_0]}$ in our main theorems.

[2] It is not obvious that the set of blowup-points has the structure of a sub-manifold-with-boundary in Cartesian coordinate space. Indeed, uncovering this structure is one of the main results of the paper. While this structure holds for open sets of solutions, including the solutions we handle in this paper, for other solutions, the set of blowup-points might fail to have the structure of a sub-manifold-with-boundary.

[3] The word "acoustic" refers to the acoustical metric $\mathbf{g}$ of definition (2.15a), which is the solution-dependent Lorentzian metric that dictates the geometry of sound waves. Throughout the paper, all Lorentzian geometric notions such as spacelike, timelike, null, etc. are with respect to $\mathbf{g}$.

properly set up the *shock development problem*, which is the problem of (locally) describing the transition of the solution from classical to weak, past the "initial singularity," which is the acoustically spacelike co-dimension 2 sub-manifold that we denote by "$\partial_-\mathcal{B}$" (the past boundary of $\mathcal{B}$, where $\partial_-\mathcal{B} = \mathcal{B}\cap\underline{\mathcal{C}}$) in the figure. In full generality, the shock development problem is open, though there has been inspiring progress, which we describe in Sect. 1.9.12. It is crucially important, for example in setting up the shock development problem, that our approach yields a complete description of the initial singularity. We highlight the following key point:

> The "initial singularity" should not be thought of as a point in spacetime, but rather the set $\partial_-\mathcal{B}$ depicted in Fig. 1A, a subset of $\mathcal{B}$ that we refer to as *the crease*. In the solution regime under study, we construct an entire connected component of $\partial_-\mathcal{B}$, and we prove that it has the structure of a 2-dimensional acoustically spacelike sub-manifold. The crease is distinguished by the following key property, tied to causality: unlike points in $\mathcal{B}\backslash\partial_-\mathcal{B}$, points in $\partial_-\mathcal{B}$ are past limit points of acoustically causal curves that are contained in a region where the solution can uniquely be extended to exist classically. More precisely, if $p \in \partial_-\mathcal{B}$, then there is a past-directed acoustically null geodesic in $\underline{\mathcal{C}}\backslash\partial_-\mathcal{B}$ (a set along which the fluid can uniquely be extended to as to be smooth) such that the null geodesic terminates at p, where the fluid's first-order partial derivatives blow up. In Fig. 1A, we depict one of these "null generators of $\underline{\mathcal{C}}$."

While we treat in detail a specific regime in our two papers – perturbations of simple isentropic plane-symmetric solutions – the methods we develop are robust and could be applied to other regimes, such as perturbations of non-vacuum steady state solutions in $\mathbb{R}^{1+3}$. We already stress that *our results apply to situations in which $\mathcal{B}$ fails to be strictly convex* (that is, whenever it fails to be strictly concave up), and that even for irrotational and isentropic solutions, substantial new ideas are needed in this case to follow the solution up to $\mathcal{B}$ and to understand its structure; see Sect. 1.3. In Fig. 1, we have depicted a $\mathcal{B}$ that, while "convex in the x^1-direction," it fails to be convex "in the (x^2, x^3)-directions." We mention here that the very recent result of Shkoller–Vicol [66] on the $2D$ isentropic Euler equations, which uses some interesting new tools, constructs an $\mathcal{O}(\epsilon)$-portion of *strictly convex* portions $\partial_-\mathcal{B}$, $\mathcal{B}$, and $\underline{\mathcal{C}}$ for some initial data with gradient of size $\frac{1}{\epsilon}$; see Sect. 1.8 for further discussion.

1.1. **Outline of the remainder of the introduction.** In Sect. 1.2, we introduce the compressible Euler equations, though not in the form we use to prove our main results. In Sect. 1.3, we highlight some of the challenges in the proofs of our main results and discuss a few of the most important new ideas we use to overcome them. In Sect. 1.4, we provide an overview of the two open problems that we resolve in the paper, and we describe how they are connected to the shock development problem. In Sect. 1.5, we state Theorem 1.4, which is a first, somewhat informal, abbreviated version of our main results; see Theorems 31.1 and 34.1 for the extended, precise statements. In Sect. 1.6, having stated Theorem 1.4, we provide some extended remarks on the ideas and methods we use in the proof. In Sect. 1.7, we describe the most relevant precursor results to this paper, focusing on those works that yielded methods that we use here. In Sect. 1.9, we discuss the history of the study of shock formation and highlight some important developments in the subject. In Sect. 1.10, we provide an overview of the main ideas of the proofs of our main results, in particular highlighting various technical issues that were not discussed earlier in the introduction. In Sect. 1.11, we provide an outline of the remainder of the paper.

1.2. **First version of the equations and local well-posedness.** Our main results concern the Cauchy problem for the $3D$ compressible Euler equations, which can be formulated as a quasilinear hyperbolic PDE system in the velocity $v : \mathbb{R}\times\Sigma \to \mathbb{R}^3$, the density $\rho : \mathbb{R}\times\Sigma \to [0,\infty)$, and the entropy $s : \mathbb{R}\times\Sigma \to \mathbb{R}$. In this paper, $\Sigma \stackrel{\text{def}}{=} \mathbb{R}\times\mathbb{T}^2$ denotes the "space manifold," that is, we assume that "space" is diffeomorphic to $\mathbb{R}\times\mathbb{T}^2$. In our setup, the Cartesian coordinates on spacetime are (t, x^1, x^2, x^3), where t is the standard Cartesian time function, x^1 denotes the standard Cartesian coordinate on $\mathbb{R}$, and (x^2, x^3) denote standard Cartesian coordinates on $\mathbb{T}^2 \stackrel{\text{def}}{=} [-\pi,\pi]^2$ (with the endpoints identified). The spatial topology $\mathbb{R}\times\mathbb{T}^2$ allows us to simplify various aspects of our approach, leading to a cleaner presentation of the analysis. However, our analysis is local in spacetime and, with modest additional effort, all our results could be extended to other spatial topologies such as $\mathbb{R}^3$. Relative to the Cartesian coordinates (t, x^1, x^2, x^3), the $3D$ compressible equations can be expressed as follows[4] (see [28] for background on the equations):

$$\mathbf{B}v^i = -\frac{\partial_i p}{\varrho}, \qquad (i = 1, 2, 3), \tag{1.1a}$$

$$\mathbf{B}\varrho = -\varrho \operatorname{div} v, \tag{1.1b}$$

$$\mathbf{B}s = 0, \tag{1.1c}$$

where p is the pressure, $\mathbf{B}$ denotes the *material vectorfield*:

$$\mathbf{B} \stackrel{\text{def}}{=} \partial_t + v^a \partial_a, \tag{1.2}$$

and div is the standard Euclidean divergence operator (see Def. 2.3). To close the equations, we assume an *equation of state* $p = p(\varrho, s)$. Our results apply for *any* sufficiently smooth equation of state – except for that of a Chaplygin gas (see Sect. 2.3.1) – with positive sound speed c defined by:

$$c \stackrel{\text{def}}{=} \sqrt{p_{;\varrho}}, \tag{1.3}$$

where $p_{;\varrho}$ is the partial derivative of the equation of state with respect to the density at fixed entropy.

We stress up front that our analysis crucially relies on a geometric reformulation of (1.1a)–(1.1c) as a system of covariant wave equations coupled to transport-div-curl equations, where the nonlinear terms exhibit remarkable null and regularity properties. The reformulation was derived in [72] (see also the precursor [51]), and we recall it in Theorem 2.15.

We assume that smooth initial data $(v, \varrho, s)|_{\Sigma_0}$ for (1.1a)–(1.1c) are prescribed along the spacelike hypersurface $\Sigma_0 \stackrel{\text{def}}{=} \{t = 0\} = \{0\} \times \mathbb{R} \times \mathbb{T}^2$. We consider only initial data such that $\varrho|_{\Sigma_0} > 0$, thereby avoiding the severe degeneracies that can occur at fluid-vacuum boundaries. It is well-known that the equations (1.1a)–(1.1c) are locally well-posed for non-vacuum initial data on $\Sigma_0 \stackrel{\text{def}}{=} \{t = 0\}$ such that $(v, \varrho, s)|_{\Sigma_0} \in H^3(\Sigma_0)$. To follow the solution to the singular boundary, we assume that the data belong to a sufficiently high order Sobolev space, where different solution variables have distinct, directionally dependent amounts of regularity. In Sect. 11, we state detailed assumptions on the state of the solution near the shock; starting the analysis "near the shock" allows us to focus on the most interesting aspect of the dynamics. The assumptions of Sect. 11 are the main ones we need to close our proof. The assumptions of Sect. 11 hold for open sets of solutions whose data on Σ_0 are close, in a high order Sobolev space, to the data of a family of "background" simple isentropic plane symmetric solutions. All of the known approaches to studying shocks away from symmetry rely on the assumption that the data belong to a high order Sobolev space. This is due to possibly singular energy estimates at the high derivative levels – even in the "good" geometric coordinate system (t, u, x^2, x^3), described below, which "hides" the singularity at the low to mid derivative levels; see Sects. 1.10.9 and (1.10.12) for further discussion of these fundamental technical issues.

1.3. **An overview of the degeneracies and difficulties in the problem.** The analysis needed to fully justify Fig. 1 is fraught with degeneracies and difficulties. While many prior works on shocks have constructed the solution in *strict subsets* of the region in Fig. 1A, our papers are the first to fully grapple with the degeneracies and construct the solution in the entire region. To handle solutions with non-zero vorticity and dynamic entropy, we rely on an arsenal of geometric and analytic techniques, developed in earlier works [4, 24, 50–52, 72], combined with key new ideas that we describe below. Here, we highlight some of the main challenges in the analysis and mention some of the methods we use to overcome them.

- (**Singularities**). The solution forms a shock along the sub-manifold-with-boundary $\mathcal{B}$. That is, some of its first-order partial derivatives with respect to the Cartesian coordinates blow up along $\mathcal{B}$ (including along the crease $\partial_{-}\mathcal{B}$), though the solution itself remains bounded. The blowup-dynamics are extremely rich: some quantities exhibit blowup of their derivatives – but only derivatives in certain directions[5] – while other quantities <u>and</u> their derivatives in all directions remain bounded. We refer to Theorem 34.1 for the details.
- (**Nonlinear geometric optics and geometric coordinates**) As in many prior works on shock formation, to follow the solution up to $\mathcal{B}$ and to obtain a precise understanding of the singularity formation, we cannot rely on the Cartesian coordinates, which are not adapted to the singularity. Instead, we rely on nonlinear geometric optics.

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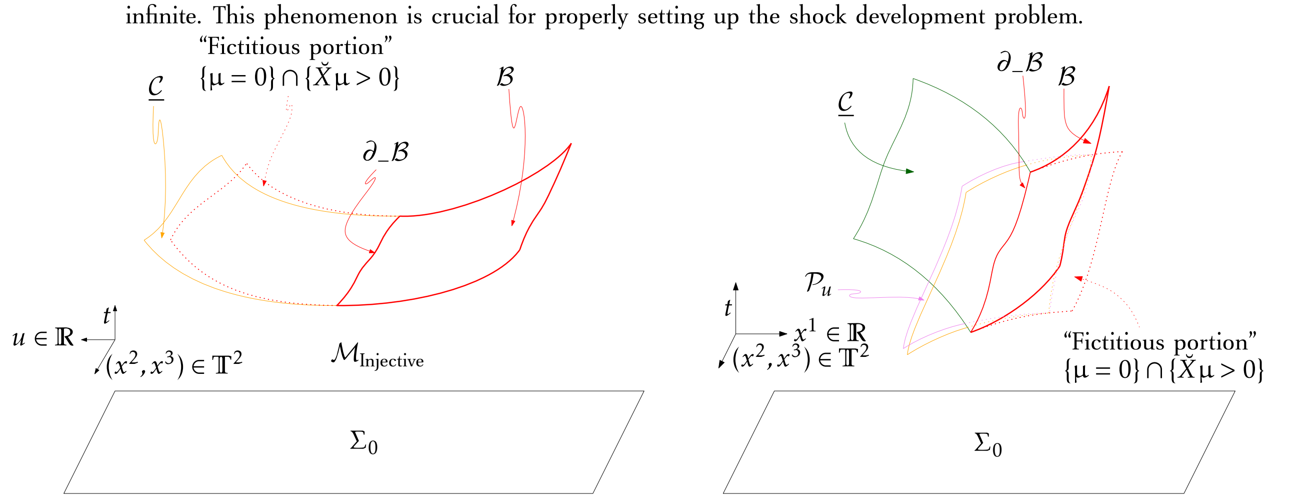
[4]Throughout, if $\mathbf{V}$ is a vectorfield and f is a scalar function, then $\mathbf{V}f \stackrel{\text{def}}{=} \mathbf{V}^\alpha \partial_\alpha f$ denotes the derivative of f in the direction of $\mathbf{V}$.

[5]Roughly, along $\mathcal{B}$, many quantities' derivatives in directions *transversal* to the characteristics blow up, while their tangential derivatives remain bounded, much like in the simple case of Burgers' equation in $1D$: $\partial_t \Psi + \Psi \partial_x \Psi = 0$.

Specifically, nonlinear geometric optics yields a "geometric coordinate system" (see Def. 3.5) – one that is **globally homeomorphic** (in the compact region under study) to the Cartesian coordinate system but **not diffeomorphic to it** up to $\mathcal{B}$ – relative to which the solution remains rather smooth; see Sect. 1.6. The key ingredient in implementing nonlinear geometric optics is an *eikonal function* u solving the acoustic *eikonal equation* $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, where the *acoustical metric* $\mathbf{g}$ is the solution-dependent Lorentzian metric (see (2.15a)) that captures the intrinsic geometry of sound waves. The level-sets of u, which we denote by $\mathcal{P}_u$, are characteristic for the compressible Euler equations; roughly, the $\mathcal{P}_u$ represent surfaces along which sound waves can propagate. In the present paper, our *geometric coordinate system* is (t, u, x^2, x^3), where u is the eikonal function and t, x^2, x^3 are standard Cartesian coordinate functions.

- (**Degeneracies in the acoustic geometry**) The fluid singularity along $\mathcal{B}$ is intimately tied to degeneracies in the acoustic geometry; roughly, along $\mathcal{B}$, the characteristic hypersurfaces $\mathcal{P}_u$ (which we also refer to as "characteristics," "null hypersurfaces," "acoustically null hypersurfaces," or "$\mathbf{g}$-null hypersurfaces"), develop *infinite density* along $\mathcal{B}$. In the present paper, the infinite density of the $\mathcal{P}_u$ is characterized by the vanishing of a function μ, the inverse foliation density, which we describe below in detail; μ is positive in the maximal development, except along $\mathcal{B}$. We depict this infinite density in Fig. 1B, where we show three distinct null hypersurfaces "piling up" along $\mathcal{B}$. It is important to appreciate that in the solution regime we are studying, *distinct characteristic hypersurfaces, viewed as sub-manifolds in Cartesian coordinate space, never actually intersect*[6] on $\mathcal{B}$, even though their density becomes infinite. This phenomenon is crucial for properly setting up the shock development problem.



(A) Singular boundary and Cauchy horizon in geometric coordinates(B) Singular boundary and Cauchy horizon in Cartesian coordinates

Figure 2. Illustrations of the main results in two coordinate systems

Our main results show that the singular boundary $\mathcal{B}$, viewed as a subset of geometric coordinate space (see Fig. 2A), is:

$$\mathcal{B} = \{\mu = 0\} \cap \{\breve{X}\mu \leq 0\}, \tag{1.4}$$

where the vectorfield $\breve{X}$, described later on (see Fig. 10 for a depiction of $\breve{X}$ in Cartesian coordinate space), is transversal to the $\mathcal{P}_u$ and satisfies $\breve{X}u = 1$. The crease $\partial_-\mathcal{B}$ is characterized by:

$$\partial_-\mathcal{B} = \{\mu = 0\} \cap \{\breve{X}\mu = 0\}. \tag{1.5}$$

From (1.4)–(1.5), the status of $\partial_-\mathcal{B}$ as a boundary of $\mathcal{B}$ is clear. In the regime under study, *the two level-sets on RHS* (1.5) *intersect transversally*, which is what gives the crease the structure of a $2D$ sub-manifold. The transversality of the intersection is a consequence of *acoustical transversal convexity*, which is mild condition satisfied by open sets of solutions and which we describe below in more detail. By doing *formal* Taylor expansions in geometric coordinates starting from the crease, one could check that the "fictitious portion" $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$ – if it existed – would lie in the timelike future of the Cauchy horizon $\underline{\mathcal{C}}$; see the dotted portion in Fig. 2A,

[6]This fact follows from the homeomorphism property of the map Υ, as described in Theorem 1.4.

which formally depicts $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$. Since $\underline{\mathcal{C}}$ lies in the causal future of the singularity along $\partial_{-}\mathcal{B}$, $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$ cannot be part of the maximal development.

We now highlight that the complementary set $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$ is a "fictitious portion" that is not part of the singular boundary or the maximal development. In the context of Fig. 2A in geometric coordinate space, the irrelevance of $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$ for $\mathcal{B}$ and the maximal development can be understood as follows: the singular boundary portion $\mathcal{B}$ in the figure cannot be extended to the left into the region where $\breve{X}\mu$ would be positive because before that region has a chance to dynamically develop, it will be cut off by a Cauchy horizon emanating from the crease $\partial_{-}\mathcal{B}$, which is the left boundary of $\mathcal{B}$ in the figure. Moreover, Figs. 2A–2B exhibit the following issue: the map from geometric coordinates to Cartesian coordinates *would have failed to be injective* on the full region depicted Fig. 2A (which includes the fictitious portion of the singular boundary) , though the map is injective on $\mathcal{M}_{\text{Injective}}$, which we define to be the region trapped in between Σ_0 and $\underline{\mathcal{C}} \cup \mathcal{B}$; see Fig. 2A. We clarify that although the present paper and our companion work [3] collectively exhibit the injectivity of the map from geometric coordinates to Cartesian coordinates on $\mathcal{M}_{\text{Injective}}$, one cannot literally prove the failure of injectivity on the extended region containing $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$ because one cannot actually construct the solution up to $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$. However, in the solution regime we study in this article, failure of injectivity of the map on the extended region (which would include $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$) could formally be shown through Taylor expansions of the solution and the acoustic geometry at the crease. As a consequence of the formal failure of injectivity, we emphasize that in Fig. 2B (which is in Cartesian coordinates), the fictitious portion $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$ (displayed in dotted lines in Fig. 2B) would have been located in a region that is already covered by geometric coordinates corresponding to the region $\mathcal{M}_{\text{Injective}}$. A related fact is that the particular characteristics depicted in Fig. 2B (we have shown two characteristic hypersurfaces and labeled one of them by "$\mathcal{P}_u$") never have the opportunity to develop infinite density because they are cut off by the Cauchy horizon $\underline{\mathcal{C}}$ before they have a chance to enter the region where they would have piled up along $\{\mu = 0\} \cap \{\breve{X}\mu > 0\}$.

- (**Null hypersurfaces and PDE energy degeneracies**). In Fig. 1A, $\mathcal{B}$ and $\underline{\mathcal{C}}$ are acoustically null hypersurfaces emanating from the crease, and in particular, $\mathcal{B}$ is ruled by acoustically null curves whose tangent vectors are denoted by L in the figure; see Prop. 33.2 for a proof of these properties of $\mathcal{B}$, and see Remarks 32.8 and Remark 33.3 for a discussion of some interesting degeneracies that occur along $\mathcal{B}$. As is well-known, any L^2-type energy that one uses to control solutions necessarily degenerates along null hypersurfaces, becoming only positive semi-definite instead of positive definite. This is a particularly challenging issue in the present context, where singularities are forming along all of $\mathcal{B}$.
- (**Regularity and rough foliations**). Many objects in the construction (in particular, the crease $\partial_{-}\mathcal{B}$) are less regular than the fluid solution, which leads to difficult regularity theory for the problem. To "detect" these rough objects as they emerge in the course of the evolution, we rely on a new family of *rough foliations* given by the level-sets of *rough time functions*, described below. The word "rough" refers to the fact that the rough time functions are also less regular than the fluid solution. There is another way in which the problem of shock formation can be viewed as a low regularity problem: the piling up of the characteristics is tied to the blowup of a Euclidean-unit-length derivative of various fluid variables in directions *transversal* to the characteristics, even though the solution remains rather smooth in directions *tangent* to the characteristics. In particular, we are forced to close the estimates knowing that with respect to the Cartesian differential structure, there will be *no differentiability* in transversal directions at the end of the classical evolution. As we already mentioned, the geometric coordinates (t, u, x^2, x^3) partially ameliorate this difficulty in the sense that the solution remains rather smooth with respect to them. Nonetheless, as in other works on shock formation, our high order geometric energies can still become singular; see Sect. 1.10.12. This is one of the main technical challenges in the PDE analysis since singular high order energy estimates make it difficult for us to prove that the solution's partial derivatives with respect to the geometric coordinates remain bounded at the lower derivative levels.
- (**One sub-manifold of $\mathcal{B}$ at a time via a family of rough time functions**). Our approach to constructing $\mathcal{B}$ is to show that it can be foliated by a family of $\mathfrak{n}$-parameterized sub-manifolds $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$ with $\mathfrak{n} \geq 0$ a real parameter, and to construct each $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$, "one $\mathfrak{n}$ at a time;" see Fig. 4. Our construction is such that the crease $\partial_{-}\mathcal{B}$ coincides with $\breve{\mathbf{T}}_{0,0}$. One might wonder why we didn't try to derive all of $\mathcal{B}$ "at the same time." From the discussion three points above, we see that that approach would have effectively required us to work with foliations of spacetime that contain or are asymptotic to level-sets of μ in regions where μ is small (recall that $\mathcal{B}$ is a portion of $\{\mu = 0\}$). The difficulty is that for real numbers $\mathfrak{m}$ small and positive, near the crease, the level-sets $\{\mu = \mathfrak{m}\}$ have a $\mathbf{g}$-timelike portion, along which top-order L^2 estimates for the solution are not available. This can formally be

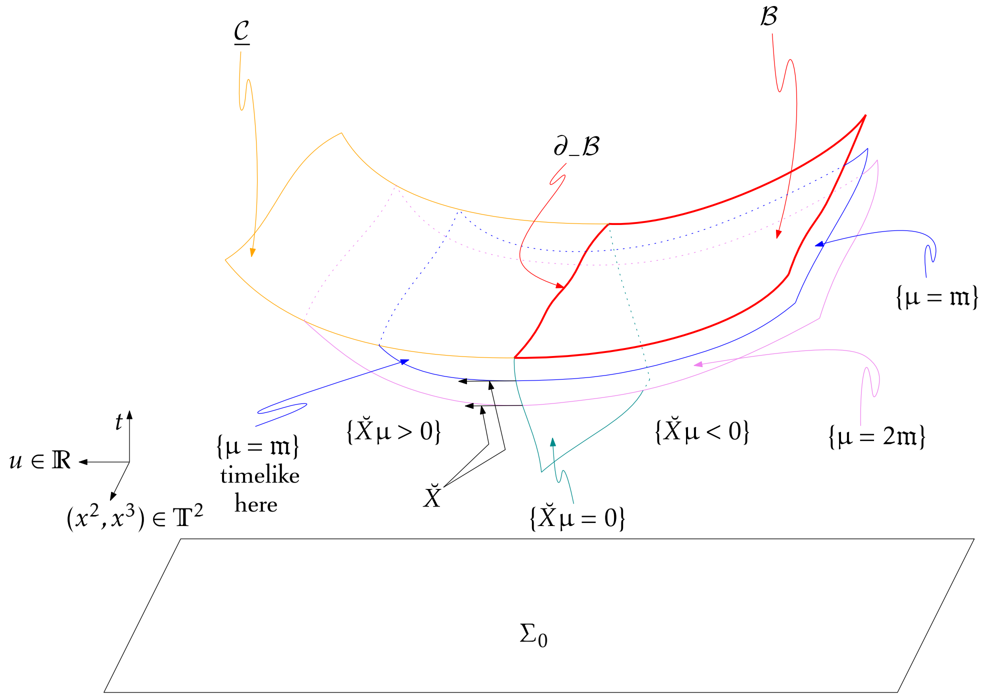


Figure 3. The singular boundary and the level-sets $\{\mu = \mathfrak{m}\}$ and $\{\mu = 2\mathfrak{m}\}$ for a small $\mathfrak{m} > 0$, in geometric coordinates

understood as the statement that in the solution regime under study, along the surface $\{\mu = 0\}$, RHS (32.31b) would be positive[7] in the regions where $\breve{X}\mu > 0$. See Fig. 3, which shows these level-sets becoming $\mathbf{g}$-timelike in the region near their intersection with the Cauchy horizon $\underline{\mathcal{C}}$. In fact, for $\mathfrak{m} > 0$ small, any surface that agreed with the level-set $\{\mu = \mathfrak{m}\}$ up to second-order along the surface $\{\breve{X}\mu = 0\}$ would suffer from the same difficulty: it would necessarily contain a $\mathbf{g}$-timelike portion, along which L^2 estimates for the solution are not available.

Remark 1.1 (Impossibility of C^2 spacelike foliations and the limited regularity of ${}^{(\text{Interesting})}\tau$)**.** The upshot is that in the solution regime under study, it is impossible to detect the entire singular boundary $\mathcal{B}$ by deriving estimates on C^2 (relative to the geometric coordinates) $\mathbf{g}$-spacelike foliations of spacetime such that $\mathcal{B}$ (including its past boundary $\partial_{-}\mathcal{B}$) is contained in the interior of one of the leaves of the foliation. This difficulty is connected to the following issue: in our main theorem, namely Theorem 34.1, the region $\mathcal{M}_{\text{Interesting}}$ that we study (which contains $\mathcal{B}$) is foliated by the level-sets of a $C^{1,1}$ (relative to the geometric coordinates) time function ${}^{(\text{Interesting})}\tau$, and *this* $C^{1,1}$ *regularity is optimal* given the shape of region; see Remark 32.10. In our approach to the PDE analysis, the $C^{1,1}$ regularity of ${}^{(\text{Interesting})}\tau$ would be *insufficient* for our proofs of some of our estimates (e.g., the co-dimension two Gauss curvature estimates we derive in Lemma 28.11). We circumvent these difficulties by avoiding deriving high order PDE estimates on the level sets of ${}^{(\text{Interesting})}\tau$; we instead derive estimates along the level sets of the time functions ${}^{(\mathfrak{n})}\tau$, which, though less regular than the fluid solution, are more regular than ${}^{(\text{Interesting})}\tau$.

To construct the sub-manifolds $\breve{\mathbf{T}}_{0,-\mathfrak{n}} \subset \mathcal{B}$ and circumvent the difficulties noted in Remark 1.1, we proceed as follows. For each $\mathfrak{n} \in [0, \mathfrak{n}_0]$, where $\mathfrak{n}_0 > 0$ is a small, data-dependent constant, we construct a *rough time function* ${}^{(\mathfrak{n})}\tau$, which is $C^{2,1}$ relative to the geometric coordinates; see Sect. 4. Note that ${}^{(\mathfrak{n})}\tau$ is one degree more differentiable than the time function ${}^{(\text{Interesting})}\tau$ mentioned above, which is sufficient for avoiding the difficulties we highlighted in Remark 1.1. Our construction of ${}^{(\mathfrak{n})}\tau$ depends on the eikonal function u, i.e., our construction of ${}^{(\mathfrak{n})}\tau$ relies on nonlinear geometric optics. For each $\mathfrak{n} \in [0, \mathfrak{n}_0]$, the level-sets of ${}^{(\mathfrak{n})}\tau$ yield a foliation of spacetime

[7] In the regime under study, the term $L\mu$ on RHS (32.31b) is strictly negative and the term $|\nabla\!\!\!/\mu|^2_{g\!\!\!/}$ is of negligible size.

by $\mathbf{g}$-spacelike hypersurfaces, and $\check{\mathbf{T}}_{0,-\mathfrak{n}} \subset \{^{(\mathfrak{n})}\tau = 0\}$. Hence, to construct $\check{\mathbf{T}}_{0,-\mathfrak{n}}$, it suffices to control the fluid up to the level-set $\{^{(\mathfrak{n})}\tau = 0\}$; the vast majority of our efforts in this paper are dedicated towards that task. The union $\bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \check{\mathbf{T}}_{0,-\mathfrak{n}}$ is the portion of $\mathcal{B}$ that we construct in our main theorem. The portion of $\mathcal{B}$ that formally corresponds to $\mathfrak{n} < 0$ never has a chance to emerge in the maximal classical development because it is cut off by the Cauchy horizon $\underline{\mathcal{C}}$. In Fig. 2A, the "irrelevant portion" of $\mathcal{B}$, corresponding to $\mathfrak{n} < 0$, is formally delineated by dotted curves (we cannot actually construct this portion, and we have displayed it only to illustrate that $\underline{\mathcal{C}}$ lies below it). We could have extended our results to handle a larger range of $\mathfrak{n}$ values, i.e., $\mathfrak{n} > \mathfrak{n}_0$; we avoided this because we would have had to modify our construction of the $^{(\mathfrak{n})}\tau$ for large $\mathfrak{n}$, which would have lengthened the paper. We refer Sect. 1.4 for a more detailed overview of our construction of the rough time functions and the $\check{\mathbf{T}}_{0,-\mathfrak{n}}$.

Remark 1.2 (The terminology "rough time function")**.** The word "rough" in "rough time function" refers to the fact that the elements of $\{^{(\mathfrak{n})}\tau\}_{\mathfrak{n}\in[0,\mathfrak{n}_0]}$ are less regular than the fluid; see Sect. 1.10.7 for further details. In the present paper, each $^{(\mathfrak{n})}\tau$ is $C^{2,1}$ (with respect to the geometric coordinates) because the fluid variables are $C^{3,1}$. The fluid variables are $C^{3,1}$ because we have only assumed limited differentiability on the fluid data in directions transversal to the characteristics $\mathcal{P}_u$, even though we assumed they are much smoother in the $\mathcal{P}_u$-tangential directions; see Sect. 11 for our data-assumptions. Despite the terminology "rough time function," if we had instead assumed that the fluid data were C^∞, then each $^{(\mathfrak{n})}\tau$ would also have been C^∞. In contrast, even with C^∞ fluid data, the time function $^{(\mathrm{Interesting})}\tau$ from Remark 1.1 would have had only $C^{1,1}$ regularity.

- (**Lack of strict convexity**). Observe that in Fig. 1A, there are points $q \in \mathcal{B}$ such that the tangent plane to $\mathcal{B}$ at q, which we denote by $T_q\mathcal{B}$ and view to be a subset of $1+3$-dimensional Cartesian coordinate space, does not lie below $\mathcal{B}$. Moreover, there are points $q \in \partial_-\mathcal{B}$ such that every (three-dimensional) Cartesian-flat plane that is acoustically causal at q and contains the two-dimensional subspace $T_q\partial_-\mathcal{B}$ (which would look one-dimensional if drawn in Fig. 1A, due to our suppression of a spatial dimension), fails to lie below $\mathcal{B}$, *even locally near* q. Let us informally refer to these phenomena as "absence of strict convexity," where here, "convexity" informally refers to "upwards bending," and "strict convexity" – though not featured Fig. 1A – would refer to "upwards bending in all directions." The absence of strict convexity poses serious technical difficulties:

  In the absence of symmetry and strict convexity, the entirety of the crease $\partial_-\mathcal{B}$ has never before been fully constructed for any open set of shock-forming solutions to any hyperbolic PDE. In particular, this aspect of our main results is new even in the case of irrotational and isentropic solutions.

  The rough time functions $^{(\mathfrak{n})}\tau$ and corresponding rough foliations allow us to derive, through a fully constructive approach, the structure of the singular boundary, even if it is not strictly convex. Instead of strict convexity, we rely on *acoustical transversal convexity* (which we refer to as transversal convexity for short), which allows us to handle, for example, perturbations of symmetric solutions, where strict convexity of the singular boundary can fail due to the approximate symmetry. Roughly, transversal convexity is a form of convexity only in a particular direction, specifically in a direction that is transversal to the level-sets of the eikonal function u. Note that transversal convexity refers to the structure of $\mathcal{B}$ viewed as an embedded sub-manifold of geometric coordinate space (as opposed to Cartesian coordinate space). In the solution regime under study, transversal convexity is captured by our data-assumption (11.18) on the inverse foliation density μ, which we are able to propagate throughout the evolution (see (18.5)). The singular boundary $\mathcal{B}$ in Fig. 2A enjoys transversal convexity (roughly, $\mathcal{B}$ is "parabolic in the u direction" at fixed (x^2, x^3) near $\partial_-\mathcal{B}$). One could check that in the solution regime under study, the transversal convexity of $\mathcal{B}$ in geometric coordinates also implies, in the Cartesian coordinate picture, the convexity of the x^1-parameterized curves in $\mathcal{B}$ along which (x^2, x^3) are fixed; while we do not directly need this "Cartesian transversal convexity" in our analysis, we have exhibited the "upwards bending[8] of $\mathcal{B}$ in the x^1-direction" in Fig. 1A. Our assumption of transversal convexity is close to optimal in the sense that without it, the qualitative character of the singular boundary can dramatically change, even for plane-symmetric solutions; see Sect. 1.6 for further discussion. We also highlight that transversal convexity was used by Christodoulou in his resolution [25] of the restricted shock development problem, which we describe below.

[8]In $1+1$ dimensions, under transversal convexity, the embedding of the singular boundary in Cartesian coordinate space can be modeled by the u-parameterized curve $t = u^2$, $x^1 = u^2 + u^3$, for $u \leq 0$. Near the origin (which models the crease) in (t, x^1)-space, this singular boundary-modeling curve is asymptotic to the graph of $t = (x^1)^2 + (x^1)^{3/2}$ for x^1 small and positive, which bends upwards (c.f. the singular boundary in Fig. 1A) and has regularity $C^{1,\frac{1}{2}}$ (c.f. the regularity of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ stated in Theorem 1.4).

- (**Degenerate wave and elliptic-hyperbolic estimates on curved domains**). To close the high order L^2 estimates, we must adapt a variety of hyperbolic energy estimates and "top-order" elliptic estimates for the vorticity, entropy, and geometry to the precise shape of $\partial_{-}\mathcal{B}$, $\mathcal{B}$, and $\underline{\mathcal{C}}$, *which are not known in advance*. The shock singularity introduces degeneracies into these estimates, and when we control the top-order derivatives of the vorticity and entropy using elliptic estimates, our handling of these degeneracies requires our observation of special cancellations within delicately constructed "elliptic-hyperbolic" identities. To exhibit the cancellations, we rely on the full nonlinear structure of the geometric formulation of compressible Euler flow provided by Theorem 2.15. In constructing the elliptic-hyperbolic identities (see Sect. 21), we rely on the framework we developed in [4]. However, for the purposes of the present paper, we had to substantially upgrade that framework to accommodate the structure of the singularity. The key new object that we use to derive the elliptic-hyperbolic identities is a well-constructed *characteristic current*, defined in Def. 21.10.

1.4. **The two open problems resolved by Theorem 1.4 and connections to the shock development problem.** In his breakthrough 2007 monograph [24], Christodoulou gave a sharp description of the stable formation of shock singularities, starting from open sets of smooth initial data, in solutions to the $3D$ irrotational and isentropic relativistic Euler equations. Together with Miao, he later extended his results to the $3D$ compressible Euler equations [28], again for irrotational and isentropic solutions. These results revealed a large subset of the maximal classical development, including a portion of the boundary.

In the wake of [24], there have been many exciting developments on the formation of shock singularities and the subsequent evolution of the solution as a weak solution, after the shock; see Sect. 1.9 for further discussion. However, two fundamental problems have remained open:

1. (**The full structure of bounded portions of the maximal classical development**). As is explained on [24, Pages 929, 968–969], Christodoulou's approach yields a union of developments of the initial data, where each of his developments can be foliated by portions of *Cartesian-flat*[9] spacelike hypersurfaces and portions of characteristic hypersurfaces. By varying the "angle of tilt" of these Cartesian-flat hypersurfaces and varying the initial data of the characteristic hypersurfaces, one obtains (see [24, Pages 929, 968–969]) "a larger part" of the maximal development. While this approach yields a sharp description of some portion of the maximal development, the precise portion that it reveals is not made explicit through the construction. Moreover, from Fig. 1A, one can infer that for some solutions, there are portions of the boundary (in particular, portions of $\partial_{-}\mathcal{B}$) that are *not accessible* through Cartesian-flat spacelike foliations. This is connected to the lack of strict convexity of the singular boundary, as we discussed in Sect. 1.3; see also Fig. 6 and, in Sect. 1.6, our discussion of the points b_1 and b_2 featured in the figure. Hence, the following problem is glaring:

   Can one derive the *full structure* of the maximal development, at least in some region of spacetime that includes a neighborhood of the boundary that contains an entire connected component of the crease?

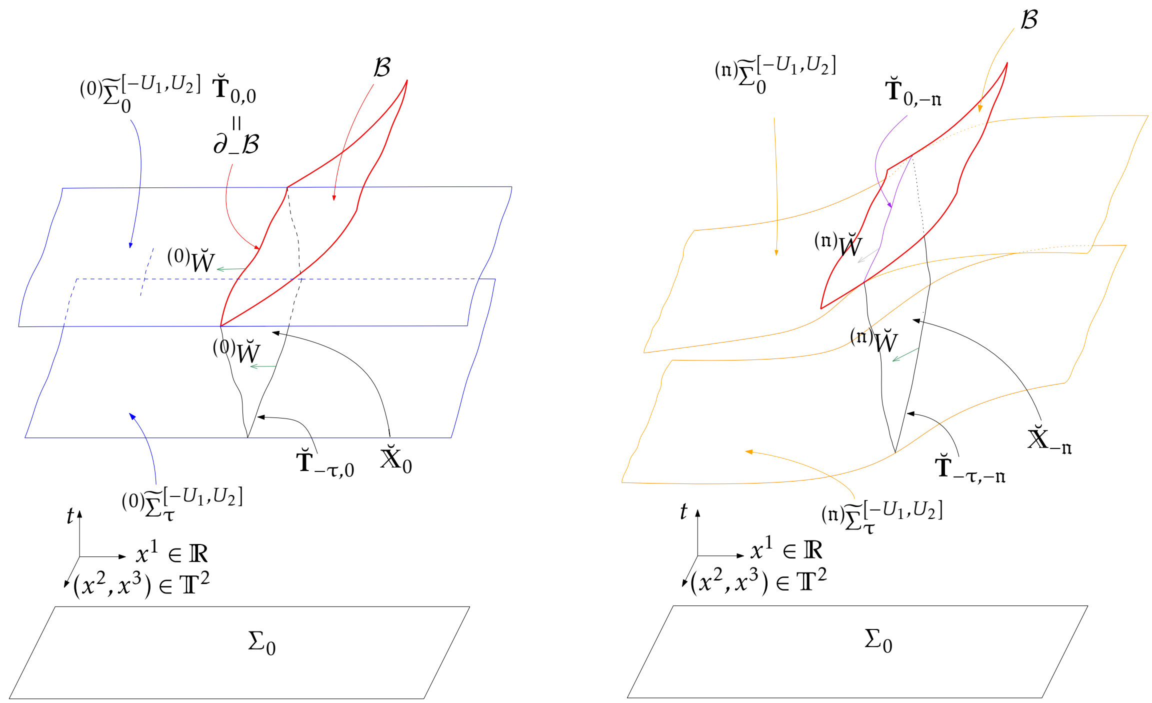
   While mathematically rich in itself, this problem is important for two other fundamental reasons: **I)** The breakthrough result [38] shows that in general, one cannot ensure uniqueness of the maximal development until one constructs it and proves that it enjoys some crucial structural properties. For quasilinear hyperbolic PDEs, the question of uniqueness of the maximal development is *global* in nature. However, the results we derive in the present article and the companion [3] exhibit a localized version of the crucial property that the maximal development "lies on one side of its boundary." In [38], the authors showed, roughly speaking, that if this property holds globally, then the maximal development is unique. **II)** As is explained in [25], the full structure of a neighborhood of the boundary of the maximal development is an essential ingredient for properly setting up the aforementioned shock development problem.
2. (**Removing the irrotationality and isentropicity assumptions**). Assuming a positive resolution to the first problem, a second one of clear physical importance stands out:

   Can one extend the result away from the irrotational and isentropic class of solutions, that is, to handle solutions with non-zero vorticity and dynamic entropy?

   It is of fundamental importance to understand such "general solutions" because vorticity and entropy will form in the weak solution (see, e.g., [25]), i.e., "after the first shock," even if the initial data are irrotational, isentropic, and C^{∞}. That is, if one aims towards developing a global-in-space-and-time theory that accommodates the formation of shocks and their subsequent interactions, *vorticity and entropy are an unavoidable aspect of the dynamics.*

[9] By a "Cartesian-flat" hypersurface, we mean a plane with respect to the standard Minkowski-rectangular coordinates on $\mathbb{R}^{1+3}$.

In the present paper and its companion [3], we resolve both problems for the $3D$ compressible Euler equations (1.1a)–(1.1c). In the notation of Fig. 1A, starting from smooth data on Σ_0, we construct the classical solution in the region lying in between Σ_0 and[10] $\underline{\mathcal{C}} \cup \partial_-\mathcal{B} \cup \mathcal{B}$. More precisely, our results apply to open sets of smooth data satisfying assumptions described below (see, for example, Theorem 1.4), and we control the solution in bounded regions of spacetime that contain *all*[11] of $\partial_-\mathcal{B}$ and a full neighborhood of it in $\underline{\mathcal{C}}$ and $\mathcal{B}$. For convenience, in the present paper, we have studied the solution only in a single, "spatially local" region. However, our approach could be used as a building block to study the solution across space, at least in regions where the solution exhibits the property of acoustical transversal convexity, mentioned in Sect. 1.3.



(A) Rough foliations adapted to the crease

(B) Rough foliations adapted to a non-crease torus $\breve{\mathbf{T}}_{0,-\mathfrak{n}} \subset \mathcal{B}$

Figure 4. Rough foliations adapted to the singular boundary, depicted in Cartesian coordinate space

In the present paper, we follow the solution up to the singular portion of the boundary, denoted by $\mathcal{B}$ in Figs. 1A and 4A, where some first-order partial derivatives of the density and velocity blow up. We again emphasize that $\mathcal{B}$ contains its past boundary, $\partial_-\mathcal{B}$, which is the aforementioned crease. In [3], we construct the Cauchy horizon, denoted by $\underline{\mathcal{C}}$ in Fig. 1A, which is an acoustically null hypersurface such that no fluid singularity forms along it, but it is nonetheless a boundary of the maximal classical development because its causal past intersects the singularity. As we mentioned earlier, our analysis in the present paper fundamentally relies on a new family of acoustically spacelike *rough foliations* that are precisely and dynamically adapted to the shape of $\mathcal{B}$, which is not known in advance. The foliations are level-sets of a one-parameter family of *rough time functions* $\{{}^{(\mathfrak{n})}\tau\}_{\mathfrak{n}\in[0,\mathfrak{n}_0]}$, where $\mathfrak{n}_0 > 0$ is a constant depending on the initial data. Each ${}^{(\mathfrak{n})}\tau$ is defined on a portion of the maximal classical development union its boundary and has a range $[\tau_0, 0]$ for some constant $\tau_0 < 0$ depending on the initial data. Moreover, ${}^{(\mathfrak{n})}\tau$ has the crucial property that its zero level-set $\{{}^{(\mathfrak{n})}\tau = 0\}$ is tangent to $\mathcal{B}$ and intersects it in a sub-manifold. More precisely, $\mathcal{B} \cap \{{}^{(\mathfrak{n})}\tau = 0\} = \breve{\mathbf{T}}_{0,-\mathfrak{n}}$ is a torus with spacetime co-dimension 2 such that the tori $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$ foliate a neighborhood of $\partial_-\mathcal{B}$ in $\mathcal{B}$, i.e., $\bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{0,-\mathfrak{n}}$ is a neighborhood of $\partial_-\mathcal{B}$ in $\mathcal{B}$. In particular, the crease $\partial_-\mathcal{B}$ is equal to $\breve{\mathbf{T}}_{0,0}$. In Fig. 4A, we exhibit two level-sets of ${}^{(0)}\tau$, where the top one contains the crease. In Fig. 4B, for some $\mathfrak{n} > 0$, we exhibit two level-sets of ${}^{(\mathfrak{n})}\tau$, where the top one contains the torus $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$.

[10] This union is not disjoint, for the crease $\partial_-\mathcal{B}$ is a past boundary of both of the closed sets $\underline{\mathcal{C}}$ and $\mathcal{B}$.

[11] More precisely, we construct an entire connected component of $\partial_-\mathcal{B}$.

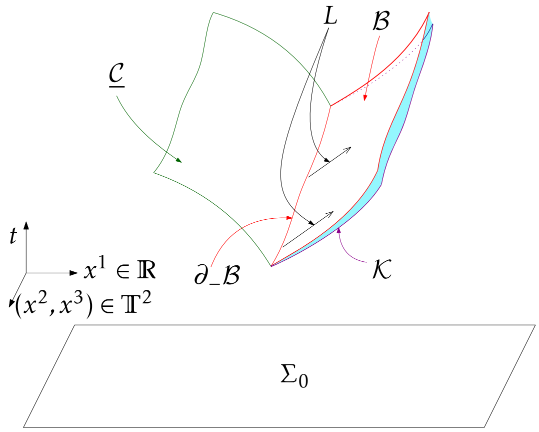


Figure 5. A localized subset of the maximal classical development and the shock hypersurface in Cartesian coordinate space

We already highlight that the hypersurfaces $\breve{\mathbb{X}}_0$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$ in Fig. 4B play a crucial role in our construction of ${}^{(0)}\tau$ and ${}^{(\mathfrak{n})}\tau$ respectively. In particular, by construction, ${}^{(0)}\tau$ and ${}^{(\mathfrak{n})}\tau$ solve transport equations with initial data given on $\breve{\mathbb{X}}_0$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$ respectively; see Sect. 4.2.1 for the details. The surfaces $\breve{\mathbb{X}}_0$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$ are transversal to $\mathcal{B}$ and intersect $\mathcal{B}$ in the tori $\breve{\mathbf{T}}_{0,0}$ and $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$ respectively. While the transversality of $\breve{\mathbb{X}}_0$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$ to $\mathcal{B}$ is crucial for our analysis, the causal structure of $\breve{\mathbb{X}}_0$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$ (i.e., whether they are acoustically timelike, spacelike or null) is *not* important because *we do not have to derive any energy estimates for the fluid* along $\breve{\mathbb{X}}_0$ or $\breve{\mathbb{X}}_{-\mathfrak{n}}$. In particular, $\breve{\mathbb{X}}_0$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$ can be acoustically timelike, spacelike, or null.

In this paper and [3], our approach relies on giving a complete description of the dynamics that in particular shows what blows up and what remains regular and yields a sharp description of the structure and regularity of $\mathcal{B}$, $\partial_{-}\mathcal{B}$, and $\underline{\mathcal{C}}$. To handle the presence of vorticity and entropy, we develop modified versions of the integral identities that we discovered in [4], adapted here to the precise structure of $\mathcal{B}$. More precisely, in the present paper, we rely on a new family of "elliptic-hyperbolic" integral identities that are adapted to the rough foliations and the structure of $\mathcal{B}$, and we develop new analytic techniques to handle the following difficulty, which permeates the paper: *the singular boundary is a degenerate acoustically null hypersurface, which leads to severe degeneracies in the estimates for the fluid variables*, especially in the top-order energy estimates for the vorticity and entropy. See Prop. 33.2 and Remark 33.3 for detailed information on the structure of the singular boundary, including its causal structure and its differential-topological properties as a subset of Cartesian coordinate space.

In the solution regime that we treat in our main results, the crease $\partial_{-}\mathcal{B}$ has the structure of a co-dimension 2 $\mathbf{g}$-spacelike sub-manifold; see Fig. 5. These structures are essential for properly setting up the shock development problem using the known techniques [25]. In fact, without these structures, it is not even clear whether the shock development problem is well-posed. In particular, the shock hypersurface (with boundary), denoted by "$\mathcal{K}$" in Fig. 5, emanates from $\partial_{-}\mathcal{B}$ and is tangent to $\mathcal{B}$ at $\partial_{-}\mathcal{B}$. Roughly, $\mathcal{K}$ is the hypersurface of discontinuity for the weak solution that develops to the future of the crease. $\mathcal{K}$ is *not* part of the constructions we provide in this paper, nor is it part of the maximal classical development. In fact, in the region between $\mathcal{K}$ and $\mathcal{B}$ which we have shaded in Fig. 5, *the classical and weak solutions disagree*; see Sect. 1.6 for further discussion of these issues. We again highlight one of the key difficulties in the problem: $\mathcal{B}$ is ruled by acoustically null curves with tangent vectors denoted by "L" in Fig. 5. This leads to rather severe degeneracies in the analysis because, as is well-known, the coercive quantities that can be used to control the solution become degenerate along null hypersurfaces. Here, the notion of null is with respect to the *acoustical metric* $\mathbf{g}$, the solution-dependent Lorentzian metric (see (2.15a)) that captures the intrinsic geometry of sound waves in the flow. See below for extended discussion of these fundamental issues.

1.5. **Abbreviated statement of the main results.** The full statement of our main results is quite lengthy, due to the intricate geometric structures and the highly tensorial nature of the singularity. In Theorem 1.4 we provide an abbreviated, slightly informal statement of the main results. In Theorems 31.1 and 34.1, we provide full statements of the main results.

We first provide two pictures illustrating the region that we study in Theorems 1.4 and 34.1. In Fig. 6, we display the region in geometric coordinates (t,u,x^2,x^3). In Fig. 7, we display the region in Cartesian coordinates (t,x^1,x^2,x^3), where in the labels, $\Upsilon(t,u,x^2,x^3)=(t,x^1,x^2,x^3)$ is the change of variables map between the two coordinate systems.

Remark 1.3 (Notation involving Υ)**.** In most of the article, we consider sets such as the singular boundary $\mathcal{B}$ to be subsets of geometric coordinate space $\mathbb{R}_t\times\mathbb{R}_u\times\mathbb{T}^2$, and we denote the image of these sets in Cartesian coordinate space $\mathbb{R}_t\times\mathbb{R}_{x^1}\times\mathbb{T}^2$ by explicitly indicating the change of variables map Υ, e.g., by $\Upsilon(\mathcal{B})$. We use this notation in particular in Fig. 7. However, in many of the other figures that depict regions in Cartesian coordinate space, such as Figs. 1 and 4, we have suppressed the map Υ so as to not clutter the figure.

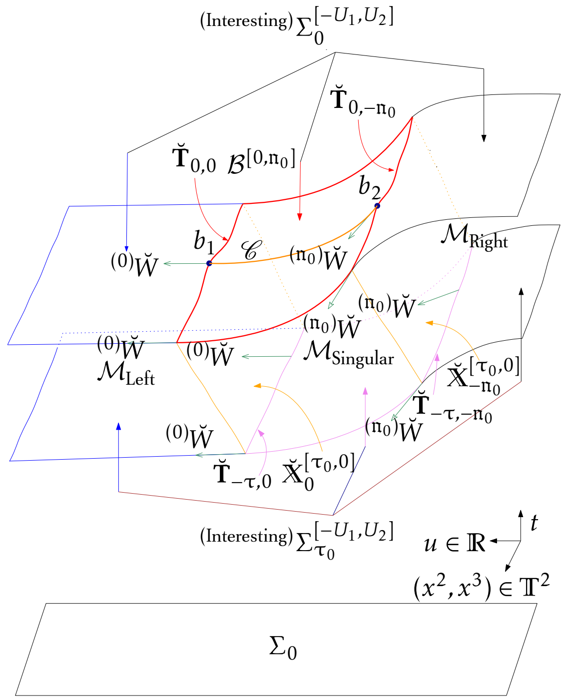


Figure 6. The region $\mathcal{M}_{\text{Interesting}}=\mathcal{M}_{\text{Left}}\cup\mathcal{M}_{\text{Singular}}\cup\mathcal{M}_{\text{Right}}$ in geometric coordinate space

Theorem 1.4 (Abbreviated statement of the main results)**.**

- *Fix any of the "admissible" background (shock-forming) simple isentropic plane-symmetric solutions that we construct in Appendix A, where we define "admissible" in Def. A.7; for each background solution, only a single Riemann invariant*[12] *is non-vanishing, and we denote it by* $\mathcal{R}_{(+)}^{\text{PS}}$*. For each such background solution, there exist numbers* $\tau_0<0$, $\mathfrak{n}_0>0$, $U_1>0$, $U_0>U_1$, *and* $U_2>0$ *such that the data of* $\mathcal{R}_{(+)}^{\text{PS}}$ *is compactly supported*[13] *in the Cauchy hypersurface portion*[14] $\Sigma_0^{[-U_1,U_2]}=\left\{(0,x^1,x^2,x^3)\mid -x^1\in[-U_1,U_2],\,(x^2,x^3)\in\mathbb{T}^2\right\}$ *in Cartesian coordinate space* $\mathbb{R}_t\times\mathbb{R}_{x^1}\times\mathbb{T}^2$.

[12]By our definition (A.1) of Riemann invariants, the fluid state $\mathcal{R}_{(+)}^{\text{PS}}=0$ corresponds to a fluid with a non-zero constant density $\bar{\varrho}>0$.

[13]We made the assumption that the initial data of the background Riemann invariant $\mathcal{R}_{(+)}^{\text{PS}}$ are compactly supported only for convenience; this assumption could be eliminated without much additional effort. In any case, the perturbed solutions that we study do not have to be compactly supported, though we only study them in a spatially compact region.

[14]The minus sign in front of the x^1 in the expression for $\Sigma_0^{[-U_1,U_2]}$ is due to the initial condition for u stated in (3.1).

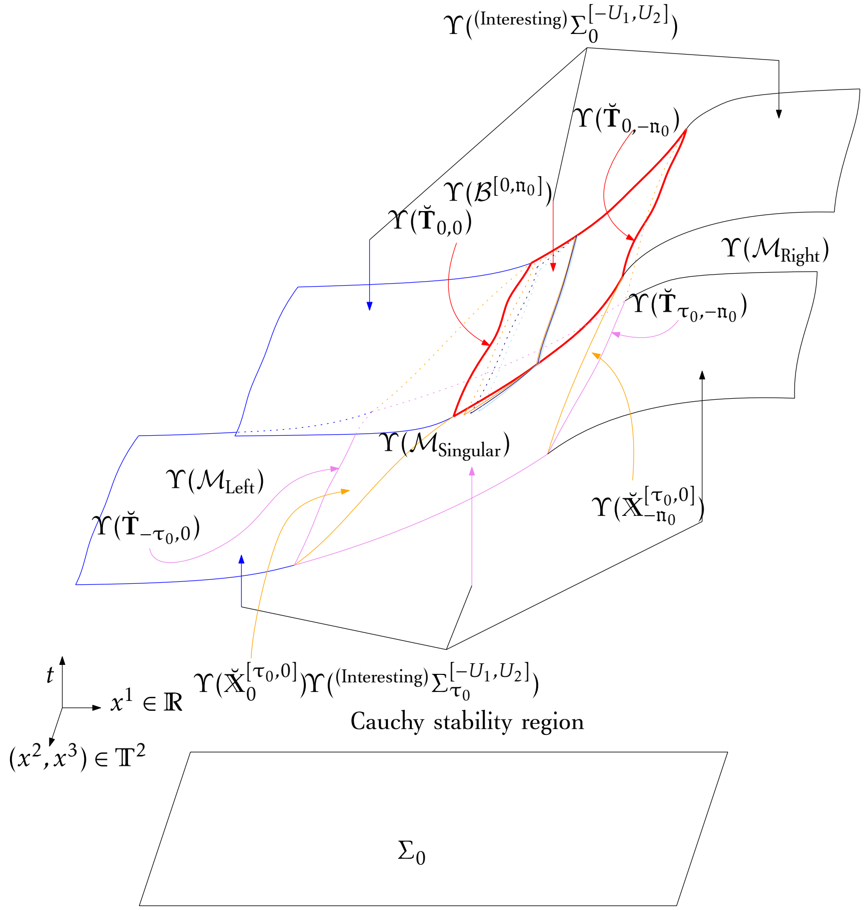


Figure 7. $\Upsilon(\mathcal{M}_{\text{Interesting}})$, i.e., the image of the region $\mathcal{M}_{\text{Interesting}}$ from Fig. 6 mapped into Cartesian coordinate space under Υ

- *Let $N_{\text{top}} \geq 24$ be an integer.*
- *Assume that the data-norm $\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ defined in* (11.4)*, which measures the $H^{N_{\text{top}}+1}(\Sigma_0^{[-U_1,U_2]})$-closeness of the perturbed data to the background data, is sufficiently small. Note that the perturbed data do not have to be compactly supported.*

Then the following conclusions hold, where the roles of τ_0 and $\mathfrak{n}_0$ are described below.

Classical existence in the Cauchy stability region. *The perturbed solution exists classically with respect to the Cartesian coordinates and remains uniformly bounded in the "Cauchy stability region" trapped in between the flat hypersurface portion $\Sigma_0^{[-U_1,U_2]}$ and the curved hypersurface portion ${}^{(\text{Interesting})}\Sigma_{\tau_0}^{[-U_1,U_2]}$, where ${}^{(\text{Interesting})}\Sigma_{\tau_0}^{[-U_1,U_2]}$ is a portion of the τ_0-level-set of the time function ${}^{(\text{Interesting})}\tau$, described below. We will not further discuss the behavior of the solution in the Cauchy stability region, noting only that it is depicted in Fig. 7.*

Acoustic geometry and regular behavior in the geometric coordinates. *There exists an **eikonal function** u solving the eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ such that the **geometric coordinates** (t,u,x^2,x^3) form a global coordinate system on the compact subset $\mathcal{M}_{\text{Interesting}} = \mathcal{M}_{\text{Left}} \cup \mathcal{M}_{\text{Singular}} \cup \mathcal{M}_{\text{Right}}$ depicted in Fig. 6. The figure is a subset of geometric coordinate space $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$, and $\mathcal{M}_{\text{Interesting}} \subset \{(t,u,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid -U_1 \leq u \leq U_2\}$. The **acoustical metric** $\mathbf{g}$ is the fluid-solution-dependent Lorentzian metric (see* (2.15a)*) that captures the intrinsic geometry of sound waves in the flow. The level-sets of u, which we denote by $\mathcal{P}_u$ and which are depicted as subsets of Cartesian coordinate space in Fig. 1B,* *are characteristic surfaces for the system* (1.1a)–(1.1c). *The fluid blowup is* not visible *in the geometric coordinates: the fluid solution and its up-to-mid-order partial derivatives in the geometric coordinate system remain bounded on* $\mathcal{M}_{\text{Interesting}}$.

The singularity is caused by the infinite density of the characteristics. *With respect to the Cartesian coordinates, there is singularity formation, described below, that coincides with the vanishing of the* ***inverse foliation density*** $\mu \overset{def}{=} -\frac{1}{(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha t \partial_\beta u}$. *The vanishing of* μ *signifies the infinite density of the level-sets of* u *(viewed as a function of the Cartesian coordinates). Within* $\mathcal{M}_{\text{Interesting}}$, *this singular behavior occurs precisely along the singular boundary portion* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *depicted in Fig. 6.*

Behavior of the change of variables map. *The change of variables map* $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ *is a* ***homeomorphism*** *from* $\mathcal{M}_{\text{Interesting}}$ *onto its image in Cartesian coordinate space. Its Jacobian determinant is* $\approx -\mu$, *and since* $\mu > 0$ *on* $\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}$, *the map is a diffeomorphism on* $\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}$.

The shock singularity in the Cartesian differential structure. *Let* Σ_t *denote the standard flat hypersurface of constant Cartesian time. There exists a* Σ_t*-tangent vectorfield* X *of Euclidean length* $\sqrt{\sum_{a=1}^3 (X^a)^2} \approx 1$ *and a constant* $C' > 0$ *such that the following occurs. Within the compact subset* $\Upsilon(\mathcal{M}_{\text{Interesting}}) = \Upsilon(\mathcal{M}_{\text{Left}} \cup \mathcal{M}_{\text{Singular}} \cup \mathcal{M}_{\text{Right}})$ *of Cartesian coordinate space depicted in Fig. 7, we have* $|X^a \partial_a v^1| \geq \frac{C'}{\mu}$ *and* $|X^a \partial_a \varrho| \geq \frac{C'}{\mu}$. *Hence, these two quantities blow up precisely on the* ***singular boundary*** *portion* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, *described below, where* μ *vanishes. The solution is smooth in* $\Upsilon\left(\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}\right)$, *and for* $\alpha = 0,1,2,3$, $i = 1,2,3$, *and* $A = 2,3$, *the fluid quantities* ϱ, v^i, s, $\text{curl}\, v^i$, $\partial_i s$, *and* $\mathbf{g}_{ab} Y^a_{(A)} \partial_\alpha v^b$ *remain bounded on all of* $\Upsilon(\mathcal{M}_{\text{Interesting}})$, *where* $Y_{(2)}$ *and* $Y_{(3)}$ *are the vectorfields from* (1.15), *and they are tangent to* $\mathcal{P}_u \cap \Sigma_t$. *Moreover, the up-to-mid-order derivatives of these quantities with respect to the* $\mathcal{P}_u$*-tangent vectorfields* $\{L, Y_{(2)}, Y_{(3)}\}$ *from* (1.15) *also remain bounded on all of* $\Upsilon(\mathcal{M}_{\text{Interesting}})$.

The structure of $\mathcal{B}^{[0,\mathfrak{n}_0]}$ ***and*** $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$. *The singular boundary* $\mathcal{B}^{[0,\mathfrak{n}_0]}$, *viewed as a subset of geometric coordinate space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$, *is contained in the top boundary of* $\mathcal{M}_{\text{Interesting}}$ *and is a subset of the level-set* $\{\mu = 0\}$, *specifically, a portion that can be realized as the limit as* $\mathfrak{m} \downarrow 0$ *of portions of the level-sets* $\{(t,u,x^2,x^3) \mid \mu(t,u,x^2,x^3) = \mathfrak{m}\}$ *that are either null or spacelike with respect to the acoustical metric* $\mathbf{g}$. $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *is an embedded* 3*-dimensional sub-manifold-with-boundary of geometric coordinate space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$. *Moreover, we can decompose* $\mathcal{B}^{[0,\mathfrak{n}_0]} = \bigcup_{\mathfrak{n} \in [0,\mathfrak{n}_0]} \check{\mathbf{T}}_{0,-\mathfrak{n}}$, *where each* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$ *is a* $C^{1,1}$, 2*-dimensional, spacelike sub-manifold of geometric coordinate space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$ *that is a graph over* $\mathbb{T}^2$ *such that* $\mu X \mu \equiv -\mathfrak{n}$ *along* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$. *In particular,* $\mu X \mu$ *is well-defined and non-zero on* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$ *when* $\mathfrak{n} > 0$, *even though* $\mu = 0$ *along* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$. *The torus* $\check{\mathbf{T}}_{0,0}$, *which we refer to as the* ***crease*** *and also denote by* $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, *is the past boundary of* $\mathcal{B}^{[0,\mathfrak{n}_0]}$.

The structure of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ ***and*** $\Upsilon(\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$. *The change of variables map* $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ *is a* ***homeomorphism*** *from the singular boundary portion* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *in geometric coordinate space onto its image* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ *in Cartesian coordinate space. Moreover,* Υ *is a diffeomorphism from* $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$ *onto its image* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$. *The image set* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ *is an embedded* 3*-dimensional sub-manifold-with-boundary in Cartesian coordinate space that has regularity*[15] $C^{1,\frac{1}{2}}$ *with respect to the Cartesian coordinates. In addition,* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$ *is a null hypersurface with respect to the acoustical metric* $\mathbf{g}$ *on* $\Upsilon(\mathcal{M}_{\text{Interesting}})$ *in the following sense:* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$ *is ruled, in a degenerate sense explained in Remark 33.3, by integral curves of the* $\mathbf{g}$*-null vectorfield* $L = L^\alpha \partial_\alpha$ *on* $\Upsilon(\mathcal{M}_{\text{Interesting}})$. *Furthermore, for* $\mathfrak{n} \in [0,\mathfrak{n}_0]$, *the images* $\Upsilon(\check{\mathbf{T}}_{0,-\mathfrak{n}})$ *of the tori* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$, *including the crease* $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]} = \check{\mathbf{T}}_{0,0}$, *are* 2*-dimensional,* $C^{1,1}$, $\mathbf{g}$*-spacelike sub-manifolds of Cartesian coordinate space* $\mathbb{R}_t \times \mathbb{R}_{x^1} \times \mathbb{T}^2$ *that are graphs over* $\mathbb{T}^2$.

Rough time functions reveal the structure of $\mathcal{B}^{[0,\mathfrak{n}_0]}$ ***and the tori foliating it***. *There exists a one-parameter family of* ***rough time functions*** $\{{}^{(\mathfrak{n})}\tau\}_{\mathfrak{n} \in [0,\mathfrak{n}_0]}$, *each with range* $[\tau_0, 0]$, *such that the level-sets* $\{{}^{(\mathfrak{n})}\tau = \tau\}$ *with* $\tau \in [\tau_0, 0)$ *do not intersect* $\mathcal{B}^{[0,\mathfrak{n}_0]}$, *while* $\{{}^{(\mathfrak{n})}\tau = 0\} \cap \mathcal{B}^{[0,\mathfrak{n}_0]} = \check{\mathbf{T}}_{0,-\mathfrak{n}}$ *is a two-dimensional submanifold that is diffeomorophic to* $\mathbb{T}^2$. *That is, each* ${}^{(\mathfrak{n})}\tau$ *reveals the structure of the torus* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$ *in* $\mathcal{B}^{[0,\mathfrak{n}_0]}$, *and* $\check{\mathbf{T}}_{0,-\mathfrak{n}} \subset \{{}^{(\mathfrak{n})}\tau = 0\}$. *Moreover, each* ${}^{(\mathfrak{n})}\tau$ *is one degree less differentiable with respect to the geometric coordinates than the fluid, and the tori* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$ *are two degrees less differentiable (viewed as submanifolds of* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$*) than the fluid.*

[15]The $C^{1,\frac{1}{2}}$ embedding of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ is provided by Prop. 33.2; it is the map $(z,x^2,x^3) \to \Upsilon \circ \mathfrak{S}(\sqrt{z},x^2,x^3)$ from the proposition. .

In addition, there exists a rough time function ${}^{(\text{Interesting})}\tau$ *with range* $[\tau_0, 0]$*, whose level-sets foliate* $\mathcal{M}_{\text{Interesting}}$*.* ${}^{(\text{Interesting})}\tau$ *is* $C^{1,1}$ *with respect to the geometric coordinates -* ***and not more regular****; see Remark 32.10. Two of its level-sets intersected with* $\{u \in [-U_1, U_2]\}$*, namely* ${}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}$ *and* ${}^{(\text{Interesting})}\Sigma_{\tau_0}^{[-U_1,U_2]}$*, are depicted in Fig. 6. Finally,* $\mathcal{B}^{[0,\mathfrak{n}_0]} \subset \{{}^{(\text{Interesting})}\tau = 0\}$*.*

Remark 1.5 (Remarks on our initial data assumptions)**.** In our main results, we have chosen to assume the smallness of the norm $\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ defined in (11.4) because this immediately allows us to conclude that our results hold for open sets of solutions whose initial data that are $H^{N_{\text{top}}+1}$-close to the data of a background solution. However, the main assumptions on the data that we use in our PDE analysis are actually stated in Sect. 11, in terms of data-size parameters that satisfy assumptions stated in Sect. 10. All these assumptions follow as consequences of the smallness of the norm $\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$. That is, the proof of our main results would go through under only the assumptions stated in Sect. 11 and the parameter-size assumptions stated in Sect. 10, modulo the remarks we make at the beginning of Sect. 27.4.

1.6. **Remarks on the main results and methods.** Before proceeding, we make a series of remarks about our main results and our proof framework, with an emphasis on the new methods we introduce in this paper, how they connect to methods developed in other papers, and how they connect to open problems.

- (**Use of acoustic geometry to detect the singularity and to "hide it" by unfolding the characteristics**). As in Christodoulou's breakthrough work [24] on irrotational and isentropic shock formation, our analysis fundamentally relies on nonlinear geometric optics (which we also loosely refer to as "the acoustic geometry"), implemented via an eikonal function u, which solves the eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$. The *acoustical metric* $\mathbf{g}$ is a Lorentzian metric whose Cartesian components $\mathbf{g}_{\alpha\beta}$ are functions of the fluid variables; see (2.15a) for the precise formula. The level-sets $\mathcal{P}_u$ of u are null hypersurfaces (also known as "characteristics" or "characteristic hypersurfaces"), and they correspond to the propagation of sound waves. Infinite density of the level-sets of u (viewed as a function of the Cartesian coordinates) signifies the formation of a shock, and in the regime under study, it coincides with the blowup of first-order partial derivatives of v and ϱ in directions *transversal* to the $\mathcal{P}_u$. As is shown in Fig. 1B, *these phenomena occur along the entire singular boundary.* Moreover, with the help of u, we can construct a "geometric coordinate system" (t, u, x^2, x^3), relative to which the solution remains rather smooth. This is crucial for our derivation of PDE estimates up to top-order.
- (**The structure of the singular boundary and the crease**). We prove that in the solution regime under study, relative to a differential structure on spacetime tied to the eikonal function, the singular boundary $\mathcal{B}$ has the structure of a $3D$ sub-manifold-with-boundary; see Fig. 5. Its past boundary $\partial_-\mathcal{B}$ is the crease, which we prove is a $2D$ acoustically spacelike sub-manifold, where our notion of "spacelike" is relative to $\mathbf{g}$. These structures are stable, and their availability is fundamental for our approach. These structures also have important implications for the shock development problem, described below.
- (**Prior works**). There are many prior works on shock formation for compressible Euler solutions, including Riemann's famous work [64] in one spatial dimension. The one-dimensional theory [33] is in a quite advanced state compared to the case of multi-dimensions. Nonetheless, in multi-dimensions, there has been dramatic progress over the last several decades, starting with Alinhac's foundational works on quasilinear wave equations [7, 8, 10, 11] and Christodoulou's breakthrough monograph on the irrotational and isentropic relativistic Euler equations [24]; see Sect. 1.9 for a discussion of some additional key developments in the history of the subject. While these results have led to a revolution in our understanding of multi-dimensional shock formation, they all were limited in one or more of the following ways: **i)** they treated only irrotational and isentropic solutions; **ii)** the methods allowed one only to follow the solution to the constant-Cartesian-time hypersurface of first blowup; **iii)** the methods applied only to fully non-degenerate singularities (see below for their definition) which, as it turns out, corresponds to understanding the blowup at the unique first (relative to Cartesian time) point that is contained inside a strictly convex crease; **iv)** the methods yielded a description only of some implicit portion of the boundary of the maximal development of the data and in particular yielded only the portion of the crease that is tangent to some *flat* spacelike hypersurface Σ such that a neighborhood of the crease lies in its future. A key point is that without strict convexity, there can be points in the crease such that no open neighborhood of them is accessible through such an approach. In particular, for the solutions we treat in our main results, the full structure of the crease is not accessible through any of these approaches; see Fig. 5.

- (**Relevance for the shock development problem**). The crease and its sub-manifold structure are crucial ingredients needed to set up the *shock development problem*, which, as we mentioned earlier, is the problem of describing the transition of the solution from classical to weak, past the initial singularity. It turns out that the crease is part of the maximal classical development *and*, once one constructs the weak solution, it will also be part of the weak solution; see [25, Section 1.5] for a detailed discussion of the connection between the maximal classical development and the weak solution. Away from symmetry, the shock development problem is an outstanding open problem. In Christodoulou's approach to studying shock developments under the simplifying assumption that the vorticity and entropy are vanishing[16] across the shock hypersurface [25], the rest of the singular boundary is also important because it plays the role of a mathematical *barrier* in the construction of the weak solution. However, aside from the crease, the singular boundary is not part of the weak solution. More precisely, in general, there is a portion of the maximal classical development that does not agree with the weak solution because the classical development does not account for the shock hypersurface. The shock hypersurface, which we denote by $\mathcal{K}$ in Fig. 5, is not part of the constructions of this paper, and we mention it to highlight that the weak solution – once it is constructed – will disagree with the classical solution in the region in between $\mathcal{B}$ and $\mathcal{K}$; we have shaded the region of disagreement in Fig. 5. We also highlight that $\mathcal{K}$ is supersonic relative to the acoustical metric in the maximal classical development, and that *$\mathcal{K}$ is "allowed" to emerge from the crease only if one imposes a weak formulation of the flow*, i.e., there is no such thing as "$\mathcal{K}$" in the classical formulation.
- (**The Cauchy horizon**). In our forthcoming paper, we will describe the emergence of a Cauchy horizon from the crease. The Cauchy horizon, denoted by $\underline{\mathcal{C}}$ in Fig. 5, is a $\mathbf{g}$-null hypersurface with past boundary equal to the crease, and it is also a crucial ingredient in the setting up the shock development problem. Like the crease, the Cauchy horizon is part of the maximal classical development and the weak solution (once one constructs it). However, in that paper, we will show that the solution remains smooth up to the Cauchy horizon (away from the crease, that is), which is in stark contrast to what happens along the singular boundary.
- (**Solution regimes other than perturbations of simple isentropic plane-waves**). Simple isentropic plane-waves, mentioned already in Theorem 1.4, are compressible Euler solutions such that $\mathcal{R}_{(+)} = \mathcal{R}_{(+)}(t, x^1)$ and $\mathcal{R}_{(-)} = v^2 = v^3 \equiv 0$, where the "almost[17] Riemann invariants" $\mathcal{R}_{(\pm)}$ are defined in Def. 2.5. Although for definiteness, we have focused on general small perturbations of such solutions, the techniques we develop here are robust and can be applied to other solution regimes of physical interest, such as the regime corresponding to small, spatially-decaying perturbations[18] of non-vacuum constant fluid states in $\mathbb{R}^{1+3}$, where dispersive effects play a fundamental role in the dynamics.
- (**Building blocks that can accommodate degeneracies**). We have focused our attention on perturbations of simple isentropic plane-symmetric waves because such solutions are of physical interest and, from the point of view of analysis, they are challenging to study because they exhibit degeneracies tied to lack of strict convexity (as we mentioned above) of their singular boundaries. This is the first paper to fully grapple with these degeneracies and as such, our results are new even in the sub-class of irrotational and isentropic solutions. The presence of these degeneracies forced us to develop techniques that we anticipate can be used, in view of finite speed of propagation, as building blocks to study the global structure of much more general shock-forming solutions across space.
- (**New, solution-dependent foliations are needed**). To access the entire crease/singular boundary, in general, one cannot exclusively rely on arguments based on analysis on regions bounded by the characteristic surfaces $\mathcal{P}_u$, surfaces Σ_t of constant Cartesian time, or, for that matter, any other family of surfaces that are "pre-specified" in the sense that they are explicitly parameterized with respect to the Cartesian coordinates (e.g., one can not generally use "tilted" spacelike hypersurfaces that are planes with respect to the Cartesian coordinates). In particular, the crease is typically not contained in a fixed $\mathcal{P}_u$ or Σ_t; see Fig. 1A. Hence, one of our key new ideas in the paper is to replace the surfaces Σ_t with better ones. That is, we construct foliations by spacelike hypersurfaces that are precisely adapted to the shape of the singular boundary (though we also fundamentally rely on foliations by the $\mathcal{P}_u$). We construct these foliations by "flowing out" (see below), along the integral curves of

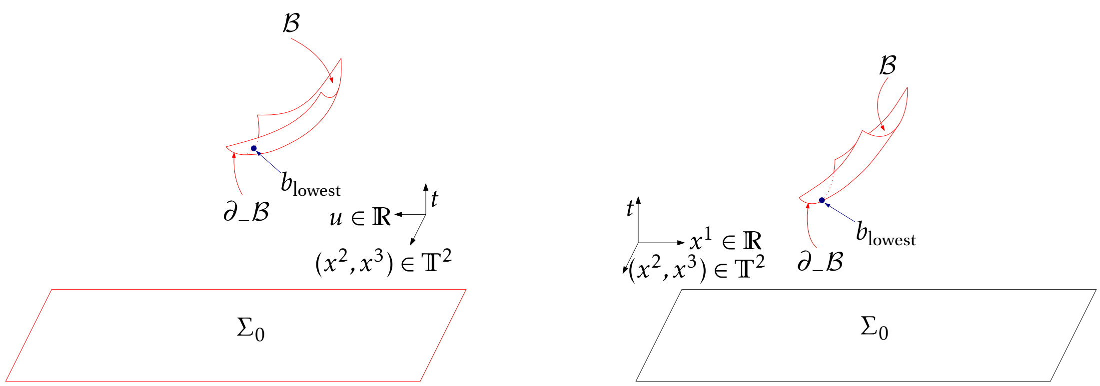
[16]The irrotational and isentropic weak solutions constructed by Christodoulou in [25] are not solutions to the compressible Euler equations because they do not respect the jump in entropy and vorticity that must occur across the shock hypersurface. Instead, they are solutions to a closely related hyperbolic PDE system that is equivalent to the compressible Euler equations for *classical* isentropic and irrotational classical solutions.

[17]For isentropic plane-symmetric solutions, $\mathcal{R}_{(\pm)}$ *are* Riemann invariants.

[18]In this regime, the dispersive tendency of sound waves competes against the transport phenomena associated to vorticity and entropy, possibly leading to exceptionally complicated long-time dynamics.

a "generating vectorfield" that is transversal to $\mathcal{P}_u$, from co-dimension 2 topological tori that are adapted to the anticipated shape of the singularity. In Fig. 4A, the generating vectorfield is denoted by ${}^{(0)}\breve{W}$, and crucially, it is also transversal to the hypersurface $\breve{\mathbb{X}}_0$, which is foliated by the tori, two of which are denoted by $\breve{\mathbf{T}}_{0,0}$ and $\breve{\mathbf{T}}_{-\tau,0}$. Similarly, in Fig. 4B, the generating vectorfield is denoted by ${}^{(\mathfrak{n})}\breve{W}$, and it is also transversal to the hypersurface $\breve{\mathbb{X}}_{-\mathfrak{n}}$, which is foliated by the tori, two of which are denoted by $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$ and $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$. While there have been many other works on the formation of shocks without symmetry assumptions, described in Sect. 1.7, those works have relied on a "background" time function (typically the Cartesian time function t) and the corresponding foliations. Note that the PDE analysis of compressible Euler flow, which inevitably involves some kind of energy estimates with respect to the spacelike foliations, *necessarily halts* when the spacelike foliations first intersect the singular boundary, because singularities form at the intersection points. This limits the portion of the singular boundary that can be detected through background foliations.

- (**Remarks on the simpler fully non-degenerate sub-regime**). Despite the previous point, we note that there is a sub-regime in which background time functions *can* be used to detect the structure of the crease (though perhaps not a full neighborhood of it in the singular boundary): the regime in which the crease and singular boundary are acoustically strictly convex (see below and Sect. 1.3). In Fig. 8, we depict an acoustically strictly convex singular boundary.



(A) An acoustically strictly convex crease and singular boundary in geometric coordinate space

(B) The same crease and singular boundary in Cartesian coordinate space

Figure 8. Acoustically strictly convex crease and singular boundary

We emphasize that symmetric solutions and their general perturbations fall outside of the acoustically strictly convex regime. Away from symmetry, the acoustically strictly convex regime was first studied by Alinhac [7, 8, 10, 11] in the context of shock-forming solutions to quasilinear wave equations, where he followed the solution to the constant-Cartesian-time hypersurface of first blowup. In particular, due to the strict convexity of the crease, the singular set within the constant-Cartesian-time hypersurface of first blowup is an *isolated point*, denoted by b_{lowest} in Fig. 8, and the isolated nature of this point was fundamental for his approach. We refer to such singularities as *fully non-degenerate*; see Sect. 1.9.3 for further discussion.

- (**Acoustical transversal convexity**) Here and throughout, by an "acoustically strictly convex" hypersurface $\mathcal{B}$, we mean that relative to a system of geometric coordinates (t, u, x^2, x^3) tied to an acoustic eikonal function u, the tangent planes to $\mathcal{B}$ lie below $\mathcal{B}$; see Fig. 8A for an example of such a surface. The key point is that acoustical strict convexity is *absent* in the singular boundaries of open sets of physically relevant shock-forming solutions, including perturbations of isentropic plane-symmetric simple waves. To prove our main results, we rely on *acoustical transversal convexity* (we will refer to it as "transversal convexity" for short), which is substantially weaker than strict convexity and it is close to optimal in the following sense: without transversal convexity, the crease could fail to have the structure of a $2D$ sub-manifold, which would drastically alter the qualitative character of the singular boundary. We highlight that transversal convexity can be ensured along a portion of the singular boundary containing the crease by making appropriate open assumption on the initial data. In the present paper, we ensure transversal convexity via the data-assumptions stated in (11.18). By an "acoustically transversally convex" hypersurface $\mathcal{B}$, we mean that for each $q \in \mathcal{B}$, relative to the geometric coordinates (t, u, x^2, x^3), there exists a *line* – as opposed to the full tangent plane – that is tangent to $\mathcal{B}$ at q such that **i)** the line lies below[19] $\mathcal{B}$ and **ii)** the line is *transversal* to the level-sets of u. In the context of Fig. 6, *lack* of strict convexity is exhibited by the observation that at b_1, the straight line that is tangent to $\breve{\mathbf{T}}_{0,0}$ (this line is not drawn in the figure) lies *above* $\mathcal{B}^{[0,\mathfrak{n}_0]}$, and similarly, at b_2, the straight line that is tangent to $\breve{\mathbf{T}}_{0,-\mathfrak{n}_0}$ lies above $\mathcal{B}^{[0,\mathfrak{n}_0]}$. In Fig. 6, our notion of transversal convexity is exhibited, for example, by the fact that along the curve $\mathscr{C}$ in $\mathcal{B}^{[0,\mathfrak{n}_0]}$ joining b_1 and b_2, the tangent lines to the curve (which are not drawn) lie *below* $\mathcal{B}^{[0,\mathfrak{n}_0]}$.

- (**Rough time functions**). Our new foliations are level-sets of an $\mathfrak{n}$-indexed family of "rough time functions," denoted by ${}^{(\mathfrak{n})}\tau$, where $\mathfrak{n}$ is a non-negative real parameter. The ${}^{(\mathfrak{n})}\tau$ solve a $\mathfrak{n}$-dependent transport equation (see (4.4a)) with coefficients and initial data (see (4.4b)) determined by the acoustic geometry (i.e., the eikonal function u and its derivatives) and the up-to-first-order derivatives of the fluid solution. Moreover, we choose the initial data surface for ${}^{(\mathfrak{n})}\tau$ to be a level-set of a first derivative of the acoustic geometry, more precisely, the $-\mathfrak{n}$ level-set of a first derivative of the inverse foliation density in a direction transversal to the characteristics. More precisely, each level-set of ${}^{(\mathfrak{n})}\tau$ is the solution a transport equation with (constant) initial data given on a topological torus that itself is equal to the intersection of a level-set of μ (recall that μ is the inverse foliation density of the characteristics) with a level-set of a transversal derivative of μ. In Fig. 4, we denote the initial data-hypersurfaces for the transport equations by $\breve{\mathbb{X}}_0$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$, we denote some tori in the data-hypersurfaces by $\breve{\mathbf{T}}_{-\tau,0}$, $\breve{\mathbf{T}}_{0,0}$, $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$, and $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$, and we denote the vectorfields that transport ${}^{(0)}\tau$ and ${}^{(\mathfrak{n})}\tau$ respectively by ${}^{(0)}\breve{W}$ and ${}^{(\mathfrak{n})}\breve{W}$. These vectorfields are tangent to the level-sets of ${}^{(0)}\tau$ and ${}^{(\mathfrak{n})}\tau$ respectively, and they are also tangent to the singular boundary along the tori $\breve{\mathbf{T}}_{0,0}$ and $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$ respectively. We emphasize that this construction is quite delicate and that **it is crucial that we consider only the case $\mathfrak{n} \geq 0$ in order to ensure that ${}^{(\mathfrak{n})}\tau$ has a $\mathbf{g}$-timelike normal or $\mathbf{g}$-null normal**; if we had tried to allow for $\mathfrak{n} < 0$, then near the crease, the level sets of ${}^{(\mathfrak{n})}\tau$ would have had a $\mathbf{g}$-timelike portion, which is not accessible through PDE analysis. For related discussion, see also the point above Remark 1.1 and the discussion in Sect. 1.3 on the "fictitious portion" of the singular boundary. In total, our construction of ${}^{(\mathfrak{n})}\tau$ is "dynamic" and "precisely solution-adapted," and we achieve it through a bootstrap argument. The singular boundary is *provably one degree less differentiable* than the fluid solution, and the same is true for our rough time functions. Moreover, the tori (such as $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$) are two degrees less differentiable. To handle this fundamental difficulty, we work with multiple coordinate systems, as we describe in the next point.
- (**Three distinct coordinate systems**). There is no known approach to studying multi-dimensional shock formation that relies only on commuting the equations with the Cartesian coordinate partial derivative vectorfields. Because of the roughness of the time functions ${}^{(\mathfrak{n})}\tau$, we cannot close our top-order energy estimates using only commutation vectorfields that are adapted[20] to their level-sets; such an approach would lead to the loss of a derivative. Hence, as in other works on shocks, we construct appropriate commutators with the help of the eikonal function u. In total, our study of the flow relies on understanding the behavior of the fluid in three coordinate systems as well as carefully controlling the relationships – which degenerate near the singular boundary – between the coordinate systems:
  1. The standard Cartesian coordinates (t, x^1, x^2, x^3), relative to which the compressible Euler equations are initially formulated and relative to which the singularity is visible.
  2. The geometric coordinates (t, u, x^2, x^3) (where u is the eikonal function), for constructing suitable commutator and multiplier vectorfields that are adapted to the singularity and have *sufficient regularity*, and relative to which the solution remains rather smooth; see Sect. (1.10.12).
  3. The adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$, where the level-sets of ${}^{(\mathfrak{n})}\tau$ are *precisely adapted to the shape* of the singular boundary.
- (**Elliptic-hyperbolic estimates for the vorticity and entropy**). It is difficult to control the top-order derivatives of the vorticity and entropy on regions that are adapted to the shape of the singular boundary. In particular, the analytic framework we use, which is based on the formulation of compressible Euler flow provided by Theorem 2.15, requires that we prove that the vorticity and entropy gradient are exactly as differentiable as the velocity and density, all the way up to the singular boundary. To this end, we develop an upgraded version of the "elliptic-hyperbolic" integral identities from [4]. The upgraded version, which we implement with the help of a new *characteristic current*, allows us to handle severe degeneracies that arise near the crease. One key degeneracy

[19] Our approach to studying the singular boundary is local, and thus, in our main results, we only need the tangent lines to locally lie below it.

[20] However, for some crucial low-derivative-level estimates, we *do* use commutators that are adapted to the level-sets of ${}^{(\mathfrak{n})}\tau$.

is that the level-sets of our time functions ${}^{(\mathfrak{n})}\tau$ *become asymptotically null* near the crease, i.e., for $\mathfrak{n}$ small and positive; see Figs. 1A and 4B. We refer to Sect. 21 for our derivation of the elliptic-hyperbolic identities.

- (**Transversal convexity**). As we have mentioned, our analysis relies on an open data-assumption that we call *acoustical transversal convexity* (which we refer to as transversal convexity for short); we refer to (11.18) for the precise, technical data-estimates that capture transversal convexity. In studying the flow, we are able to propagate the transversal convexity all the way up to the singular boundary, which in particular ensures that the level-sets $\{\mu = 0\}$ and $\{\breve{X}\mu = 0\}$ on RHS (1.5) (which defines the crease) intersect transversally, As we mentioned already in Sect. 1.6, our assumption of transversal convexity is weaker than the assumption that the graph of $\{\mu = 0\}$, viewed as a subset of geometric coordinate space, is strictly convex, and our weaker assumption is what allows us, for example, to handle perturbations of symmetric solutions. Equivalently, under our approach, transversal convexity allows us to understand the structure of the crease and the singular boundary for singularities that are more general than the fully non-degenerate ones featured in Fig. 8, including the perturbations of simple isentropic plane-waves that we treat in detail in our main theorems, as is depicted in Fig. 1A.

  Our transversal convexity assumption is close to optimal in the sense that without it, *the qualitative character of the singular boundary can dramatically change*, e.g., without transversal convexity, the crease could fail to have the structure of a $2D$ sub-manifold, and then one could not even properly set up the shock development problem using the known approach [25]; see the next point. The qualitative change of the structure of the crease in the absence of transversal convexity can readily be seen in the context of the simple isentropic plane-symmetric solutions that we study in Appendix A. More precisely, using the explicit formulas provided by Cor. A.3, one can construct simple isentropic plane-symmetric fluid initial data such that transversal convexity fails and such that the crease $\{\mu = 0\} \cap \{\breve{X}\mu = 0\}$, viewed as subset of $1 + 3$-dimensional geometric coordinate space, fails to have the structure of a $2D$ sub-manifold. For example, one can any consider smooth initial data such that the data-term $\frac{d}{du}\mathfrak{H}[\mathring{\mathcal{R}}^{\text{PS}}_{(+)}(u)]$ on RHS (A.23b) achieves its negative minimum along a closed interval I of u-values. It is then easy to see, using (A.23b), (A.24), and the fact that $\breve{X} = \frac{\partial}{\partial u}$ for simple isentropic plane-symmetric solutions, that $\breve{X}\mu = 0$ at all points in $\Sigma_{T_{\text{Shock}}}$ where μ vanishes, where T_{Shock} is the Cartesian time of first blowup. In particular, the level-sets $\{\mu = 0\}$ and $\{\breve{X}\mu = 0\}$ *coincide* on the interval of u values I within $\Sigma_{T_{\text{Shock}}}$, signifying a dramatic failure of transversal convexity. Moreover, using the characterization (1.5) of the crease, we see that the crease, viewed as a subset of $1 + 3$-dimensional geometric coordinate space, is the following set: $\left\{(t, u, x^2, x^3) \mid t = T_{\text{Shock}}, u \in I, (x^2, x^3) \in \mathbb{T}^2\right\}$, which is a $3D$ sub-manifold-with-boundary (in particular, it is not a $2D$ sub-manifold).

- (**Homeomorphism property of the change of variables map**). Finally, we highlight that – thanks to the transversal convexity – we are able to prove that the change of variables map $\Upsilon(t, u, x^2, x^3) \stackrel{\text{def}}{=} (t, x^1, x^2, x^3)$ from geometric to Cartesian coordinates is a homeomorphism all the way up to the singular boundary, and it is a diffeomorphism away from the singular boundary; see Prop. 33.1. In particular, Υ is a bijection on the region $\mathcal{M}_{\text{Interesting}}$ depicted in Fig. 6, and *the level-sets of u never actually intersect* $\mathcal{M}_{\text{Interesting}}$ (even along the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]}$!), despite the fact that the density of the level-sets of u in Cartesian coordinate space becomes infinite along the singular boundary. Under transversal convexity, the map $u \to -u^3$ provides a crude scalar caricature of the degenerate behavior of the x^1 Cartesian coordinate along a constant Cartesian time slice Σ_t that happens to be tangent to the crease. **These phenomena are not just a mathematical curiosity**; the bijective property of Υ up to and including the singular boundary is crucial for formulating the shock development problem, and this property can dramatically fail without transversal convexity. As in the previous point, this difficulty can readily be seen in the context of the simple isentropic plane-symmetric solutions that we study in Appendix A. For example, in the example discussed in the previous point, using that $\frac{\partial}{\partial u}x^1 = -\mu$ in plane symmetry, one finds that $x^1 = x^1(t, u, x^2, x^3)$ is constant along the entire crease $\left\{(t, u, x^2, x^3) \mid t = T_{\text{Shock}}, u \in I, (x^2, x^3) \in \mathbb{T}^2\right\}$, thereby exhibiting a dramatic failure of the injectivity of Υ. In the Cartesian coordinate space picture in $1 + 1$ dimensions (i.e., ignoring the x^2 and x^3 directions), the shock formation in this example corresponds to a continuum of characteristic curves all intersecting in a single point in Cartesian coordinate space. This is in stark contrast to the behavior of the characteristic hypersurfaces on the singular boundary in the transversally convex regime, as we show in Prop. 33.2 and Fig. 14.

1.7. **Works building up towards Theorem 1.4.** In this section, we describe some key prior works that developed some of the technology we use here. While we have already mentioned some of them, here we provide additional details. We refer to Sect. 1.9 for a more extensive (though far from comprehensive) discussion of the history of works on shock formation.

1.7.1. *Christodoulou's framework from* [24]. Our proof of Theorem 1.4 relies on the framework of nonlinear geometric optics developed by Christodoulou [24] in his groundbreaking proof of stable shock formation for the irrotational and isentropic relativistic Euler equations in three spatial dimensions. In this setting, the equations reduce to a scalar quasilinear wave equation for a potential function Φ:

$$(\mathbf{g}^{-1})^{\alpha\beta}(\boldsymbol{\partial}\Phi)\partial_\alpha\partial_\beta\Phi = 0. \tag{1.6}$$

In (1.6), $\mathbf{g}_{\alpha\beta}$ is the *acoustical metric*, a Lorentzian metric whose rectangular component functions $\mathbf{g}_{\alpha\beta}$ are nonlinear functions of $\boldsymbol{\partial}\Phi \stackrel{\text{def}}{=} (\partial_t\Phi, \partial_1\Phi, \partial_2\Phi, \partial_3\Phi)$ depending on the equation of state: $\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}(\boldsymbol{\partial}\Phi)$. To implement nonlinear geometric optics, Christodoulou constructed an *eikonal function* u, that is, a solution to the *eikonal equation*

$$(\mathbf{g}^{-1})^{\alpha\beta}(\boldsymbol{\partial}\Phi)\partial_\alpha u\partial_\beta u = 0. \tag{1.7}$$

The level-sets of u, denoted by $\mathcal{P}_u$, are characteristic for equation (1.6). Notice that the principal coefficients on LHS (1.7) depend on $\boldsymbol{\partial}\Phi$ and thus the evolution of u is coupled to that of Φ, signifying dynamic nature of the geometry and the quasilinear nature of the flow. In [24], Christodoulou used u to construct a system of *geometric coordinates* $(t, u, \vartheta^1, \vartheta^2)$ on spacetime such that the solution remains rather smooth relative to these coordinates. The formation of the singularity can be recovered as a degeneracy between the geometric coordinates and the standard ones, (t, x^1, x^2, x^3). The degeneracy is signified by the vanishing of the *inverse foliation density* $\upmu$, defined by:

$$\upmu \stackrel{\text{def}}{=} -\frac{1}{(\mathbf{g}^{-1})^{\alpha\beta}(\boldsymbol{\partial}\Phi)\partial_\alpha t\partial_\beta u}. \tag{1.8}$$

In the region of classical existence, one has $\upmu > 0$, and when $\upmu \to 0$, the density of the $\mathcal{P}_u$ in (t, x^1, x^2, x^3)-coordinate space becomes infinite, signifying the "piling up" of the characteristics. In Fig. 1B, we show three distinct characteristic hypersurfaces piling up along the singular boundary, along which $\upmu$ vanishes. The hard part of the proof is to derive energy estimates in regions where $\upmu$ is small. We refer readers to Sect. 1.9 for further discussion on Christodoulou's framework.

1.7.2. *Wave equations beyond fluid mechanics.* The wave equations studied by Christodoulou in [24] were Euler–Lagrange equations that were invariant with respect to the Poincaré group. In [69], we extended the results of [24] to apply to all wave equations of type (1.6) and of type:

$$\square_{\mathbf{g}(\Psi)}\Psi = \mathfrak{Q}(\boldsymbol{\partial}\Psi, \boldsymbol{\partial}\Psi) \tag{1.9}$$

that fail to satisfy Klainerman's null condition [44], where $\square_{\mathbf{g}(\Psi)}$ is the covariant wave operator of $\mathbf{g}(\Psi)$ (see Def. 2.13) and $\mathfrak{Q}$ is a null form relative to $\mathbf{g}$ (see Def. 2.14). The null form structure on RHS (1.9) is crucial for showing that in the context of shock formation, $\mathfrak{Q}(\boldsymbol{\partial}\Psi, \boldsymbol{\partial}\Psi)$ is an error term that has a negligible effect on the dynamics.

1.7.3. *Nearly simple plane-symmetric waves.* In [73], the second author and his collaborators extended the methods of [24] and [69] to prove stable shock formation for a large class of quasilinear wave equations on the spacetime $\mathbb{R} \times \Sigma$, where $\Sigma = \mathbb{R} \times \mathbb{T}$ was the two-dimensional spatial manifold. The initial data we treated were analogs of the data from Theorem 1.4. More precisely, the data were (asymmetric) perturbations of simple plane-symmetric waves, which are solutions that depend only on $(t, x^1) \in \mathbb{R} \times \mathbb{R}$ and which feature a wave moving only in one direction[21] (say to the right). As in the relativistic case, for irrotational and isentropic solutions to the $2D$ compressible Euler equations, the dynamics reduces to a quasilinear wave equation of type (1.6). Hence, as a special case, the results of [73] yielded stable shock formation for nearly simple and isentropic plane-symmetric solutions to the $2D$ irrotational and isentropic compressible Euler equations.

[21] In plane symmetry, one can study the flow by constructing Riemann invariants, in which case simple plane-symmetric waves would feature only a single non-zero Riemann invariant. See Appendix A, in which we use Riemann invariants to construct the background solutions whose perturbations we study in our main results.

1.7.4. *A new formulation of the flow and stable shock formation in the presence of vorticity and entropy.* In [51], the second author and Luk developed a new formulation of barotropic[22] compressible Euler flow with remarkable regularity and null structures, which in many regimes allows one to study the flow as if it was a perturbation of the quasilinear wave equation (1.9). In [72], the second author derived a similar new formulation for all equations of state in which the pressure is a function of the density and entropy, thus allowing one to incorporate thermodynamic effects into the framework of [24]. In particular, the equations of [72] include a system of transport-div-curl equations for the vorticity and entropy, which allows one to propagate a gain of one derivative for these quantities relative to standard estimates. In this article, we use the equations of [72] to prove our main results, and the gain in regularity is crucial for our approach. In Theorem 2.15, we recall the new formulation of the flow derived in [72]. In [37], we derived a similar new formulation for the relativistic Euler equations.

In [50], the second author and Luk used the equations of [51] and the technology of [24, 69, 73] to prove the first stable shock formation result for the compressible Euler equations with vorticity. In $2D$, the authors treated open sets of initial data with vorticity that are close to the data of an irrotational simple plane-wave solution. The main theorem yielded the full structure of the set of blowup-points within the constant-time hypersurface $\Sigma_{T_{\text{Shock}}}$ of first blowup; in the context of Fig. 1A, the authors understood the structure of $\Sigma_{T_{\text{Shock}}} \cap \mathcal{B}$.

In [52], the second author and Luk used the equations of [72] to extend the results of [50] to the $3D$ case in the presence of vorticity and entropy. The proof was much more difficult than the $2D$ case because the regularity theory for the vorticity and entropy relied on elliptic estimates on constant-Cartesian-time hypersurfaces Σ_t, which are difficult to derive near the shock. In particular, the authors used the elliptic estimates to handle the vorticity stretching term in the equations, which, as is well-known, vanishes for $2D$ barotropic solutions, i.e., in the $2D$ barotropic case, (1.11b) simplifies to $\mathbf{B}\Omega = 0$. These elliptic estimates relied on the transport-div-curl equations for the vorticity and entropy that we derived in [72] (i.e., equations (1.11c)–(1.11d)), and the approach yielded elliptic estimates only along complete, flat hypersurfaces of constant time. As in the $2D$ case, this approach yielded the full structure of the set of blowup-points within the constant-time hypersurface $\Sigma_{T_{\text{Shock}}}$ of first blowup, i.e., in the context of Fig. 1A, the singular boundary portion $\Sigma_{T_{\text{Shock}}} \cap \mathcal{B}$. As we have already mentioned, in order to study the flow beyond the Cartesian-flat hypersurface $\Sigma_{T_{\text{Shock}}}$ and to understand a larger portion of $\mathcal{B}$, one needs additional ingredients, including the identities that we describe in Sect. 1.7.5.

1.7.5. *Remarkable localized integral identities.* In [4], we derived new localized, geometric integral identities for solutions to the $3D$ compressible Euler equations. The identities allow us to derive elliptic estimates for the vorticity and entropy not just on regions bounded by constant-Cartesian-time hypersurfaces Σ_t (as described above), but rather on *arbitrary* spacetime regions that are bounded by spacelike or null hypersurfaces. We use this crucial ingredient in the present paper to derive top-order vorticity and entropy estimates up to the singular boundary.

1.8. **A different approach.** Very recently, Shkoller–Vicol [66] have introduced an important new method for making progress on the open problems described in Sect. 1.4. For some open sets of initially smooth, shock-forming solutions of the $2D$ isentropic Euler equations with spatial topology $\mathbb{T}^2$, they construct a small portion of the crease $\partial_{-}\mathcal{B}$, the singular boundary $\mathcal{B}$ and the Cauchy horizon $\underline{\mathcal{C}}$, which they call "curve[23] of pre-shocks", "downstream surface", and "upstream surface", respectively.

Their work uses Arbitrary-Lagrangian-Eulerian (ALE) coordinates, which are closely related to the acoustical geometric coordinates used in the current paper. Roughly ALE coordinates are equivalent to working with[24] Υ^{-1}, where the change of variables map Υ was introduced at the beginning of Sect. 1.5. Importantly, the paper [66] introduces a new tool in the study of shock formation that is more efficient with respect to regularity. More precisely, the authors differentiate the equations one time with a Cartesian derivative before differentiating with geometric coordinates, which allows them to close the energy estimates for the acoustic geometry without having to use Alinhac's Nash–Moser estimates or the kind of renormalizations that have previously been employed in the context of Christodoulou's geometric energy method; see Sect. 1.9.4 for further discussion. While the approach of first differentiating *some* solution variables one time with a Cartesian derivative has previously been employed in the study of multidimensional shock formation [50, 70], [66] is the first result to use this approach for the shock-forming variables. To implement their approach, the authors rely on a novel, quasilinear version of Riemann invariants for the once-differentiated solution variables, as well as well as some well-designed integration by parts that take into account the precise structure of the equations satisfied by the quasilinear

[22]Barotropic equations of state are such that the pressure p can be expressed as a function of the density ϱ, i.e., with no dependence on s.

[23]Note that in $1+2$ dimensions, the co-dimension two nature of the $\partial_{-}\mathcal{B}$ implies that it is one dimensional.

[24]Alinhac also worked with Υ^{-1} in his works [7, 8, 10, 11] on shock formation for quasilinear wave equations.

Riemann invariants. We also point out that the elliptic-hyperbolic estimates described in Sect. **??** were not needed in [66], in part due to the absence of vorticity stretching in $2D$ and their isentropicity assumptions. It would be interesting to discover if the gain in regularity for the transport variables that we rely on in the general $3D$ case can be derived through the Riemann invariant method introduced in [66].

We now describe the structure and portion of the crease, singular boundary, and Cauchy horizon constructed in [66]. The portion of $\mathcal{B}$ constructed in [66] is strictly convex in the sense explained above in Sects. 1.3 and 1.6 and depicted in Fig. 8B; see also [66, Figure 4 (Center Panel), Section 6.6]. This is a consequence of the authors' "*non-degeneracy condition*" [66, Page 8], which is equivalent to the *fully non-degenerate* singularities described in Sect. 1.6. With $\Sigma_{T_{\text{Shock}}}$ denoting the constant-Cartesian-time hypersurface of first blowup present in this regime, the portions of $\partial_{-}\mathcal{B}, \mathcal{B}, \underline{\mathcal{C}}$ constructed in [66] are contained in $\mathbb{T}^2 \times [T_{\text{Shock}}, T_{\text{Shock}} + \mathcal{O}(\epsilon)]$, where ϵ is small, the initial gradient of the shock-forming variable is of size $\frac{1}{\epsilon}$, and these submanifolds have convexity that is lower-bounded by a positive ϵ-independent constant; see [66, Sect. 4.1] and [66, Condition (v), Page 29]. It would be interesting to adapt the approach of [66] to obtain more of the boundary of the maximal development, e.g., the full structure of a connected component of the crease. We also highlight that it remains an open problem to derive the full maximal development (across space and time) for shock-forming solutions.

1.9. **Additional history and results tied to shocks and singularities.** A fundamental issue, discovered by Riemann [64] in the context of one spatial dimension, is that initially smooth compressible Euler solutions can develop shock singularities in finite time. Roughly, shocks are singularities such that ρ, v, s remain bounded, but some first derivative of ρ and v blows up. Riemann's analysis relied on his discovery of Riemann invariants for isentropic solutions in $1D$. Relative to Riemann invariants, the equations reduce to a quasilinear system for two transport equations with distinct characteristic directions; see Appendix A. The proof of the blowup of the solution's first derivatives then follows from differentiating the equations to obtain a Riccati-type structure, much like in the model case of Burgers' equation $\partial_t \Psi + \Psi \partial_x \Psi = 0$.

1.9.1. *The advanced state of the* $1D$ *theory and the key difference with multi-dimensions.* In $1D$, for a large class of quasilinear hyperbolic systems (such as strictly hyperbolic systems), there is an advanced theory capable of describing the global behavior of solutions, including the formation of shocks and the subsequent interactions of the shock waves. We refer readers to the compendium [33] for a history of the subject and a comprehensive introduction to the main $1D$ techniques.

A principal reason for the advanced state of the $1D$ theory is that the equations are well-posed for initial data in appropriate bounded variation (BV) spaces. The state of affairs is dramatically different in multi-dimensions; Rauch's fundamental work [62] showed that in multi-dimensions, quasilinear hyperbolic systems are typically *ill-posed* for data in BV spaces. In fact, the only known well-posedness results in multi-dimensions are for initial data in L^2-type Sobolev spaces. For this reason, in multi-dimensions, one is forced to derive energy estimates, which can be incredibly difficult in regions containing singularities; this is the main reason why the theory of multi-dimensional shock waves is so much less developed compared to the $1D$ case.

We also highlight that for the relativistic Euler equations in $1D$, we recently provided [2] a complete description of a localized subset of the maximal development for some initially smooth, isentropic plane-symmetric shock-forming solutions that enjoy the acoustical transversal convexity described in Sect. 1.3. In particular, we showed that a singular boundary and a Cauchy horizon emerge from the first singularity. See also [66, Appendix A] for an analogous result for the compressible Euler equations in $1D$.

1.9.2. *The first blowup-result in multi-dimensions without symmetry: proof by contradiction.* Providing a *constructive* proof of shock formation in higher dimensions turns out to be a very hard problem. In a nutshell, the reason is that away from $1D$, it seems necessary to carefully track the evolution of characteristic hypersurfaces, which is much more difficult compared to the $1D$ case. The characteristic geometry (e.g., the eikonal function u in the context of the present article) is much more difficult to construct and control, and, crucially, all proofs of even local well-posedness rely on energy estimates in L^2-based Sobolev spaces, which are difficult to derive near singularities. In [67], Sideris proved an influential, *non-constructive* stable blowup-result in $3D$, the first one for multi-dimensional compressible Euler flow. He assumed that the equation of state is barotropic (see Footnote 22) and that it satisfies a convexity assumption. His proof applied to a large set of data, but it did not reveal the nature of the blowup; his arguments relied on virial-type identities, and he showed blowup through a contradiction-argument.

1.9.3. *Alinhac's constructive proof of the formation of isolated singularities.* The first constructive results on shock formation in multidimensions were by Alinhac, who used nonlinear geometric optics (i.e., he constructed characteristic surfaces) and Nash–Moser estimates to prove stable small-data shock formation for a class of quasilinear wave equations of the form:

$$(\mathbf{g}^{-1})^{\alpha\beta}(\boldsymbol{\partial}\Phi)\partial_\alpha\partial_\beta\Phi = 0, \tag{1.10}$$

whenever they fail to satisfy the first[25] null condition in $2D$ [7, 8], whenever they fail to satisfy the second null condition in $2D$ [10], and whenever they fail to satisfy the null condition in $3D$ [10], where the $3D$ null condition was identified by Klainerman in [44]; see also Alinhac's lecture notes for an overview of his approach [11]. He used the Nash–Moser estimates to avoid derivative loss in his control of the regularity of the characteristic surfaces. His proof applied to open sets of *fully non-degenerate* initial data, which he showed lead to the formation of *fully non-degenerate* singularities in which $\boldsymbol{\partial}^2\Phi$ blows up while Φ and $\boldsymbol{\partial}\Phi$ remain bounded. Roughly, Alinhac's fully non-degenerate initial data were such that the singular boundary has the strictly convex structure depicted in Fig. 8, and his approach allowed him to follow the solution up to the lowest point on the singular boundary, but not further. By a "fully non-degenerate" singularity, we mean that the singularity is an isolated point in the constant-Cartesian-type hypersurface of first blowup. In Fig. 8, we denote this lowest point by "b_{lowest}." Alinhac's framework has also been applied to other wave equations; see, for example, [34–36].

1.9.4. *Christodoulou's breakthrough on irrotational, isentropic shock formation.* In [24], Christodoulou proved a stable shock formation result for open sets of irrotational and isentropic initial solutions to the relativistic Euler equations in three spatial dimensions. In this context, one can study the flow with the help of a potential function Φ, and relativistic Euler flow reduces to a quasilinear wave equation of type (1.10). Aside from a single exceptional equation of state corresponding to the graph of a timelike minimal surface in Minkowski space, all wave equations of irrotational and isentropic relativistic fluid mechanics fail to satisfy the null condition, i.e., the basic mechanism driving shock formation is present in all equations but one. The data that Christodoulou treated were compact perturbations of non-vacuum constant state data, and his shock formation results revealed the instability of these states (due to shock formation in perturbed solutions) under irrotational and isentropic perturbations. To study the solution, Christodoulou used a refined version of nonlinear geometric optics, based on techniques that he co-developed with Klainerman in their proof of the stability of the Minkowski space [23]. Specifically, Christodoulou constructed an eikonal function u, i.e., a solution to the eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}(\boldsymbol{\partial}\Phi)\partial_\alpha u \partial_\beta u = 0$, and he showed that the wave equation solution's first Cartesian derivatives, $\boldsymbol{\partial}\Phi$, remain quite smooth relative to the geometric coordinates. The level-sets $\mathcal{P}_u$ of u are characteristic hypersurfaces for the wave equation. The surfaces depend on the solution itself, reflecting the quasilinear nature of the flow.

As we mentioned already, as in the present paper, in Christodoulou's framework, the formation of the shock corresponds to the vanishing of the inverse foliation density μ. In the region of classical existence, one has $\mu > 0$, and when $\mu \to 0$, the density of the $\mathcal{P}_u$ in Cartesian coordinate space becomes infinite, signifying the piling up of the characteristics. The blowup of $\boldsymbol{\partial}^2\Phi$ is a consequence of the degeneracy between the Cartesian and a system[26] of "geometric coordinates" $(t, u, \vartheta^1, \vartheta^2)$, where we schematically represent the degeneracy caused by the vanishing of μ as follows: $\boldsymbol{\partial} \sim \frac{1}{\mu}\frac{\partial}{\partial u}$; the blowup of $\boldsymbol{\partial}^2\Phi$ then follows from this relation, from proving that $\mu \to 0$ in finite time, and from proving a lower of the form $\left|\frac{\partial}{\partial u}\partial\Phi\right| \gtrsim 1$.

Importantly, Christodoulou's proof relied on geometric energy estimates relative to foliations of spacetime by the $\mathcal{P}_u$ and flat Cartesian time slices Σ_t, rather than Nash–Moser estimates. To control the geometry without derivative loss, he relied on techniques and renormalized quantities that have their roots in [26, 45]. Crucially, Christodoulou's geometric approach allowed him to prove shock formation for a larger class of singularities than the fully non-degenerate ones treated by Alinhac. He was able to show that blowup occurs at one or more points, even for solutions whose singular boundaries do not have to enjoy the strict convexity displayed in Fig. 8. However, in the absence of strict convexity, the full structure of the maximal development was not revealed. Interestingly, the work [24] also yielded a sharp conditional global existence result, which showed that irrotational, isentropic near-constant-state solutions are global unless a shock forms.

[25] In $2D$, for small-data solutions to (1.10) to be global, two null conditions must be satisfied, one tied to the structure of the quadratic nonlinearities and the second tied to the structure of the cubic nonlinearities. These $2D$ null conditions and their relevance for small-data global existence were discovered by Alinhac [9].

[26] In [24], the "angular" coordinate functions ϑ^1 and ϑ^2 were constructed so as to be constant along the integral curves of the null generator L. In contrast, in the present paper, in the role of the "angular" coordinate functions, we use the standard Cartesian coordinates x^2 and x^3, which are not typically constant along the integral curves of L. This minor difference turns out to have no substantial effect on the analysis.

1.9.5. *Shock-formation works that built upon Christodoulou's framework.* Many authors have used Christodoulou's approach to prove stable shock formation for various quasilinear hyperbolic PDEs. For example, there are stable shock formation results for:

- A larger class of wave equations [59, 69].
- Solution regimes that are different than the small, compactly supported data regime: [41, 59].
- Solutions that exist classically precisely on a past-infinite half-slab [58].
- Various systems involving multiple speeds of propagation, some with symmetry [12, 29], and some without [70, 71].

1.9.6. *Shock formation in the presence of vorticity and entropy.* The aforementioned paper [50] was to first to prove stable shock formation for open sets of $2D$ compressible Euler solutions with vorticity under an arbitrary[27] barotropic equation of state. The initial data were not required to be fully non-degenerate, i.e., transversal convexity did not play a role in the proof of blowup. The main theorem followed the solution to the constant-time hypersurface of first blowup, and it gave a complete description of what blows up and what does not, as well as a precise description of the set of singular points at the time of first blowup, i.e., in the context of Fig. 1A, a complete description of $\Sigma_{T_{\text{Shock}}} \cap \mathcal{B}$. Since the geometric setup relied on the methods of [24], the singularities were allowed to be more general than the fully non-degenerate ones described in Sect. 1.9.3.

The recent work [52] extended [50] to the case of the $3D$ compressible Euler equations under an arbitrary (Footnote 27 applies here as well) equation of state with vorticity and entropy. As in [50], stable shock formation was proved in [52] without transversal convexity, although additional information on the Hölder regularity of the solution with respect to the Cartesian coordinates was derived[28] in a sub-regime of solutions that *do* have transversal convexity.

1.9.7. *A new approach to proving the formation of fully-non-degenerate shock singularities via self-similarity.* In [16, 17], the authors developed an interesting new approach for proving the formation of shock singularities in $3D$ compressible Euler solutions under an adiabatic equation of state without symmetry, and with vorticity (and also entropy in [17]). The approach allows one to follow the solution to the time of first blowup, and the singularities produced are isolated within the constant-time hypersurfaces of first blowup. That is, in the context of Fig. 8, the approach allows one to follow the solution up to the point b_{lowest}. Such singularities are analogs of the non-degenerate singularities that Alinhac studied [7, 8, 10, 11] in the case of quasilinear wave equations. The framework of [16, 17] relied on modulation parameters to show that for open sets of smooth data with large gradients, a singularity develops in a short time, and it is a perturbation of a self-similar Burgers'-type shock.

The aforementioned $3D$ works were preceded by the work [18] in $2D$ azimuthal symmetry with vorticity. In the recent work [15], in the same symmetry class, the authors constructed shock-forming solutions whose cusp-like spatial behavior (with respect to the standard coordinates) is *non-generic*; such solutions are unstable. In the language of the present paper, such solutions do not exhibit the quantitative transversal convexity (18.5) that we exploit in our main theorem.

1.9.8. *Self-similar blowup for non-hyperbolic PDEs.* Notably, there are *non-hyperbolic* PDES with solutions that exhibit self-similar blowup modeled on a self-similar Burgers'-type shock. Examples include the Euler–Poison system [61], Burgers' equation with transverse viscosity [32], the Burgers–Hilbert equations [76], the fractal Burgers equation [46], and various dispersive or dissipative perturbations of the Burgers equation [60]; see also [30, 31]. It would be interesting to investigate the extent to which the set of singular points can be understood.

1.9.9. *Implosion singularities.* The recent breakthrough work [57] in spherical symmetry showed that for the compressible Euler equations *and* the compressible Navier–Stokes equations, under adiabatic equations of state $p = \varrho^{\gamma}$ with $\gamma > 1$, outside of a countable set of γ-values, there exist C^{∞} initial data with density tending to 0 at spatial infinity such that the corresponding solution's density blows up at the center of symmetry in finite time. In fact, there are infinitely many such singularity-forming solutions, collectively exhibiting a discrete sequence of blowup-rates. These "implosion singularities" are much more severe than shocks. The methods of [57] suggest that the implosion singularity might enjoy co-dimension stability under perturbations of the initial data without symmetry. In [13], the results of [57] were extended to all $\gamma > 1$, and in the case $\gamma = \frac{7}{5}$, implosion singularity formation was shown for Navier–Stokes solutions for some initial data with a non-zero limiting density at spatial infinity. See also the recent paper [19] for a proof of implosion singularity formation for some solutions with spatial topology $\mathbb{R}^3$ or $\mathbb{T}^3$ *without symmetry assumptions*.

[27] As in our main results, the Chaplygin gas equation of state is exceptional and is not known to lead to shock formation.

[28] With transversal convexity, the solution was shown to enjoy $C^{1/3}$-Hölder regularity with respect to the Cartesian coordinates up to the singularity, while not enjoying $C^{(1/3)^+}$-Hölder regularity. Providing a more detailed treatment of this sub-regime was inspired by Hölder regularity results derived in [16–18] for some solutions with fully non-degenerate singularities.

1.9.10. *Rarefaction waves.* In [5, 6], Alinhac used a Nash–Moser iteration scheme to prove local existence and uniqueness of rarefaction wave solutions for a large class of multi-dimensional hyperbolic systems that includes scalar conservation laws and the compressible Euler equations as special cases. The solutions are multi-dimensional analogs of solutions to the well-known Riemann problem in $1D$, in which the initial data are piecewise smooth and discontinuous, and the initial discontinuity is immediately smoothed out by the flow. The recent works [53, 54] have sharpened Alinhac's results for the $2D$ isentropic compressible Euler equations for a family of irrotational data. Specifically, in [53, 54], for discontinuous irrotational data that are (asymmetric) perturbations of plane-symmetric data for a corresponding $1D$ Riemann problem, the authors proved that the corresponding $2D$ irrotational and isentropic rarefaction solution is a perturbation of the standard $1D$ rarefaction wave solution to the Riemann problem. The approach of [53, 54] circumvents loss of derivatives in the corresponding linearized problem, which allows the authors to avoid Alinhac's Nash–Moser estimates. [53, 54] also provides a sharp description of the characteristic geometry in the problem.

1.9.11. *Inviscid limits.* A physically important and mathematically interesting problem is to study the relationship between the formation of shock singularities in classical solutions to hyperbolic PDEs and the behavior of classical solutions when a small amount of viscosity is added to the equations. Of particular interest is to understand the zero-viscosity limiting behavior of classical solutions. The recent paper [21], which concerns Burgers' equation in $1D$ and its viscous analog, is the only work to date that yields rigorous information about the zero-viscosity limit of classical solutions all the way up until the time of first singularity formation for the inviscid solution. See also the earlier work [20], which uses formal asymptotic expansions to connect the behavior of viscous solutions to the inviscid one near the shock. The main results of [21] show that the $1D$ viscous Burgers' equation solution can be decomposed into a singular piece and a smoother piece, and that the viscous solution converges to the singular piece in $\|\cdot\|_{L^\infty}$ as the viscosity vanishes (the L^∞ norm is shown to be bounded from above by the viscosity parameter to a positive power). Here, the $\|\cdot\|_{L^\infty}$ norm is over the entire region of classical existence of the inviscid solution, i.e., the L^∞-convergence holds all the way until the time of first blowup for the solution to the inviscid Burgers' equation. The results of [21] apply for non-degenerate initial data, where the notion of non-degeneracy in [21] is essentially equivalent to the transversal convexity assumption satisfied by the solutions featured in our main theorems. An important open problem is to extend the results of [21] to the compressible Euler equations (where the corresponding viscous model is the compressible Navier–Stokes equations) and to multiple spatial dimensions.

1.9.12. *Progress on the shock development problem.* In Majda's celebrated works [55, 56], he proved linear stability and local well-posedness results for a class of weak solutions to the $3D$ compressible Euler equations arising from a set of discontinuous initial data. More precisely, he assumed that the data on $\mathbb{R}^3$ were piecewise smooth and jumped across a smooth two-dimensional hypersurface such that the jumps are consistent with well-known Rankine–Hugoniot jump conditions. He also assumed the data satisfy a certain "stability condition." His main result was the construction of a local, unique weak solution in a subset of $\mathbb{R}^{1+3}$, and a corresponding three-dimensional shock hypersurface, across which the solution jumps in accordance with the Rankine–Hugoniot jump conditions. This is known as the *shock front problem*.

An important problem, distinct from the shock front problem, is to determine whether/how Majda's discontinuous initial conditions can develop from an initially smooth solution. That is, one would like to describe the *transition* of compressible Euler solutions from being smooth, to developing a "first singularity" – which in the language of the present paper is the crease – and finally to becoming a weak solution that develops a shock hypersurface (emanating from the crease), across which the solution jumps. This is known as the *shock development problem*. In Fig. 5, we denote the shock hypersurface by "$\mathcal{K}$," and we show its emergence from the crease $\partial_-\mathcal{B}$. We stress that Majda's works did not study the flow in a neighborhood of a crease $\partial_-\mathcal{B}$, but rather started from initial conditions on a flat hypersurface Σ_{t_0} such that $\Sigma_{t_0} \cap \mathcal{K}$, which is a co-dimension one hypersurface in Σ_{t_0} across which the data jump, is already assumed to exist. A crucial issue in the study of the shock development problem is that the jump conditions imply that initially irrotational and isentropic smooth solutions will develop dynamic entropy and non-trivial vorticity as the solution jumps across the shock hypersurface. This means that it is not possible to solve the true shock development problem in the class of irrotational and isentropic solutions. The full problem in $3D$ without symmetry assumptions remains open, but there has been inspiring progress in recent years:

- In [47], under a convexity assumption on the nonlinearity, Lebaud solved the shock formation and shock development problem for the p-system in $1D$, which is a 2×2 hyperbolic system that admits a pair of Riemann invariants. She assumed that before the shock, one of the Riemann invariants is constant (so that the pre-shock dynamics are that of a simple wave), and she made a non-degeneracy assumption on the initial data that is an analog of the transversal convexity we exploit in our main results. In [22], the authors proved a similar result, but without assuming that one of the Riemann invariants is constant before the shock. In [43], the authors proved a related result for scalar conservation laws, their main new contribution being that they treated initial data satisfying a weaker non-degeneracy assumption.

- In [27], Christodoulou–Lisibach used Riemann invariants and a pair of eikonal functions (one "ingoing" and the other "outgoing") to solve the shock development problem for spherically symmetric, barotropic solutions to the relativistic Euler equations. Their proof worked for all equations of state and for solution regimes such that the *determinism condition* holds. Roughly, this condition posits that the shock hypersurface that emerges from the singularity should be: **i)** supersonic (i.e., spacelike) with respect to the acoustical metric corresponding to the pre-shock state of the solution (which, in the literature, is often called the "state ahead of the shock"), and **ii)** subsonic (i.e., timelike) with respect to the acoustical metric corresponding to the post-shock state of the solution (which, in the literature, is often called the "state behind the shock"). Such solutions are considered to be the physical ones.[29] Mathematically, the problem comprised a system of quasilinear transport equations for a piecewise smooth solution coupled to the free boundary problem of tracking the location of the shock hypersurface, which emanates from the crease. They solved the problem through an iteration scheme that relied on the assumption that the crease exhibits transversal convexity and the availability of estimates that stem from already having access to the maximal development (which Christodoulou constructed in [24]). Relative to geometric coordinates analogous to the ones that we use in our main results, the data were assumed to be smooth along the crease and Cauchy horizon, and this allowed them to use Taylor expansions to approximate the expected location of the shock hypersurface and the behavior of the solution along it; these Taylor expansions form a crucial ingredient in controlling the iterates.
- In his breakthrough monograph [25], Christodoulou extended the methods from [27] and solved the "restricted" shock development problem *without symmetry assumptions* for the compressible Euler equations and relativistic Euler equations in an arbitrary number of spatial dimensions. The word "restricted" means that he studied only irrotational and isentropic solutions, and that he ignored the jump in entropy and vorticity across the shock hypersurface, thereby producing a weak solution to a hyperbolic PDE system that approximates the real one. Christodoulou assumed that along the crease, the solution satisfies the same kind of transversal convexity that the solutions produced by our main results enjoy. In particular, this class of singularities is more general than the fully non-degenerate ones treated by Alinhac, as we discussed in Sect. 1.9.3. The main new difficulty in [25] is deriving energy estimates for the solution and the acoustic geometry. In particular, relative to a system of geometric coordinates, the high order energies exhibit degenerate behavior, much like the high order energies in the present paper.
- In [42], Huicheng–Lu solved the shock development problem for a class of first-order, scalar, divergence-form hyperbolic equations in $2D$, starting from initial singularities that are fully non-degenerate, i.e., such that the crease is strictly convex, as in Fig. 8.
- In [14], Buckmaster–Drivas–Shkoller–Vicol extended the methods from [16–18] to solve the shock development problem for the $2D$ compressible Euler equations in azimuthal symmetry with vorticity and entropy under the adiabatic equations of state $p = \frac{1}{\gamma}\varrho^{\gamma}\exp(s)$ for initial singularities that are fully non-degenerate with respect to variations in the radial and angular variable. In particular, [14] provides the first solution to a compressible Euler shock development problem with vorticity. As in [27], the problem comprised transport equations coupled to the issue of tracking the location of the free boundary, and it was solved via an iteration scheme based on Taylor expansions relative to a coordinate system in which the solution is rather smooth. A new feature compared to [27] is that there is another characteristic direction in [14], corresponding to the transporting of vorticity and entropy by the material derivative vectorfield.

We also highlight that recently, there have been other interesting works on weak solutions to the compressible Euler equations in $1D$ without entropy. In [48], Lisibach proved local existence for the shock reflection (off of a wall) problem in plane-symmetry. In [49], he studied the interaction of two shocks in plane-symmetry and proved local existence of a weak solution near the interaction point. See also [75], in which Wang proved the same result in spherical symmetry.

1.10. **Ideas behind the proof of the main results.** In this section, we provide a more detailed overview of some key ideas behind the proof of our main results.

[29]In particular, in [27], it was shown that in the full shock development problem without symmetry, there is a regime in which the determinism condition is equivalent to the entropy increasing via a jump across the shock hypersurface.

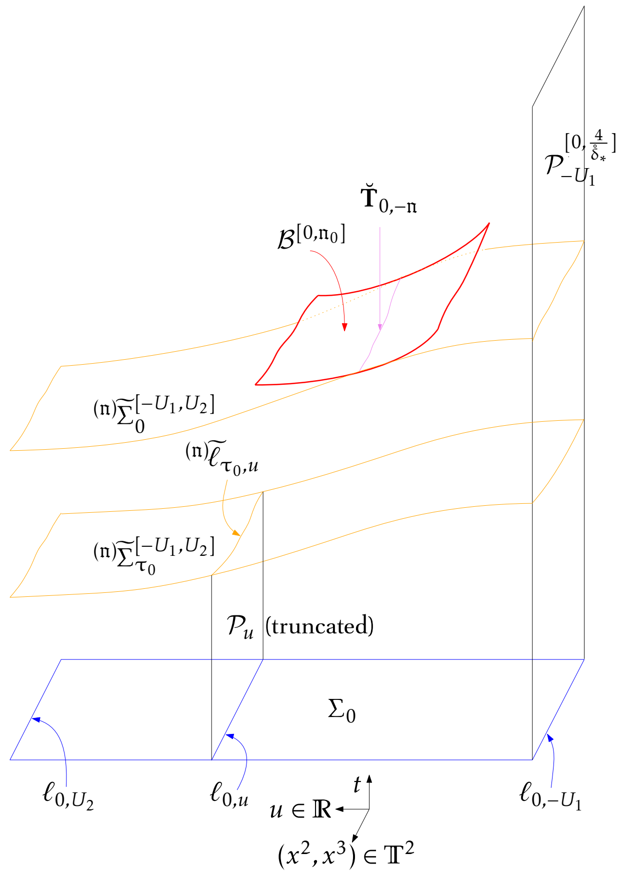


Figure 9. data-hypersurfaces in geometric coordinate space

1.10.1. *Almost Riemann invariants.* To study perturbations of simple isentropic plane-symmetric waves, we find it convenient to replace the scalar functions ρ (where ρ is the logarithmic density from Def. 2.4) and v^1 with the "almost[30] Riemann invariants" (see Def. 2.5) $\mathcal{R}_{(+)}$, $\mathcal{R}_{(-)}$ which carry the same information.

Here and in the rest of the paper, $\vec{\Psi} \stackrel{\text{def}}{=} (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$ are the fluid "wave-variables" and $\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}(\vec{\Psi})$ denotes the acoustical metric (see definition (2.15a)), which drives the propagation of sound waves.

1.10.2. *"Late-time" assumptions on the data for perturbations of simple isentropic plane-waves.* We consider the "bona fide" initial data to be the data specified along a portion of the flat spacelike hypersurface Σ_0 and a portion of a null hypersurface, denoted by $\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}$ in Fig. 9 (i.e., we are studying the Cauchy problem for spacelike-characteristic data), where the parameter $\mathring{\delta}_* > 0$ (see (11.6)) depends on the data and is such that the Cartesian time of first blowup is approximately $\mathring{\delta}_*^{-1}$. However, the structures we need to detect the singular boundary appear only late in the evolution, and we therefore find it convenient to state our data-assumptions on a "late" hypersurface portion of constant rough time, denoted by ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$ in Fig. 9, that will end up being close to the singular boundary $\mathcal{B}^{[0,\mathfrak{n}_0]}$. That is, we find it convenient to state the assumptions on the spacelike level-sets ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]} \stackrel{\text{def}}{=} \{{}^{(\mathfrak{n})}\tau = \tau_0\} \cap \{-U_1 \leq u \leq U_2\}$ of the rough time functions, which we describe in detail in Sect. 1.10.7. In Sect. 11, we state all of these data-assumptions. In Appendix B, we use Cauchy stability arguments to show that these assumptions are satisfied by perturbations of the bona fide data on Σ_0 corresponding to a class of shock-forming simple isentropic plane-wave solutions. In Appendix A, we use standard arguments to construct these simple isentropic plane-wave solutions.

[30]For plane-symmetric isentropic solutions, $\mathcal{R}_{(+)}$, $\mathcal{R}_{(-)}$ are Riemann invariants, but away from symmetry, they are not.

1.10.3. *The geometric div-curl-transport formulation of the flow.* To derive estimates for the fluid variables, we fundamentally rely on a geometric reformulation of the flow, which was derived in [72] and which we restate below in Theorem 2.15. To aid our discussion in the remainder of the introduction, here we recall the formulation in schematic form, with unimportant lower order terms omitted:

$$\square_{\mathbf{g}(\vec{\Psi})}\Psi = (\mathcal{C},\mathcal{D}) + \mathfrak{Q}(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}) + (\Omega,S)\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{1.11a}$$

$$\mathbf{B}(\Omega,S) = (\Omega,S)\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{1.11b}$$

$$\mathbf{B}(\mathcal{C},\mathcal{D}) = \mathfrak{Q}(\boldsymbol{\partial}\vec{\Psi},\partial\Omega) + \mathfrak{Q}(\boldsymbol{\partial}\vec{\Psi},\partial S) + S\cdot\mathfrak{Q}(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi}) + S\cdot S\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{1.11c}$$

$$(\operatorname{div}\Omega,\operatorname{curl}S) = \partial\vec{\Psi}. \tag{1.11d}$$

In (1.11a)–(1.11d), $\Omega = \frac{(\operatorname{curl} v)^i}{\exp(\rho)}$ is the specific vorticity (here, ρ is the logarithmic density, defined in Sect. 2.3.1) and $S^i = \partial_i s$ is the entropy gradient vectorfield. The *modified fluid variables* $\mathcal{C}$ and $\mathcal{D}$ are defined in Def. 2.7 and satisfy $\mathcal{C} \sim \operatorname{curl}\Omega$ and $\mathcal{D} \sim \operatorname{div} S$, where "$\sim$" denotes equality up to lower order (in the sense of regularity) factors and terms. Moreover, $\square_{\mathbf{g}(\vec{\Psi})} f$ is the covariant wave operator of $\mathbf{g}(\vec{\Psi})$ acting on the scalar function f (see Def. 2.13), $\boldsymbol{\partial} f = (\partial_t f, \partial_1, \partial_2 f, \partial_3 f)$ is the array of Cartesian coordinate spacetime partial derivatives of f, and $\partial f = (\partial_1, \partial_2 f, \partial_3 f)$ is the array of Cartesian spatial partial derivatives of f. The "$\mathfrak{Q}$" are *null forms relative to* $\mathbf{g}$ (see Def. 2.14); see Sect. 1.10.11 for a discussion of the crucial role they play in the proof.

We highlight that when deriving top-order L^2 estimates for the specific vorticity and entropy gradient, we crucially rely on the transport-div-curl equations (1.11c)–(1.11d) to propagate sufficient regularity for the source terms $(\mathcal{C},\mathcal{D})$ on RHS (1.11a); see Sect. 27 for the details. That is, even though these terms formally satisfy $\mathcal{C} \sim \operatorname{curl}\Omega$ and $\mathcal{D} \sim \operatorname{div} S$, we cannot control $(\mathcal{C},\mathcal{D})$ using only the transport equation (1.11b). The reason is that (1.11b) has a source term depending on $\boldsymbol{\partial}\vec{\Psi}$, and thus, by estimating only the transport equation, we could prove only that $\operatorname{curl}\Omega, \operatorname{div} S$ are as regular as $\boldsymbol{\partial}^2\vec{\Psi}$, which would be insufficient regularity for treating $\operatorname{curl}\Omega$ and $\operatorname{div} S$ as source terms in the wave equation (1.11b).

1.10.4. *Three kinds of coordinate systems.* The proof of our main results is based on understanding the behavior of the solution relative to three coordinate systems as well as understanding the degenerate (near the shock) transformation properties of the coordinate systems.

1. (**The Cartesian coordinates** (t, x^1, x^2, x^3)). These are the fundamental coordinates relative to which the compressible Euler equations (1.1a)–(1.1c) are posed. In the solution regime under study, the singularity along $\mathcal{B}$ coincides with the blowup of $|\partial_1 \mathcal{R}_{(+)}|$, where $\mathcal{R}_{(+)}$ is the "almost Riemann invariant" introduced in Def. 2.5.
2. (**The geometric coordinates** (t, u, x^2, x^3)). As in the other works described in Sect. 1.7, u is an eikonal function, that is, a solution to the eikonal equation:

$$(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \tag{1.12}$$

where $\mathbf{g} = \mathbf{g}(\vec{\Psi})$ is the acoustical metric, a fluid-dependent Lorentzian metric that governs the propagation of sounds waves; see definition (2.15a). Its level-sets $\mathcal{P}_u$ are characteristic for the compressible Euler equations (1.1a)–(1.1c), and its gradient vectorfield $-(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta u$ is parallel to the vectorfield L appearing throughout the article. In this paper, we refer to the $\mathcal{P}_u$ as "characteristics," "null hypersurfaces," "acoustically null hypersurfaces," or "$\mathbf{g}$-null hypersurfaces." We refer to (t, u, x^2, x^3) as the "geometric coordinates." We prove that *the solution remains rather smooth* relative to the geometric coordinates. However, our high order geometric energies can blow up as the shock forms (i.e., on $\mathcal{B}$), which introduces severe technical difficulties into the PDE analysis. This is tied to the crucial fact that the geometric coordinates are *not diffeomorphic* to the Cartesian ones up to $\mathcal{B}$; see Prop. 33.1.
3. • (**The adapted rough coordinates** $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$). These are a new ingredient, fundamental for our main results. The *rough time functions* ${}^{(\mathfrak{n})}\tau$ are a one-parameter family of time coordinates, indexed by $\mathfrak{n} \in [0, \mathfrak{n}_0]$, where $\mathfrak{n}_0 > 0$ is a real number depending on the solution regime under study. The *rough adapted coordinates* $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$ are the corresponding one-parameter family of coordinate systems. The key virtue of the ${}^{(\mathfrak{n})}\tau$ is that their level-sets are *good spacelike hypersurfaces* in geometric coordinate space that, unlike the constant-Cartesian-time hypersurfaces Σ_t, are adapted to the shape of $\mathcal{B}$. The price one pays is that the ${}^{(\mathfrak{n})}\tau$ have limited regularity (see Sect. 1.10.7) and that when $\mathfrak{n} > 0$ is small, the level sets of ${}^{(\mathfrak{n})}\tau$ can be "almost null" near $\partial\_\mathcal{B}$, leading to degeneracies in the PDE estimates; see Fig. 4B. The ${}^{(\mathfrak{n})}\tau$ are constructed (see Sect. 4) so that their range is $[\tau_0, 0]$ for some constant $\tau_0 < 0$, such that the fluid data are "known" on the level-set

$\{{}^{(\mathfrak{n})}\tau = \tau_0\}$ (see Sect. 1.10.2), and such that $\{{}^{(\mathfrak{n})}\tau = 0\}$ intersects $\mathcal{B}$ precisely in one embedded, spacelike, two-dimensional torus, denoted by $\check{\mathbf{T}}_{0,-\mathfrak{n}}$. The union of the $\mathfrak{n}$-indexed tori foliates $\mathcal{B}$; see Fig. 4.

- (**The adapted rough coordinates** $({}^{(\text{Interesting})}\tau, u, x^2, x^3)$). With the help of the family of coordinate systems $\{({}^{(\mathfrak{n})}\tau, u, x^2, x^3)\}_{\mathfrak{n}\in[0,\mathfrak{n}_0]}$, we construct another rough time function, denoted by ${}^{(\text{Interesting})}\tau$, whose level sets foliate a region $\mathcal{M}_{\text{Interesting}}$ containing the entire singular boundary portion under study. ${}^{(\text{Interesting})}\tau$ is a $C^{1,1}$ function of the geometric coordinates (t, u, x^2, x^3), *and this is its optimal regularity regardless of the smoothness of the data* (see Remark 32.10). In particular, the singular boundary portion under study is a sub-manifold-with-boundary that is contained in the level-set $\{{}^{(\text{Interesting})}\tau = 0\}$; see Fig. 6, which displays $\mathcal{M}_{\text{Interesting}}$ as well as two level-sets of ${}^{(\text{Interesting})}\tau$, denoted by ${}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}$ and ${}^{(\text{Interesting})}\Sigma_{\tau_0}^{[-U_1,U_2]}$.

1.10.5. *The vanishing of* μ *signifies the singularity formation.* As in [24], the formation of the shock singularity is precisely characterized by the vanishing of the inverse foliation density μ, defined in (1.8), where in the present context, $\mathbf{g}$ is the acoustical metric. We recall that the vanishing of μ signifies the infinite density of the characteristics; see Fig. 1B, where μ vanishes along $\mathcal{B}$ and the characteristics pile up there. The blowup of some Cartesian partial derivative of the almost Riemann invariant $\mathcal{R}_{(+)}$ then follows from proving a bound of the schematic form $\mu|X\mathcal{R}_{(+)}| \gtrsim 1$ in a neighborhood of the points where μ vanishes, where X is a vectorfield that is transversal to the $\mathcal{P}_u$, that is L^∞ close to the Cartesian partial derivative vectorfield ∂_1, and that has Euclidean length approximately equal to 1. This is essentially the same blowup-mechanism as in the irrotational and isentropic case, as we described in Sect. 1.9.4. Here we highlight that the vectorfield $\check{X} \overset{\text{def}}{=} \mu X$, which is a geometric replacement for the partial derivative vectorfield $\frac{\partial}{\partial u}$ (in the geometric coordinate system), can be used to derive *regular estimates* for the solution's transversal derivatives: all fluid quantities and geometric tensors Q that we use to study the solution satisfy $|\check{X}Q| \lesssim 1$ up to the singularity; the μ-weight in the definition of $\check{X}$ precisely cancels out the singular behavior of $|XQ|$ as $\mu \downarrow 0$. See Sect. 1.10.9 for further discussion.

1.10.6. *The causal structure of the singular boundary.* The discussion in Sect. 1.10.5 suggests that the singular boundary should be the entire hypersurface $\{\mu = 0\}$. However, as we already discussed in Sect. 1.3, this is not true, for only a subset of $\{\mu = 0\}$ is part of the maximal development. One reason is that formally, for the solutions under study, $\{\mu = 0\}$ is the limit of hypersurfaces $\{\mu = \mathfrak{m}\}$ as $\mathfrak{m} \downarrow 0$, and for $\mathfrak{m} > 0$, these hypersurfaces have a spacelike part, a null part, and timelike part. For $\mathfrak{m} > 0$, our analysis yields access to the portions of $\{\mu = \mathfrak{m}\}$ that are $\mathbf{g}$-spacelike or $\mathbf{g}$-null, but not necessarily the $\mathbf{g}$-timelike portion. Hence, the portion of $\{\mu = 0\}$ that makes up the singular boundary can be viewed as the limit as $\mathfrak{m} \downarrow 0$, of the portion of $\{\mu = \mathfrak{m}\}$ that is $\mathbf{g}$-spacelike or $\mathbf{g}$-null. From a careful analysis of the causal structure of the hypersurfaces $\{\mu = \mathfrak{m}\}$, which we provide in Lemma 32.6, we find that the accessible portion of $\{\mu = 0\}$ that arises in the limit $\mathfrak{m} \downarrow 0$ is $\{\mu = 0\} \cap \{\check{X}\mu \leq 0\}$, where the vectorfield $\check{X}$ is the same one described in Sect. 1.10.5. More precisely, our results are local in spacetime, and our main theorem yields the structure of the singular boundary portion $\{\mu = 0\} \cap \{-\mathfrak{n}_0 \leq \check{X}\mu \leq 0\}$, which we denote by $\mathcal{B}^{[0,\mathfrak{n}_0]}$ in Fig. 6. We also recall (see Sect. 1.3) that the complementary set $\{\mu = 0\} \cap \{\check{X}\mu > 0\}$ is a "fictitious portion" that is not part of the maximal development (and hence cannot be constructed uniquely from the initial data). From a different perspective, in the context of Fig. 1A, the fictitious portion $\{\mu = 0\} \cap \{\check{X}\mu > 0\}$ is "cut off" by the Cauchy horizon $\underline{\mathcal{C}}$, i.e., in the maximal classical globally hyperbolic development, $\underline{\mathcal{C}}$ develops "before" $\{\mu = 0\} \cap \{\check{X}\mu > 0\}$ has a chance to form.

In view of the above discussion, in our main theorem, we construct the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]} \overset{\text{def}}{=} \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \{\mu = 0\} \cap \{\check{X}\mu = -\mathfrak{n}\}$; see Fig. 6. Our aforementioned *transversal convexity* assumption on the initial data, which we are able to propagate throughout the evolution (see (18.5)), ensures that the hypersurfaces $\{\mu = 0\}$ and $\{\check{X}\mu = -\mathfrak{n}\}$ intersect transversally in embedded, two-dimensional $\mathbf{g}$-spacelike tori that we denote by $\check{\mathbf{T}}_{0,-\mathfrak{n}}$. In particular, we show that $\mathcal{B}^{[0,\mathfrak{n}_0]}$ is an embedded three-dimensional sub-manifold-with-boundary that is foliated by the tori $\check{\mathbf{T}}_{0,-\mathfrak{n}}$. Its past boundary, denoted by $\partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$, is the torus $\check{\mathbf{T}}_{0,0}$, a set that we have also been referring to as the crease.

We next highlight that the crease plays a distinguished role in the shock development problem described in Sect. 1.9.12. Once that problem is solved, in the weak solution, the hypersurface of discontinuity $\mathcal{K}$ will emanate from the crease; see Fig. 5. In particular, the crease is a crucial component of the "data" for the shock development problem.

1.10.7. *The one-parameter family of rough time functions* $\{{}^{(\mathfrak{n})}\tau\}_{\mathfrak{n}\in[0,\mathfrak{n}_0]}$. We follow the solution up to the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]} = \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \check{\mathbf{T}}_{0,-\mathfrak{n}}$ by constructing a one-parameter family of time functions $\{{}^{(\mathfrak{n})}\tau\}_{\mathfrak{n}\in[0,\mathfrak{n}_0]}$ and controlling the solution on the near-zero level-sets of each ${}^{(\mathfrak{n})}\tau$. The ${}^{(\mathfrak{n})}\tau$ are constructed such that their range is $[\tau_0, 0]$ for some small parameter $\tau_0 < 0$ and such that $\check{\mathbf{T}}_{0,-\mathfrak{n}} \subset \{{}^{(\mathfrak{n})}\tau = 0\}$. We construct ${}^{(\mathfrak{n})}\tau$ by setting it equal to μ on the hypersurface

$\{\breve{X}\mu = -\mathfrak{n}\}$ and then transporting it along the flow of a well-constructed vectorfield that is transversal to $\{\breve{X}\mu = -\mathfrak{n}\}$. Our construction ensures that the level-sets of ${}^{(\mathfrak{n})}\tau$ are $\mathbf{g}$-spacelike in the region of classical existence, that they are tangent to $\mathcal{B}^{[0,\mathfrak{n}_0]}$, and that they lie below $\mathcal{B}^{[0,\mathfrak{n}_0]}$. In Fig. 4B, we denote the hypersurface $\{\breve{X}\mu = -\mathfrak{n}\}$ by $\breve{\mathbb{X}}_{-\mathfrak{n}}$ and we denote the well-constructed transversal vectorfield by ${}^{(\mathfrak{n})}\breve{W}$. See Sect. 4 for the details on the construction of ${}^{(\mathfrak{n})}\tau$.

A key feature of our construction is that for $\tau \in [\tau_0, 0]$, we have:

$$\min_{\{{}^{(\mathfrak{n})}\tau=\tau\}} \mu = -\tau, \tag{1.13}$$

and the minimum value is achieved precisely along the set $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}} = \{{}^{(\mathfrak{n})}\tau = \tau\} \cap \{\breve{X}\mu = -\mathfrak{n}\}$. Thus, for $\mathfrak{m} \in [0, -\tau_0]$, we have $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} = \{\mu = \mathfrak{m}\} \cap \{\breve{X}\mu = -\mathfrak{n}\}$. Our high order energy estimates feature crucial factors of $1/\mu$, and by (1.13), in adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$, such factors are bounded in magnitude on $\{{}^{(\mathfrak{n})}\tau = \tau\}$ by $\leq \frac{1}{|\tau|}$. Thus, our energy estimates on the level-sets of ${}^{(\mathfrak{n})}\tau$ feature difficult factors of $\frac{1}{|\tau|}$, which leads to singular estimates at the high derivative levels. While related difficulties were present in all the works that we cited above on shock formation without symmetry assumptions, the estimate (1.13) allows for a simplified approach to handling the degeneracy; it directly connects the degeneracy to a coordinate function. We refer to Sect. 29.7.1 for our detailed analysis of the factors of $\frac{1}{|\tau|}$ and the way they affect our energy estimates, and to Sect. 1.10.12 for an overview of these estimates.

We emphasize the following key point:

> The rough time functions ${}^{(\mathfrak{n})}\tau$ are precisely adapted to the structure of the singularity and, in particular, are not more regular than μ. Since it turns out that μ is one degree less differentiable[31] than the fluid variables $\vec{\Psi}$, we cannot use commutation vectorfields adapted to the level-sets of ${}^{(\mathfrak{n})}\tau$ to control the solution up to top-order; that approach would lead to derivative loss. For this reason, as in other works on shock formation, we use the eikonal function u to construct[32] commutation vectorfields that allow us to control the solution.

Finally, we point out that it is not too difficult to construct the ${}^{(\mathfrak{n})}\tau$; the difficult part of the analysis is controlling the fluid solution and the acoustical geometry all the way up the hypersurface $\{{}^{(\mathfrak{n})}\tau = 0\}$.

1.10.8. *The bootstrap assumptions.* To prove our main results, we fix $\mathfrak{n} \in [0, \mathfrak{n}_0]$ and make an elaborate set of bootstrap assumptions on the slab $[\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2] \times \mathbb{T}^2$ corresponding to the adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$ (the corresponding region in geometric coordinate space is denoted by ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,U_2]}$ in the bulk of the paper). More precisely, on the slab, our bootstrap assumptions describe the following:

- The behavior of μ, including the key property of transversal convexity; see Sect. 12.2.1.
- The rough time function ${}^{(\mathfrak{n})}\tau$; see Sect. 12.2.2.
- The properties of various change of variables maps; see Sect. 12.2.3.
- The structure and locations of various embedded sub-manifolds; see Sect. 12.2.4.
- The size of the Cartesian coordinates t and x^1 on the slab; see Sect. 12.2.5.
- The "soft" regularity properties of various quantities on the closure of the slab; see Sect. 12.2.6.
- The L^∞-size of the fluid variables and their $\mathcal{P}_u$-tangential derivatives up to mid-order; see Sect. 12.3.1.
- The L^∞-size of the transversal and mixed tangential-transversal derivatives of the fluid variables and the acoustic geometry; see Sect. 12.3.2.
- The size of our L^2-type energies and null-fluxes for the wave-variables $\vec{\Psi}$ up to top-order; see Sect. 24.3.

Our primary analytic tasks in the paper are to derive strict improvements of the bootstrap assumptions by making suitable assumptions on the data. Once this has been accomplished, a standard continuity argument, provided by Prop. 31.2 and the proof of Theorem 31.1, implies that the solution exists classically on the maximal slab $[\tau_0, 0) \times [-U_1, U_2] \times \mathbb{T}^2$ and satisfies the bootstrap assumptions there. Hence, Theorem 31.1 yields the main results at fixed $\mathfrak{n}$, which in turn form the main ingredients in the proof of the central theorem on the singular boundary, Theorem 34.1. In Sect. 12.5, we provide a road map, indicating the spots in the article where we derive improvements of the bootstrap assumptions.

[31] In particular, in view of the transport equation (3.44), we see that μ cannot be more regular than the source term $\breve{X}\vec{\Psi}$. This is optimal, as can already be seen in plane-symmetry via the explicit formula (A.24).

[32] It does not seem possible to adequately control the solution by commuting the equations with the standard Cartesian partial derivatives ∂_α; the Cartesian vectorfields are not adapted to the singularity, and that approach would lead to singular error terms that we have no obvious means to control.

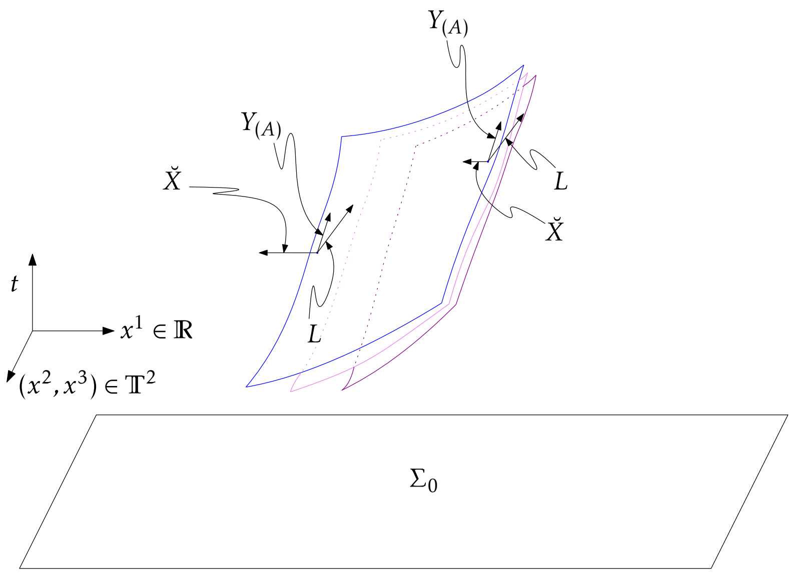


Figure 10. The commutation vectorfields in Cartesian coordinates

1.10.9. *Regular estimates with respect to the geometric coordinates on the rough foliations.* A key idea of the proof, going back to [24], is to prove that the solution remains rather smooth relative to the geometric coordinates (t, u, x^2, x^3). We implement this by following the approach of [24, 50, 52, 73] and using the eikonal function to construct a set of *commutation vectorfields*:

$$\mathscr{Z} \stackrel{\text{def}}{=} \{L, \check{X}, Y_{(2)}, Y_{(3)}\} \tag{1.14}$$

that are adapted to the characteristics; see Fig. 10 for a picture of these vectorfields (with one spatial dimension suppressed), and see Sect. 3 for our detailed construction of the elements of $\mathscr{Z}$. In (1.14) and throughout, L is a $\mathbf{g}$-null vectorfield (i.e., $\mathbf{g}(L, L) = 0$) that is proportional to the (geodesic) gradient vectorfield $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta$, normalized by $Lt = 1$, and tangent to the characteristics $\mathcal{P}_u$. $Y_{(2)}, Y_{(3)}$ are geometric replacements for ∂_2 and ∂_3 that span the tangent space of the *acoustic tori* $\ell_{t,u} \stackrel{\text{def}}{=} \mathcal{P}_u \cap \Sigma_t$. The vectorfields $\{L, Y_{(2)}, Y_{(3)}\}$ span the tangent space of $\mathcal{P}_u$ at each of its points. $\check{X}$ is tangent to the flat Cartesian hypersurface Σ_t, $\mathbf{g}$-orthogonal to $\ell_{t,u}$, and normalized[33] by $\check{X}u = 1$. Note that $\check{X}$ is transversal to the characteristics. This is important because all the degeneracies in the problem of shock formation are tied to the behavior of transversal derivatives. We also note that, in accordance with the discussion in Sect.,1.10.5, we have $\check{X} = \upmu X$, where $\mathbf{g}(X, X) = 1$; see Lemma 3.9.

As in [52], we use the full set of commutators $\mathcal{Z}$ to derive L^∞ and Hölder estimates for solutions, while for deriving energy estimates, we only need to use the following $\mathcal{P}_u$-tangent subset:

$$\mathscr{P} \stackrel{\text{def}}{=} \{L, Y_{(2)}, Y_{(3)}\}. \tag{1.15}$$

In particular, our results show that, based on our assumptions on the data, the up-to-fourth order derivatives of $\vec{\Psi}$, Ω, and S with respect to the elements of $\mathcal{Z}$ are bounded in L^∞. This is equivalent to the L^∞-boundedness of the up-to-fourth order derivatives of these quantities with respect to the geometric coordinate partial derivatives. Similar results hold for the up-to-third order derivatives of $\mathcal{C}$ and $\mathcal{D}$, and due to our assumptions on the data, all fluid variables enjoy additional regularity in directions tangent to the characteristics $\mathcal{P}_u$. Similar results – though at different regularity levels – also hold for the rough time functions ${}^{(\mathfrak{n})}\tau$ and various embeddings of various sub-manifolds into geometric or Cartesian coordinate space, such as the hypersurfaces $\check{\mathbf{X}}_{-\mathfrak{n}} \stackrel{\text{def}}{=} \{\check{X}\upmu = -\mathfrak{n}\}$ and the $\upmu$-*adapted tori* $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} \stackrel{\text{def}}{=} \{\upmu = \mathfrak{m}\} \cap \{\check{X}\upmu = -\mathfrak{n}\}$.

[33]In this paper, this normalization condition is equivalent to $\mathbf{g}(\check{X}, \check{X}) = \upmu^2$.

To derive the L^∞ and Hölder estimates, we make "fundamental" L^∞ bootstrap assumptions for the fluid wave-variables and their $\mathscr{P}$-derivatives up to mid-order, as we described in Sect. 1.10.8. By combining our fundamental bootstrap assumptions with some auxiliary ones, we commute all relevant equations with the elements of $\mathscr{Z}$ and treat them as transport equations with derivative-losing source terms. This allows us to derive L^∞ and Hölder estimates for the solution's transversal and mixed transversal-tangential derivatives and to control the embeddings mentioned in the previous paragraph. We carry out this analysis in Sects.14–17. In Sect. 18, we derive related, but much sharper, pointwise estimates for μ at the low derivative levels, which in particular yield the crucial estimate (1.13). Near the end of the paper, we use our energy estimates and Sobolev embedding to improve the fundamental bootstrap assumptions; see Sect. 30.

Finally, we will briefly discuss the regularity of the Cartesian components of the elements of $\mathscr{Z}$. They have regularity at the schematic level ∂u, and it turns out that by using renormalizations (which we refer to a "modified quantities") and elliptic estimates on co-dimension 2 hypersurfaces (specifically, the *rough tori* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u} \overset{\text{def}}{=} \{{}^{(\mathfrak{n})}\tau = \tau\} \cap \mathcal{P}_u$), which are techniques that originated in [24, 26, 45], we can show that the elements of $\mathscr{Z}$ have just enough regularity to allow us to derive energy estimates up to top-order. We construct the modified quantities in Sect. 19, we derive the elliptic estimates in Sect. 28, and we derive the "final" L^2 estimates involving these quantities Sect. 29.

1.10.10. *Smooth geometry versus rough geometry and elliptic estimates for* χ. Although regularity considerations force us to commute the equations with elements of $\mathscr{Z}$ and $\mathscr{P}$ (e.g., the adapted rough coordinate partial derivative vectorfields do not have sufficient regularity for commuting the equations up to top-order), in order to detect the precise structure of the singular boundary, we must derive estimates on the rough hypersurfaces $\{{}^{(\mathfrak{n})}\tau = \tau\}$, the characteristics $\mathcal{P}_u$, and the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u} \overset{\text{def}}{=} \{{}^{(\mathfrak{n})}\tau = \tau\} \cap \mathcal{P}_u$. For this reason, throughout the analysis, we have to control the geometry of these surfaces (e.g., the Gauss curvature of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$) by quantitatively relating it to the elements of $\mathscr{Z}$ and $\mathscr{P}$ and their derivatives. A particularly noteworthy manifestation of this issue is the following: we must control the top-order derivatives of the null second fundamental form $\chi = \frac{1}{2}\not{\mathcal{L}}_L \not{g}$, where $\not{g}$ is the Riemannian metric induced by $\mathbf{g}$ on $\ell_{t,u} = \Sigma_t \cap \mathcal{P}_u$. Here, $\not{\mathcal{L}}_L \not{g}$ denotes Lie differentiation with respect to L followed by $\mathbf{g}$-orthogonal projection onto $\ell_{t,u}$. Note that the "acoustic torus" $\ell_{t,u}$ is *not* adapted (or necessarily fully contained in) the rough spacetime regions under study. Even though χ is $\ell_{t,u}$-tangent, to avoid derivative loss in the top-order estimates on the rough spacetime regions, we have to derive elliptic estimates for χ on the *rough tori* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. To close these elliptic estimates, we must suitably relate χ to a tensor on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and control the Gauss curvature of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ (viewed as a subset of spacetime equipped with the metric $\mathbf{g}$), all while being mindful about crucial factors of μ, whose vanishing signifies the shock singularity. This delicate analysis is located in Sect. 28, where we construct two distinct frames on $\mathcal{P}_u$, one adapted to the acoustic tori $\ell_{t,u}$ and one adapted to the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, and we control the relationship between the two frames as a key step in obtaining the elliptic estimates.

1.10.11. *Derivative-quadratic terms and null forms.* As in many of the aforementioned works on shock formation, in the present paper, we crucially rely on the fact that all the derivative-quadratic inhomogeneous terms in the system (1.11a)–(1.11d) are null forms relative to $\mathbf{g}$. From the point of view of analysis, the crucial point is that when expanded relative to (say) the commutator frame (1.14), the μ-weighted[34] null forms satisfy, schematically, $\mu\mathfrak{Q}(\partial\vec{\Psi}, \partial\vec{\Psi}) = \breve{X}\vec{\Psi} \cdot P\vec{\Psi} + \mu P\vec{\Psi} \cdot P\vec{\Psi}$, where P schematically denotes elements of the $\mathcal{P}_u$-tangent subset (1.15); see Lemma 9.3 for a more detailed statement concerning the precise null forms we encounter in our analysis. In particular, in the expansion of $\mu\mathfrak{Q}(\partial\vec{\Psi}, \partial\vec{\Psi})$, terms proportional to $\breve{X}\vec{\Psi} \cdot \breve{X}\vec{\Psi}$ are completely absent. This is crucial because, if present, signature considerations would imply that such terms would have to be accompanied by a factor of μ^{-1} (i.e., the term would be proportional to $\mu^{-1}\breve{X}\vec{\Psi} \cdot \breve{X}\vec{\Psi}$), and the factor of μ^{-1} (which blows up as $\mu \to 0$) would have obstructed our philosophy of deriving regular estimates for the solution's $\mathcal{Z}$-derivatives. Similar remarks apply to all the other null forms in (1.11a)–(1.11d).

The upshot is the following: all of the derivative-quadratic terms in the system (1.11a)–(1.11d) that drive the formation of the shock are "hidden" in the definition of the covariant wave operator $\square_{\mathbf{g}(\vec{\Psi})}$ on LHS (1.11a) and become visible when one expands $\square_{\mathbf{g}(\vec{\Psi})}\Psi$ relative to the Cartesian coordinates (say, via the formula (2.20)). The virtue of working with $\square_{\mathbf{g}(\vec{\Psi})}$ is that there is an advanced machinery for deriving geometric commutator and multiplier energy estimates for such operators.

[34] In our analysis, we introduce a μ weight into various equations and estimates.

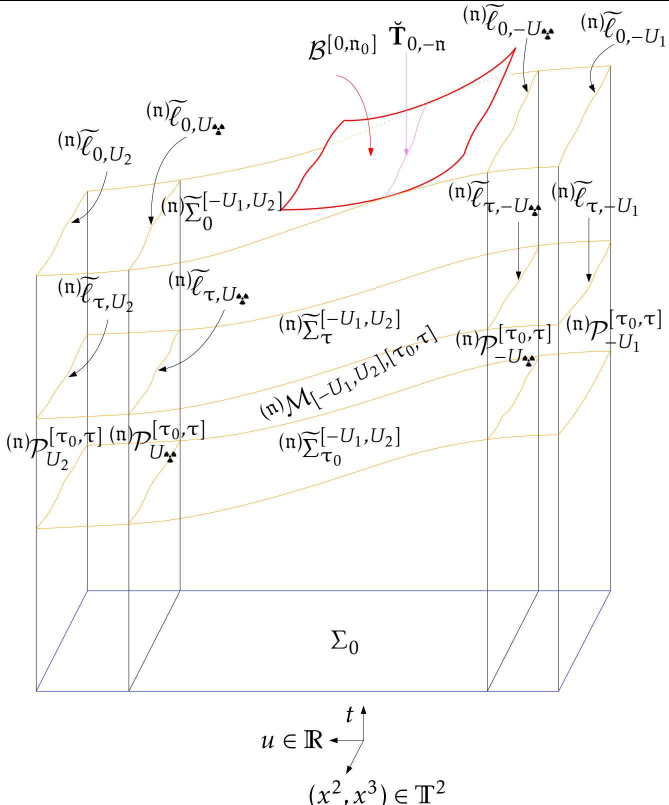


Figure 11. Regions in geometric coordinate space on which we derive estimates

1.10.12. *Geometric wave and transport energy estimates on the rough foliations, with singular high order behavior.* To derive L^2 estimates up to top-order, we commute equations[35] (1.11a)–(1.11d) with the elements of $\mathscr{P}$ (see (1.15)) and derive energy and elliptic estimates on regions of the form $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}$, which are bounded by rough hypersurfaces (i.e., level sets of the rough time function) on the top and bottom and null hypersurfaces on the sides; see Fig 11.

Our elliptic estimates, which are localized in spacetime, come in two flavors: those for the acoustic geometry, as we mentioned in Sect. 1.10.10, and "elliptic-hyperbolic" estimates for some of the fluid variables. The localized elliptic-hyperbolic estimates for the fluid variables are a difficult new feature of the present work, and we discuss them in Sect. 1.10.13; here we discuss the energy estimates for the wave equations (1.11a), which in reality are coupled to the elliptic estimates. More precisely, for solutions Ψ to (1.11a), we construct coercive energies $\mathbb{E}[\Psi](\tau,u)$ on rough hypersurface portions $^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}$ and null fluxes $\mathbb{F}[\Psi](\tau,u)$ on null hypersurface portions $^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau)}$ (see Def. 4.11 for definitions of these portions), and we derive suitable energy identities by using a well-known framework based on the energy-momentum tensor for wave equations and the well-chosen multiplier vectorfield $\breve{T} = (1+2\mu)L + 2\breve{X}$; see Prop. 20.9. We note in passing that we derive similar energy estimates for the transport equations (1.11b)–(1.11c), and we will not discuss them in this introduction, aside from mentioning that the $\mathbf{g}$-timelike property $\mathbf{g}(\mathbf{B},\mathbf{B}) = -1$ (see Lemma 3.9) is crucial for those estimates, as it ensures that $\mathbf{B}$ *is transversal to the characteristics.*

As we mentioned in Sect. 1.10.9, to control the top-order derivatives of the acoustic geometry without derivative loss, we rely on renormalizations, which we implement by constructing appropriate "modified quantities," originating in [24, 26, 45, 69]; see Sect. 19. As in other works on shock waves without symmetry, this renormalization leads to top-order energy identities for the wave-variables Ψ involving singular terms. Our rough foliations are constructed so that the singular terms are bounded by factors of $\frac{1}{|\tau|}$, and the resulting top-order energy-flux inequalities can be caricatured as

[35] More precisely, to avoid uncontrollable error terms, we weight these equations by a factor of μ before commuting.

follows:

$$\begin{aligned} &\mathbb{E}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)+\mathbb{F}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)+\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)\\ &\leq C\mathring{\epsilon}^2+\boxed{A}\int_{\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{E}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau',u)\,\mathrm{d}\tau'+\cdots, \end{aligned} \tag{1.16}$$

where LHS (1.16) is the sum of the top-order energy $\mathbb{E}$, the null flux $\mathbb{F}$, and the following *coercive* spacetime integral (see Defs. 20.7–20.8 for the precise definitions), where $\not{g}$ is the Riemannian metric induced by $\mathbf{g}$ on $\ell_{t,u}=\Sigma_t\cap\mathcal{P}_u$:

$$\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)\overset{\mathrm{def}}{=}-\int_{(\tau',u',x^2,x^3)\in[\tau_0,\tau)\times[-U_1,u]\times\mathbb{T}^2}\mathbf{1}_{[-U_{\clubsuit},U_{\clubsuit}]}(u')({}^{(\mathfrak{n})}\widetilde{L}\mu)|\not{d}\mathcal{P}^{N_{\mathrm{top}}}\Psi|_{\not{g}}^2\,\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}u'\mathrm{d}\tau'. \tag{1.17}$$

In (1.16), $\mathring{\epsilon}\geq 0$ is the small size of the perturbation of the initial data away from a background plane-symmetric solution, $\tau_0<0$ denotes the initial rough time (at which the data perturbation is assumed to be small), and $\tau\in[\tau_0,0]$. Moreover, $A>0$ is a *universal constant* (which we have placed in a box to highlight its importance) that is independent of the equation of state, and $\cdots$ denotes similar[36] or easier error terms or terms that can be handled with the help of the elliptic-hyperbolic estimates described in Sect. 1.10.13. In the spacetime integral $\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)$ on LHS (1.16), $|\not{d}\mathcal{P}^{N_{\mathrm{top}}}\Psi|_{\not{g}}^2$ denotes the square norm of the derivatives of $\mathcal{P}^{N_{\mathrm{top}}}\Psi$ in directions tangent to the acoustic tori $\ell_{t',u'}=\Sigma_{t'}\cap\mathcal{P}_{u'}$, while $\mathbf{1}_{-U_{\clubsuit},U_{\clubsuit}]}(u')$ is the characteristic function of a small interval $[-U_{\clubsuit},U_{\clubsuit}]$ of u-values near the crease, and the factor ${}^{(\mathfrak{n})}\widetilde{L}\mu$ is a null derivative of μ that is *quantitatively negative* in this interval. Hence, in view of the minus sign in (1.17), we see that *the overall sign of the spacetime integral on LHS* (1.16) *is positive*, and this integral is crucially used in the proof to absorb some of the error terms in "$\cdots$" on RHS (1.16). We also note in passing that some of the error terms in "$\cdots$" are non-singular terms that can be handled with the help of the null fluxes $\mathbb{F}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)$, and that some of these terms, when treated with Grönwall's inequality in u, allow for the possibility of exponential growth in u, which is permissible within the scope of our approach (u is confined to a compact set). The key point is that by applying Grönwall's inequality to (1.16), we deduce (ignoring the "$\cdots$" terms here) the following *singular* (as $\tau\uparrow 0$) top-order energy estimate:

$$\mathbb{E}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)+\mathbb{F}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)+\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau,u)\lesssim\mathring{\epsilon}^2|\tau|^{-A}. \tag{1.18}$$

Thus, even though we have used geometrically defined commutators to try to turn the problem of shock formation into a "regular" problem, a remnant of the singularity survives[37] at the top-derivative level, i.e, our "unfolding" of the characteristics does not completely "hide" the singularity. This calls into question the basic philosophy of the approach.

However, two crucial structural features of the problem rescue the situation. First, the universal constant A – and hence the singular factor $|\tau|^{-A}$ on RHS (1.18) – is *independent* of N_{top}. This means that we can choose N_{top} to be large without increasing the top-order singularity strength. Second, below top-order, one can avoid using the renormalization procedure to control the acoustic geometry. This leads to the loss of one derivative in the energy estimate hierarchy, but the loss of one derivative is permissible below top-order. The price one pays is that this procedure couples the below-top-order estimates to the singular top-order ones. At one derivative below the top-order, the corresponding energy-null flux inequality can be caricatured[38] as follows:

$$\begin{aligned} &\mathbb{E}[\mathcal{P}^{N_{\mathrm{top}}-1}\Psi](\tau,u)+\mathbb{F}[\mathcal{P}^{N_{\mathrm{top}}-1}\Psi](\tau,u)+\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}-1}\Psi](\tau,u)\\ &\leq C\mathring{\epsilon}^2+C\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{E}^{1/2}[\mathcal{P}^{N_{\mathrm{top}}-1}\Psi](\tau',u)\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{E}^{1/2}[\mathcal{P}^{N_{\mathrm{top}}}\Psi](\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau'+\cdots. \end{aligned} \tag{1.19}$$

By applying Grönwall's inequality to (1.19) and accounting for the singular top-order behavior stated in (1.18), we deduce that:

$$\mathbb{E}[\mathcal{P}^{N_{\mathrm{top}}-1}\Psi](\tau,u)+\mathbb{F}[\mathcal{P}^{N_{\mathrm{top}}-1}\Psi](\tau,u)+\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}-1}\Psi](\tau,u)\lesssim\mathring{\epsilon}^2|\tau|^{-(A-2)}. \tag{1.20}$$

[36]In reality, in the energy estimates, there are other difficult error terms that have to be handled with integration by parts with respect to $\frac{1}{L^{(\mathfrak{n})}\tau}L$, and these lead to difficult singular boundary terms that we control in Lemma 29.12.

[37]These singular top-order estimates are solely a product of our need to derive energy estimates, and in particular, this difficulty does not arise in $1D$, where one can use transport estimates to close the problem.

[38]In reality, the top-order and below-top-order energy estimates are highly coupled and have to be derived simultaneously through an intricate version of Grönwall's lemma, which we provide in Sect. 29.7.1.

(1.20) shows that if we descend one derivative level below top-order, then the energy becomes less singular by a factor of $|\tau|^2$. As in [24, 50, 52, 69], one can continue the descent, proving that:

$$\mathbb{E}[\mathcal{P}^{N_{\mathrm{top}}-2}\Psi](\tau,u)+\mathbb{F}[\mathcal{P}^{N_{\mathrm{top}}-2}\Psi](\tau,u)+\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}-2}\Psi](\tau,u)\lesssim \mathring{\epsilon}^2|\tau|^{-(A-4)}, \tag{1.21}$$

$\cdots$, and finally arriving at the *non-singular* estimate:

$$\mathbb{E}[\mathcal{P}^{N_{\mathrm{top}}-\frac{A}{2}}\Psi](\tau,u)+\mathbb{F}[\mathcal{P}^{N_{\mathrm{top}}-\frac{A}{2}}\Psi](\tau,u)+\mathbb{K}[\mathcal{P}^{N_{\mathrm{top}}-\frac{A}{2}}\Psi](\tau,u)\lesssim \mathring{\epsilon}^2. \tag{1.22}$$

The crucial non-singular energy estimate (1.22) is what allows us to show, via Sobolev embedding, that the solution is bounded with respect to the geometric coordinates (t,u,x^2,x^3) at derivative levels $\approx N_{\mathrm{top}}-\frac{A}{2}$ and below.

Finally, we note that a related energy estimate hierarchy holds for the other fluid variables (including Ω, S, $\mathcal{C}$, $\mathcal{D}$) and the acoustic geometry (including quantities such as μ and χ), and that in practice, all these estimates are coupled (though in certain spots in our bootstrap argument, we exploit weak coupling, which allows us to derive some estimates before deriving others). We refer to Sect. 24 for detailed statements of the full hierarchy of energy estimates.

1.10.13. *Localized elliptic-hyperbolic fluid variable estimates via the characteristic current.* As we mentioned earlier, to close the top-order energy estimates for the vorticity and entropy on the rough domains ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u']}=\{\tau_0\leq {}^{(\mathfrak{n})}\tau\leq \tau\}\cap\{-U_1\leq u\leq u'\}$, we cannot rely exclusively on the transport equation (1.11b); that approach would lead to the loss of a derivative in our scheme. Instead, we use the full structure of the transport-div-curl system (1.11b)–(1.11d). We also have to show that the corresponding estimates, which are singular, are compatible with the blowup-rates of the high order wave and transport energies, described in Sect. 1.10.12. We used a version of this approach in [52], where we followed the solution precisely to the constant-Cartesian-time hypersurface of first blowup. The elliptic estimates in [52] were much simpler because we derived them only on the flat hypersurfaces Σ_t, whose geometry is trivial, and because we did not try to derive the localized structure of the singular boundary; this allowed us to close the proof by deriving the elliptic estimates only across all of space, thereby (also exploiting the assumption of compactly supported data in [52]) avoiding the difficult spatial boundary terms that we encounter in the present work.

To derive the desired estimates, we use the technology of [4], which allows one to combine the Euclidean div-curl system (1.11c)–(1.11d) with the transport equations (1.11b) to derive "elliptic-hyperbolic identities." These identities yield spacetime integrals that provide – up to error integrals that must be controlled – a sufficient amount of Sobolev regularity on *any* spacetime region that is globally hyperbolic with respect to $\mathbf{g}$ (in particular, on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u']}$). See Prop. 21.14 for the precise elliptic-hyperbolic identity that we use to close the top-order estimates.

Compared to [4], there are three new aspects of our elliptic-hyperbolic identities and estimates:

- The elliptic-hyperbolic identities depend, roughly, on certain curvature components of the boundaries of the domain. Some of these components become very singular as $\mu\to 0$, and we need to ensure that the singularity strength is compatible with the blowup-rates of the high order wave and transport energies. In Prop. 23.4, we provide pointwise estimates guaranteeing that indeed, all of the error terms in the identities are controllable within the scope of our approach.
- We derive a new version of the identities from [4] based on applying the divergence theorem to a well-constructed *characteristic current* $\mathscr{J}^{\alpha}$, which is tangent to the characteristic hypersurfaces $\mathcal{P}_u$; see Def. 21.10. Because of the $\mathcal{P}_u$-tangency, when we apply the divergence theorem to $\mathscr{J}^{\alpha}$ on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u']}$, *there are no boundary integrals along the lateral boundaries* $\mathcal{P}_u$. This allows us to completely avoid error integrals on $\mathcal{P}_u$ that feature the top-order derivatives of μ; the point is that it is not possible to derive top-order L^2 estimates for μ on $\mathcal{P}_u$ because it satisfies a transport equation $L\mu=\cdots$, where L is *tangent* to the $\mathcal{P}_u$. We also highlight that, unlike the currents in [4], our characteristic current here does not involve the future-directed normal ${}^{(\mathfrak{n})}\hat{N}$ to the constant-rough-time hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u']}$. Avoiding ${}^{(\mathfrak{n})}\hat{N}$-dependent terms is advantageous because the derivatives of ${}^{(\mathfrak{n})}\hat{N}$ become very singular as $\mu\to 0$, and it is not at all clear that such terms would have been compatible the blowup-rates of the high order wave and transport energies.
- Our first attempt at deriving elliptic-hyperbolic identities yields error integrals along the top boundary portion ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u']}$ that contain dangerous terms with *insufficient regularity*. It turns out that the dangerous terms can be handled via a subtle and technical argument, captured in divergence-form in Lemma 21.13. Roughly, by integrating by parts on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u']}$ and using the structure of the dangerous terms and the compressible Euler equations, we can replace the dangerous terms with controllable ones. However, this procedure leads to co-dimension-two boundary integrals on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u'}$ and ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,-U_1}$ which, from the point of view of regularity, have the status of top-order terms; the presence of these co-dimension-two integrals is a major difference from [52], where

we integrated across all of space. The key point is that, as in [4], the rough tori boundary integrals are either controlled by the data or *enter with a good sign*, which is crucial for closing the top-order regularity theory; see the tori integrals in the identity (21.63).

1.11. **Outline of the remainder of the paper.**

- In Sect. 2, we define the fluid variables that we use in our analysis and recall the geometric formulation of the flow derived in [72].
- In Sects. 3–6, we derive basic properties and identities (not yet estimates) tied to the eikonal function u, the rough time function ${}^{(\mathfrak{n})}\tau$, the corresponding geometries, and changes of variables between various coordinate systems and vectorfields.
- In Sect. 7, we construct the "ingoing" $\mathbf{g}$-null vectorfield $\underline{L}$, which is transversal to the characteristics $\mathcal{P}_u$ and complements the $\mathbf{g}$-null vectorfield L, which generates the $\mathcal{P}_u$. We use $\underline{L}$ in Sect. 21, when we derive the "elliptic-hyperbolic" identities that we will use to control the top-order derivatives of Ω and S. We clarify that, while we use $\underline{L}$ to construct various tensors and derive identities, we do not use it to derive estimates, i.e., we do not "integrate in the $\underline{L}$ direction."
- In Sect. 8, we define various norms, area forms, and volume forms. We also introduce notation for various strings of commutation vectorfields.
- In Sect. 9, we introduce schematic notation and derive various identities in schematic form. These will be used throughout the remainder of the paper.
- In Sects. 10–11, we list various parameters corresponding to the solutions under study and state our assumptions on the initial data.
- In Sect. 12, we state all our bootstrap assumptions – except for the energy bootstrap assumptions.
- In Sect. 13, we use the bootstrap assumptions to derive preliminary pointwise, commutator, and differential operator comparison estimates.
- In Sect. 14, we use the bootstrap assumptions to analyze the data-hypersurface $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}})}$ for the rough time function ${}^{(\mathfrak{n})}\tau$, which solves the transport ${}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\tau = 0$ with data prescribed on $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}})}$. We also derive basic properties of the flow map of ${}^{(\mathfrak{n})}\breve{W}$.
- In Sect. 15, we use the bootstrap assumptions to derive estimates for ${}^{(\mathfrak{n})}\tau$. We also show that various quantities extend to the closure of the bootstrap region as elements of various Hölder spaces. Finally, we study the properties of the map ${}^{(\mathfrak{n})}\Phi(\tau,u,x^2,x^3) = (\mu,\breve{X}\mu,x^2,x^3)$, which is important for understanding the structure of the singular boundary.
- In Sect. 16, we use the bootstrap assumptions to control the flow map of the $\mathbf{g}$-null vectorfield ${}^{(\mathfrak{n})}\widetilde{L} = \frac{1}{L^{(\mathfrak{n})}\tau}L$, which is the principal operator in many of the transport equations that we will later study.
- In Sect. 17, we derive L^∞ estimates that yield improvements of many of our quantitative bootstrap assumptions.
- In Sect. 18, we derive sharp estimates for μ. These are crucial for the energy estimates and for understanding the structure of the singular boundary. We also study the homeomorphism and diffeomorphism properties of the change of variables map $(t,u,x^2,x^3) \to (t,x^1,x^2,x^3)$. Finally, we derive pointwise estimates for various geometric quantities that are tied to the rough acoustic geometry.
- In Sect. 19, we construct the modified quantities that we use to control the acoustic geometry in L^2 without derivative loss.
- In Sect.20, we construct some basic ingredients needed for the hyperbolic energy estimates. In particular, we define energies and null-fluxes, derive energy–null-flux identities for solutions to wave equations and transport equations, and exhibit the coerciveness of the energies and null fluxes.
- In Sect. 21, we derive the "elliptic-hyperbolic" identities that we will use to control the top-order derivatives of Ω and S.
- In Sect. 22, we commute the wave equations satisfied by the wave-variables $\vec{\Psi} = (\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s)$ up to top-order and derive pointwise estimates for the inhomogeneous terms. These are a preliminary ingredient in our L^2 analysis of the inhomogeneous terms.
- In Sect. 23, we provide an analog of Sect. 22 for the transport-variables. That is, we commute the transport equations satisfied by Ω, S, $\mathcal{C}$, and $\mathcal{D}$ up to top-order and derive pointwise estimates for the inhomogeneous terms. The estimates in this section are preliminary ingredients for our derivation of energy and elliptic estimates.

- In Sect. 24, we state all of our a priori energy estimates. We also state bootstrap assumptions for the energies of the "wave-variables" $\vec{\Psi} \stackrel{\text{def}}{=} (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$. The proof of the energy estimates occupies a substantial portion of the remainder of the paper, all the way through Sect. 29.
- In Sect. 25, we derive preliminary L^2 estimates for the below-top-order derivatives of the eikonal function quantities μ, L^i, χ, and $\mathrm{tr}_{\not{g}}\chi$. We also derive preliminary L^2 for $\vec{\Psi}$ that lose one derivative.
- In Sect. 26, we use the wave energy bootstrap assumptions to derive below-top-order energy estimates for Ω, S, $\mathcal{C}$, and $\mathcal{D}$.
- In Sect. 27, we use the elliptic-hyperbolic identities of Sect. 21, the wave energy bootstrap assumptions, and the below-top-order energy estimates of Sect. 26 to derive the top-order "elliptic-hyperbolic" energy estimates Ω, S, $\mathcal{C}$, and $\mathcal{D}$.
- In Sect. 28, we derive general elliptic estimates on the rough tori, which we will use in Sect. 29.3 to control the top-order derivatives of the acoustic geometry along the rough foliations.
- In Sect. 29, we derive up-to-top-order energy estimates for the wave-variables and the acoustic geometry. This completes the proof of the energy estimates stated in Sect. 24 and in particular yields a strict improvement of our wave energy bootstrap assumptions.
- In Sect. 30, we use the energy estimates and Sobolev embeddings to derive L^∞ estimates that yield strict improvements of the remaining quantitative bootstrap assumptions. This closes the bootstrap argument and completes our proof of a priori estimates.
- In Sect. 31, we use the a priori estimates and a continuation principle to show that we can extend the solution all the way up to the level-set $\{{}^{(\mathfrak{n})}\tau = 0\}$, which contains the two-dimensional torus $\check{\mathbf{T}}_{0,-\mathfrak{n}}$, which in turn is contained in the singular boundary. We provide these results as Theorem 31.1, which is the first main theorem of the paper. This theorem provides a development of the data containing the portion of the singular boundary that is "accessible" via the foliation of spacetime by the level-sets of ${}^{(\mathfrak{n})}\tau$.
- In Sect. 32, we study the union of the developments as $\mathfrak{n}$ varies, and we define an interesting sub-region, $\mathcal{M}_{\text{Interesting}}$, which contains a portion of the singular boundary, namely $\mathcal{B}^{[0,\mathfrak{n}_0]}$, and its past boundary $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]} = \check{\mathbf{T}}_{0,0}$, the crease. We also construct a new rough time function ${}^{(\text{Interesting})}\tau$, whose level-sets foliate $\mathcal{M}_{\text{Interesting}}$. Finally, we derive various quantitative and qualitative properties of various geometric objects tied to $\mathcal{M}_{\text{Interesting}}$ and ${}^{(\text{Interesting})}\tau$.
- In Sect. 33, we study the homeomorphism and diffeomorphism properties of the change of variables map $\Upsilon(t, u, x^2, x^3) = (t, x^1, x^2, x^3)$ on $\mathcal{M}_{\text{Interesting}}$. We also exhibit the properties of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, i.e., the embedding of the singular boundary in Cartesian coordinate space.
- In Sect. 34, we state and prove Theorem 34.1, which is the main result of the paper. The theorem shows that $\mathcal{M}_{\text{Interesting}}$ contains the portion $\mathcal{B}^{[0,\mathfrak{n}_0]}$ of the singular boundary and the crease, and it gives a detailed description of the solution in the different coordinate systems as well as the change of variables maps. The theorem is essentially a conglomeration of results derived earlier in the paper.

2. Basic setup, compressible Euler flow, and its geometric reformulation

In this section, we first introduce some basic notational conventions and definitions. We then provide a standard first-order quasilinear hyperbolic formulation of compressible Euler flow. Next, we define a series of additional fluid variables and geometric tensors associated to the flow. Finally, we recall the new formulation of the flow derived in [72]. More precisely, we use a slightly modified version of the formulation in [72] that is adapted to the nearly plane-symmetric solutions featured in our main results. The only difference with [72] is that here, we replace the density and velocity component v^1 with our "almost Riemann invariants" $\mathcal{R}_{(+)}$ and $\mathcal{R}_{(-)}$, which are useful for capturing smallness in the regime under study. In total, the new formulation comprises covariant wave equations, which govern the propagation of sound waves, coupled to systems of transport-div-curl systems, which drive the evolution of the vorticity and entropy.

2.1. **Basic notation and conventions.**

The precise definitions of some of the concepts referred to here are provided later in the article.

- (**Cartesian coordinates**) Our analysis takes place on subsets of the spacetime manifolds $\mathbb{R}\times\Sigma$, where $\Sigma \stackrel{\text{def}}{=} \mathbb{R}\times\mathbb{T}^2$ is the spatial manifold. We fix a standard Cartesian coordinate system $\{x^\alpha\}_{\alpha=0,1,2,3}$ on $\mathbb{R}\times\Sigma$, where $t \stackrel{\text{def}}{=} x^0 \in \mathbb{R}$ is the time coordinate and $(x^1,x^2,x^3)\in\Sigma$ are the spatial coordinates (where (x^2,x^3) are standard coordinates[39] on $\mathbb{T}^2$). By a *plane-symmetric* solution, we mean one whose fluid variables are independent of (x^2,x^3) in this coordinate system. We sometimes refer to t as the "Cartesian time function." $\Sigma_{t'} \stackrel{\text{def}}{=} \{(t,x^1,x^2,x^3)\in\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid t=t'\}$ denotes the standard flat hypersurface of constant Cartesian time.

- (**Cartesian coordinate partial derivatives**) We use the notation $\{\partial_\alpha\}_{\alpha=0,1,2,3}$ (or $\partial_t \stackrel{\text{def}}{=} \partial_0$) to denote the Cartesian coordinate partial derivative vectorfields.
- (**Lowercase Greek index conventions**) Lowercase Greek spacetime indices α, β, etc. correspond to the Cartesian coordinate spacetime coordinates and vary over $0,1,2,3$. All lowercase Greek indices are lowered and raised with the acoustical metric $\mathbf{g}$ (see definition (2.15a)) and its inverse $\mathbf{g}^{-1}$, and *not with the Minkowski metric*. Throughout the article, if ξ is a type $\binom{m}{n}$ spacetime tensorfield, then unless we indicate otherwise, in our identities and estimates, $\left\{\xi^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}\right\}_{\alpha_1,\cdots\alpha_m,\beta_1,\cdots,\beta_n=0,1,2,3}$ **denotes its components with respect to the Cartesian coordinates**. This is important because some of our identities and estimates hold only with respect to the Cartesian coordinates.
- (**Lowercase Latin index conventions**) Lowercase Latin spatial indices a, b, etc. correspond to the Cartesian spatial coordinates and vary over $1,2,3$. Much like in the previous point, if ξ is a type $\binom{m}{n}$ Σ_t-tangent tensorfield (see Def. 3.3), then unless we indicate otherwise, in our identities and estimates, $\left\{\xi^{a_1\cdots a_m}_{b_1\cdots b_n}\right\}_{a_1,\cdots,a_m,b_1,\cdots,b_n=1,2,3}$ **denotes its components with respect to the Cartesian spatial coordinates**.
- (**Uppercase Latin index conventions**) Uppercase Latin spatial indices A,B, etc. correspond to the coordinates (x^2,x^3) on $\mathbb{T}^2$ and vary over $2,3$. In particular, if V is a vectorfield, and $A\in\{2,3\}$, then $V^A = V^\alpha\partial_\alpha x^A$, where (x^2,x^3) are the standard Cartesian coordinates on $\mathbb{T}^2$.
- (**Tilded indices**) We use tilded indices such as $\widetilde{\alpha}$ in the same way as their non-tilded counterparts.
- (**Einstein summation**) We use Einstein's summation convention in that repeated indices are summed, e.g., $L^AX^A \stackrel{\text{def}}{=} L^2X^2+L^3X^3$.
- (**Use of "$\cdot$"**) We sometimes use "$\cdot$" to denote the natural contraction between two tensors. For example, if ξ is a spacetime one-form and V is a spacetime vectorfield, then $\xi\cdot V \stackrel{\text{def}}{=} \xi_\alpha V^\alpha$. At other times, we use "$\cdot$" to schematically denote products, e.g., $A_1\cdot A_2\cdot A_3$ is a trilinear form in A_1,A_2,A_3.
- (**Tensor contractions**) If V and W are vectorfields, then $V_W \stackrel{\text{def}}{=} V^\alpha W_\alpha = \mathbf{g}_{\alpha\beta}V^\alpha W^\beta$. If ξ is a one-form and V is a vectorfield, then $\xi_V \stackrel{\text{def}}{=} \xi_\alpha V^\alpha$. We use similar notation when contracting higher-order tensorfields against vectorfields. For example, if ξ is a type $\binom{0}{2}$ tensorfield and V and W are vectorfields, then $\xi_{VW} \stackrel{\text{def}}{=} \xi_{\alpha\beta}V^\alpha W^\beta$.
- (**Commutator of operators**) If Q_1 and Q_2 are two operators, then $[Q_1,Q_2] \stackrel{\text{def}}{=} Q_1Q_2-Q_2Q_1$ denotes their commutator.
- (**Constants**) We establish conventions for constants (such as "C") in Sect. 10.3.

2.2. **Basic differential operators.** In our analysis, we will encounter many kinds of differential operators. Here, we define some basic operators.

Definition 2.1 (Gradient one-form of a scalar function)**.** If f is a scalar function, then $\mathrm{d}f$ denotes the gradient one-form associated to f, e.g., $(\mathrm{d}f)_\alpha \stackrel{\text{def}}{=} \mathrm{d}f\cdot\partial_\alpha = \partial_\alpha f$.

Definition 2.2 (Vectorfield derivative of scalar functions)**.** If V is a vectorfield and f is a scalar function, then $Vf \stackrel{\text{def}}{=} V^\alpha\partial_\alpha f = V\cdot\mathrm{d}f$ denotes the derivative of f in the direction V.

Definition 2.3 (Euclidean divergence and curl)**.** div and curl respectively denote the Euclidean spatial divergence and curl operators. That is, given a Σ_t-tangent vectorfield $V=V^a\partial_a$, we define, relative to the Cartesian spatial coordinates, $\mathrm{div}\,V$ and $\mathrm{curl}\,V$ to be the following scalar function and Σ_t-tangent vectorfield:

$$\mathrm{div}\,V \stackrel{\text{def}}{=} \partial_aV^a, \qquad (\mathrm{curl}\,V)^i \stackrel{\text{def}}{=} \epsilon_{iab}\partial_aV^b, \tag{2.1}$$

where ϵ_{iab} is the fully antisymmetric symbol normalized by $\epsilon_{123}=1$.

2.3. **A first-order formulation involving the logarithmic density.**

[39] While the coordinates x^2,x^3 on $\mathbb{T}^2$ are only locally defined, the corresponding partial derivative vectorfields ∂_2,∂_3 can be extended so as to form a global smooth frame on $\mathbb{T}^2$. Similar remarks apply to the one-forms $\mathrm{d}x^2,\mathrm{d}x^3$ These simple observations are relevant for this paper because when we derive estimates, the coordinate functions x^2,x^3 themselves are never directly relevant; what matters are estimates for the components of various tensorfields with respect to the frame $\{\partial_t,\partial_1,\partial_2,\partial_3\}$ and the basis dual co-frame $\{\mathrm{d}t,\mathrm{d}x^1,\mathrm{d}x^2,\mathrm{d}x^3\}$, which are everywhere smooth.

2.3.1. *The logarithmic density, assumptions on the equation of state, and normalizations.* We find it convenient to work with the logarithmic density featured in the next definition, rather than the density. In the rest of the paper,

$$\overline{\varrho} > 0 \tag{2.2}$$

denotes a fixed constant "background density."

Definition 2.4 (Logarithmic density)**.** We define the *logarithmic density* ρ as follows:

$$\rho \overset{\text{def}}{=} \ln(\varrho/\overline{\varrho}). \tag{2.3}$$

In the rest of the paper, we view the speed of sound c (which is defined in (1.3)) to be a function of (ρ, s). Note that by (1.3) and the chain rule, we have $c(\rho,s) = \sqrt{(\overline{\varrho})^{-1}\exp(-\rho)p;_{\rho}}$, where $p;_{\rho} \overset{\text{def}}{=} \frac{\partial p}{\partial \rho}$ denotes the partial derivative of the equation of state with respect to the logarithmic density at fixed s.

Notation 2.1 (Partial differentiation with respect to state-space variables)**.** In accordance with the above notation, for any scalar function $f = f(\rho, s)$, we use the notation $f_{;\rho} \overset{\text{def}}{=} \frac{\partial f}{\partial \rho}$ to denote the partial derivative of f with respect to the logarithmic density at fixed s. Similarly, we denote the partial derivative of f with respect to s at fixed ρ by $f_{;s} \overset{\text{def}}{=} \frac{\partial f}{\partial s}$. We also write $f_{;\rho;s} \overset{\text{def}}{=} \frac{\partial^2 f}{\partial s \partial \rho}$, and we use similar notation for other higher partial derivatives of f with respect to ρ, s.

To ensure that shocks occur for solutions near static isentropic fluid states with constant density $\overline{\varrho} > 0$, we assume the following non-degeneracy condition:

$$\overline{c}^{-1}\overline{c_{;\rho}} + 1 \neq 0, \tag{2.4}$$

where LHS (2.4) is defined to be the constant obtained by evaluating $c^{-1}c_{;\rho} + 1$ at $\rho = s \equiv 0$. Equation (2.4) ensures that the null condition fails to hold for perturbations of the background solution $\rho = s \equiv 0$; see Sect. 3.13. Our main results hold for all equations of state except for that of a Chaplygin gas, namely $p = C_0 - C_1\exp(-\rho)$, where $C_0 \geq 0$ and $C_1 > 0$ are constants. This equation of state is degenerate in the following sense: $c^{-1}c_{;\rho} + 1 \equiv 0$.

By rescaling Cartesian time if necessary, we can assume the following convenient normalization condition:

$$c(\rho = 0, s = 0) = 1. \tag{2.5}$$

2.3.2. *A first-order formulation involving the logarithmic density.* From definition (2.3), equations (1.1a)–(1.1c), and the chain rule, it follows that relative to the standard Cartesian coordinates on $\mathbb{R} \times \Sigma$, the compressible Euler equations can be expressed as the following system in ρ, v, and s:

$$\mathbf{B}v^i = -c^2\delta^{ia}\partial_a\rho - \exp(-\rho)\frac{p_{;s}}{\overline{\varrho}}\delta^{ia}\partial_a s, \tag{2.6a}$$

$$\mathbf{B}\rho = -\operatorname{div} v, \tag{2.6b}$$

$$\mathbf{B}s = 0. \tag{2.6c}$$

2.4. **The almost Riemann invariants.** To study solutions close to simple isentropic plane-symmetric solutions, we find it convenient to replace ρ and v^1 with a pair of "almost Riemann invariants," denoted by $\mathcal{R}_{(+)}$ and $\mathcal{R}_{(-)}$. In this paper, simple isentropic plane-symmetric solutions are, by definition, such that $\mathcal{R}_{(+)}$ is a function of only (t, x^1) and $\mathcal{R}_{(-)} = s = v^2 = v^3 \equiv 0$.

Definition 2.5 (The almost Riemann invariants)**.** We define the *almost Riemann invariants*[40] *away from symmetry* $\mathcal{R}_{(\pm)}$ as follows:

$$\mathcal{R}_{(\pm)} \overset{\text{def}}{=} v^1 \pm F(\rho, s), \qquad \text{where } F(\rho, s) \overset{\text{def}}{=} \int_0^{\rho} c(\rho', s)\,d\rho'. \tag{2.7}$$

Remark 2.6 (**Clarification on our approach to estimating** ρ **and** v^1)**.** We have introduced $\mathcal{R}_{(\pm)}$ because they are convenient for studying perturbations of simple isentropic plane-waves (for which only $\mathcal{R}_{(+)}$ is non-vanishing); $\mathcal{R}_{(\pm)}$ allow us to capture various kinds of smallness of the perturbations. It is well-known that for isentropic plane-symmetric solutions, one can use $\{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}\}$ as the unknowns in place of $\{\rho, v^1\}$; see Appendix A. Away from symmetry, a similar remark also holds for our almost Riemann invariants, provided we take into account the entropy. Specifically, from

[40]Compare $\mathcal{R}_{(\pm)}$ with the true Riemann invariants for the plane-symmetric solutions given by (A.1).

(2.5) and definition (2.7), it follows that $v^1 = \frac{1}{2}(\mathcal{R}_{(+)} + \mathcal{R}_{(-)})$, and that when ρ, v^1, and s are sufficiently small (as is captured by the smallness parameters $\mathring{\alpha}$ and $\mathring{\epsilon}$ that we introduce in Sect. 10), we have (via the implicit function theorem) $\rho = (\mathcal{R}_{(+)} - \mathcal{R}_{(-)}) \cdot \widetilde{F}(\mathcal{R}_{(+)} - \mathcal{R}_{(-)}, s)$, where $\widetilde{F}$ is a smooth function. This allows us to control ρ and v^1 in terms of $\mathcal{R}_{(+)}$, $\mathcal{R}_{(-)}$, and s. Throughout the article, we use this observation without explicitly pointing it out. In particular, even though many of the equations that we study explicitly involve ρ and v^1, it should be understood that we always estimate these quantities in terms of the "wave-variables" $\mathcal{R}_{(+)}$, $\mathcal{R}_{(-)}$, and s, which are featured in the array (2.11a) defined below.

2.5. **The higher order fluid variables.** The "higher order" fluid variables in the next definition appear in Theorem 2.15, which provides the formulation of compressible Euler flow that we use throughout our analysis.

Definition 2.7 (The higher order fluid variables)**.**

1. We define the *specific vorticity* to be the Σ_t-tangent vectorfield whose Cartesian spatial components are:

$$\Omega^i \stackrel{\text{def}}{=} \frac{(\operatorname{curl} v)^i}{\exp(\rho)} = \frac{\epsilon_{ijk}\delta^{jl}\partial_l v^k}{\exp(\rho)}, \tag{2.8}$$

   where δ^{jl} is the Kronecker delta.

2. We define the *entropy gradient* to be the Σ_t-tangent vectorfield whose Cartesian spatial components are:

$$S^i \stackrel{\text{def}}{=} \delta^{ia}\partial_a s = \partial_i s. \tag{2.9}$$

3. We define the *modified fluid variables* to be the Σ_t-tangent vectorfield $\mathcal{C}$ and the scalar function $\mathcal{D}$ whose Cartesian spatial components are:

$$\mathcal{C}^i \stackrel{\text{def}}{=} \exp(-\rho)(\operatorname{curl}\Omega)^i + \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^a\partial_a v^i - \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}(\operatorname{div} v)S^i, \tag{2.10a}$$

$$\mathcal{D} \stackrel{\text{def}}{=} \exp(-2\rho)\operatorname{div} S - \exp(-2\rho)S^a\partial_a\rho. \tag{2.10b}$$

2.6. **Arrays of fluid variables and array norm notation.** We provide the next definition for notational convenience.

Definition 2.8 (The fluid variable array $\vec{\Psi}$ and the partial array $\vec{\Psi}_{(\text{Partial})}$)**.** We define the *array of wave*[41] *variables* as follows:

$$\vec{\Psi} \stackrel{\text{def}}{=} (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4) \stackrel{\text{def}}{=} (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s). \tag{2.11a}$$

We define the *partial array of wave-variables* by:

$$\vec{\Psi}_{(\text{Partial})} \stackrel{\text{def}}{=} (\Psi_1, \Psi_2, \Psi_3, \Psi_4) = (\mathcal{R}_{(-)}, v^2, v^3, s). \tag{2.11b}$$

We view $\vec{\Psi}$ to be an array of scalar functions Ψ_ι, where $\iota = 0, \cdots, 4$. We will not attribute any tensorial structure to the labeling index ι besides simple contractions, denoted by $\diamond$, corresponding to the chain rule; see Def. 2.12.

In the next definition, we introduce notation for norms of arrays.

Definition 2.9 (Norm conventions with arrays)**.**

- Given the fluid variable array $\vec{\Psi}$ from Def. 2.8, we define:

$$|\vec{\Psi}| \stackrel{\text{def}}{=} \max_{\iota\in\{0,\cdots,4\}} |\Psi_\iota|. \tag{2.12}$$

  For any norm $\|\cdot\|$ on scalar functions that appears in the paper, we set:

$$\|\vec{\Psi}\| \stackrel{\text{def}}{=} \max_{\iota\in\{0,\cdots,4\}} \|\Psi_\iota\|. \tag{2.13}$$

  We use a similar convention for Ω: $|\Omega| = \max_{a=1,2,3}|\Omega^a|$, and similarly for $\vec{\Psi}_{(\text{Partial})}$, S, $\mathcal{C}$, etc.
- We use the following convention when taking norms of more than one variable at a time:

$$\|(\Omega, S)\| \stackrel{\text{def}}{=} \max\{\|\Omega\|, \|S\|\}. \tag{2.14}$$

[41]These "wave-variables" solve wave equations; see Theorem 2.15.

2.7. **The acoustical metric and related geometric objects.** In the following definition, we introduce the acoustical metric and its inverse. This Lorentzian[42] metric drives the propagation of sound waves and is necessary to reveal the full geometry of the singular boundary.

Definition 2.10 (The acoustical metric)**.** Relative to the Cartesian coordinates (t, x^1, x^2, x^3), we define the acoustical metric $\mathbf{g}$ and the inverse acoustical metric $\mathbf{g}^{-1}$ as follows, where the material derivative vectorfield $\mathbf{B}$ is defined in (1.2) and the speed of sound c is defined in (1.3):

$$\mathbf{g} = -\mathrm{d}t \otimes \mathrm{d}t + c^{-2} \sum_{a=1}^{3} (\mathrm{d}x^a - v^a \mathrm{d}t) \otimes (\mathrm{d}x^a - v^a \mathrm{d}t), \tag{2.15a}$$

$$\mathbf{g}^{-1} = -\mathbf{B} \otimes \mathbf{B} + c^2 \sum_{a=1}^{3} \partial_a \otimes \partial_a. \tag{2.15b}$$

Straightforward calculations yield that $\mathbf{g}_{\alpha\gamma}(\mathbf{g}^{-1})^{\gamma\beta} = \delta_\alpha^\beta$, where δ_α^β is the Kronecker delta, i.e., $\mathbf{g}^{-1}$ is indeed the inverse of $\mathbf{g}$. In the remainder of the article, we silently lower and raise lowercase Greek indices with $\mathbf{g}$ and $\mathbf{g}^{-1}$, e.g., $V^\alpha = (\mathbf{g}^{-1})^{\alpha\beta} V_\beta$.

In our forthcoming analysis, the undifferentiated quantities v^i and $c-1$ will be small, where we quantify their smallness via the parameters $\mathring{\alpha}$ and $\mathring{\epsilon}$, which we introduce in Sect. 10. Hence, in view of (2.15a), we find it convenient to introduce the following decomposition:

$$\mathbf{g}_{\alpha\beta}(\vec{\Psi}) = m_{\alpha\beta} + \mathbf{g}_{\alpha\beta}^{(\mathrm{Small})}(\vec{\Psi}), \tag{2.16}$$

where $m_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric and $\mathbf{g}_{\alpha\beta}^{(\mathrm{Small})}(\vec{\Psi})$ is a smooth function of $\vec{\Psi}$ satisfying:

$$\mathbf{g}_{\alpha\beta}^{(\mathrm{Small})}(\vec{\Psi} = 0) = 0. \tag{2.17}$$

The scalar functions $G_{\alpha\beta}^\iota$ in the following definition will appear as coefficients in many of the equations that we study.

Definition 2.11 ($\vec{\Psi}$-derivatives of $\mathbf{g}$)**.** Viewing the Cartesian component functions $\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}(\vec{\Psi})$ as functions of the wave-variables, for $\alpha, \beta = 0, 1, 2, 3$ and $\iota = 0, 1, 2, 3, 4$, we define:

$$G_{\alpha\beta}^\iota(\vec{\Psi}) \stackrel{\mathrm{def}}{=} \frac{\partial}{\partial \Psi_\iota} \mathbf{g}_{\alpha\beta}(\vec{\Psi}), \tag{2.18a}$$

$$\vec{G}_{\alpha\beta} = \vec{G}_{\alpha\beta}(\vec{\Psi}) \stackrel{\mathrm{def}}{=} \Big(G_{\alpha\beta}^0(\vec{\Psi}), G_{\alpha\beta}^1(\vec{\Psi}), G_{\alpha\beta}^2(\vec{\Psi}), G_{\alpha\beta}^3(\vec{\Psi}), G_{\alpha\beta}^4(\vec{\Psi}) \Big). \tag{2.18b}$$

For each fixed $\iota \in \{0, \cdots, 4\}$, we view $\{G_{\alpha\beta}^\iota\}_{\alpha,\beta=0,\cdots,3}$ to be the Cartesian components of the spacetime tensorfield "G^ι." Similarly, we view $\{\vec{G}_{\alpha\beta}\}_{\alpha,\beta=0,\cdots,3}$ to be the Cartesian components of the array-valued spacetime tensorfield $\vec{G}$.

Definition 2.12 (Operators involving $\vec{\Psi}$)**.** Let V_1, V_2 be vectorfields, and let D be a differential operator. We define:

$$D\vec{\Psi} \stackrel{\mathrm{def}}{=} (D\Psi_0, D\Psi_1, D\Psi_2, D\Psi_3, D\Psi_4), \qquad \vec{G}_{V_1 V_2} \diamond D\vec{\Psi} \stackrel{\mathrm{def}}{=} \sum_{\iota=0}^{4} G_{\alpha\beta}^\iota V_1^\alpha V_2^\beta D\Psi_\iota. \tag{2.19}$$

2.8. **Covariant wave operator and g-null forms.** In this section, we provide some definitions that we need to state Theorem 2.15, which provides the geometric formulation of compressible Euler flow that we use throughout our analysis.

We start by recalling the standard definition of the covariant wave operator $\square_{\mathbf{g}}$.

Definition 2.13 (Covariant wave operator of the acoustical metric)**.** The covariant wave operator $\square_{\mathbf{g}}$ of the acoustical metric $\mathbf{g} = \mathbf{g}(\vec{\Psi})$ acts on scalar-valued functions φ as follows:[43]

$$\square_{\mathbf{g}} \varphi \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{|\det \mathbf{g}|}} \partial_\alpha \Big\{ \sqrt{|\det \mathbf{g}|} (\mathbf{g}^{-1})^{\alpha\beta} \partial_\beta \varphi \Big\}. \tag{2.20}$$

We now recall the definition of a standard null form with respect to the acoustical metric ("**g**-null form" for short).

[42] By "Lorentzian," we mean that viewed as a quadratic form, the symmetric 4×4 matrix $(\mathbf{g}_{\alpha\beta})_{\alpha,\beta=0,1,2,3}$ has signature $(-,+,+,+)$.

[43] The formula (2.20) holds relative to arbitrary coordinates.

Definition 2.14 (Standard $\mathbf{g}$-Null forms)**.** Let φ and $\widetilde{\varphi}$ be scalar functions. We define $\mathfrak{Q}^{(\mathbf{g})}(\partial\varphi,\partial\widetilde{\varphi})$ to be the following derivative-quadratic term:

$$\mathfrak{Q}^{(\mathbf{g})}(\partial\varphi,\partial\widetilde{\varphi}) \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\widetilde{\varphi}. \tag{2.21a}$$

For $0 \leq \alpha < \beta \leq 3$, we define $\mathfrak{Q}_{\alpha\beta}(\partial\varphi,\partial\widetilde{\varphi})$ to be the following derivative-quadratic term:

$$\mathfrak{Q}_{\alpha\beta}(\partial\varphi,\partial\widetilde{\varphi}) \stackrel{\text{def}}{=} \partial_\alpha\varphi\partial_\beta\widetilde{\varphi} - \partial_\beta\varphi\partial_\alpha\widetilde{\varphi}. \tag{2.21b}$$

In the rest of the paper, we use the terminology *null form relative to* $\mathbf{g}$ or $\mathbf{g}$*-null form* to denote any linear combination of the standard null forms (2.21a)–(2.21b) with (possibly solution dependent) coefficients that are controllable under the scope of our approach.

2.9. **The geometric wave-transport-divergence-curl formulation of the compressible Euler equations.** Our main results fundamentally rely on the following formulation of the compressible Euler equations, first derived in [72].

Theorem 2.15 (The geometric wave-transport-divergence-curl formulation of the compressible Euler equations)**.** *Let $\overline{\varrho} > 0$ be any constant background density,*[44] *and assume that $(\uprho, v^1, v^2, v^3, s)$ is a solution to the compressible Euler equations* (2.6a)–(2.6c) *in three spatial dimensions under an arbitrary equation of state $p = p(\varrho, s)$ with positive sound speed c (see* (1.3)*). Let $\mathbf{B}$ be the material derivative vectorfield defined in* (1.2)*, let $\mathbf{g}$ be the acoustical metric from Def. 2.10, let $\square_{\mathbf{g}}$ be the corresponding covariant wave operator from Def. 2.13, let $\mathcal{R}_{(\pm)}$ be the almost Riemann invariants from Def. 2.5 (see Remark 2.6 concerning their significance for this paper), let $F = F(\uprho, s)$ be the function from Def. 2.5, and let Ω, S, $\mathcal{C}$, and $\mathcal{D}$ be the higher order variables from Def. 2.7. Then the scalar-valued functions $\uprho$, v^i, $\mathcal{R}_{(\pm)}$, s, Ω^i, S^i, $\mathrm{div}\,\Omega$, $\mathcal{C}^i$, $\mathcal{D}$, and $(\mathrm{curl}\,S)^i$, $(i = 1, 2, 3)$, also solve the following equations, where ϵ_{ijk} is the fully antisymmetric symbol normalized by $\epsilon_{123} = 1$, and* ***the Cartesian component functions v^i are treated as scalar-valued functions under covariant differentiation on LHS*** (2.22a)*:*

Covariant wave equations.

$$\square_{\mathbf{g}(\vec{\Psi})} v^i = -c^2\exp(2\uprho)\mathcal{C}^i + \mathfrak{Q}^i_{(v)} + \mathfrak{L}^i_{(v)}, \tag{2.22a}$$

$$\square_{\mathbf{g}(\vec{\Psi})} \mathcal{R}_{(\pm)} = -c^2\exp(2\uprho)\mathcal{C}^1 \pm \left\{ F_{;s}c^2\exp(2\uprho) - c\exp(\uprho)\frac{p_{;s}}{\overline{\varrho}} \right\}\mathcal{D} + \mathfrak{Q}_{(\pm)} + \mathfrak{L}_{(\pm)}, \tag{2.22b}$$

$$\square_{\mathbf{g}(\vec{\Psi})} \uprho = -\exp(\uprho)\frac{p_{;s}}{\overline{\varrho}}\mathcal{D} + \mathfrak{Q}_{(\uprho)} + \mathfrak{L}_{(\uprho)}, \tag{2.22c}$$

$$\square_{\mathbf{g}(\vec{\Psi})} s = c^2\exp(2\uprho)\mathcal{D} + \mathfrak{L}_{(s)}. \tag{2.22d}$$

Transport equations.

$$\mathbf{B}\Omega^i = \mathfrak{L}^i_{(\Omega)}, \tag{2.23a}$$

$$\mathbf{B}s = 0, \tag{2.23b}$$

$$\mathbf{B}S^i = \mathfrak{L}^i_{(S)}. \tag{2.23c}$$

Transport-divergence-curl system for the specific vorticity.

$$\mathrm{div}\,\Omega = \mathfrak{L}_{(\mathrm{div}\,\Omega)}, \tag{2.24a}$$

$$\mathbf{B}\mathcal{C}^i = \mathfrak{M}^i_{(\mathcal{C})} + \mathfrak{Q}^i_{(\mathcal{C})} + \mathfrak{L}^i_{(\mathcal{C})}. \tag{2.24b}$$

Transport-divergence-curl system for the entropy gradient.

$$\mathbf{B}\mathcal{D} = \mathfrak{M}_{(\mathcal{D})} + \mathfrak{Q}_{(\mathcal{D})}, \tag{2.25a}$$

$$(\mathrm{curl}\,S)^i = 0. \tag{2.25b}$$

[44]Recall that $\uprho$ depends on $\overline{\varrho}$; see Def. 2.4.

*Above, the main terms $\mathfrak{M}_{(\mathcal{C})}^i$ and $\mathfrak{M}_{(\mathcal{D})}$ in the transport equations for the modified fluid variables are the **null forms relative to** $\mathbf{g}$ (see Def. 2.14) defined by:*[45]

$$\begin{aligned}\mathfrak{M}_{(\mathcal{C})}^i &\overset{def}{=} -2\delta_{jk}\epsilon_{iab}\exp(-\rho)(\partial_a v^j)\partial_b\Omega^k + \epsilon_{ajk}\exp(-\rho)(\partial_a v^i)\partial_j\Omega^k \\ &\quad + \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\left\{(\mathbf{B}S^a)\partial_a v^i - (\mathbf{B}v^i)\partial_a S^a\right\} \\ &\quad + \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\left\{(\mathbf{B}v^a)\partial_a S^i - (\mathbf{B}S^i)\partial_a v^a\right\},\end{aligned} \tag{2.26a}$$

$$\mathfrak{M}_{(\mathcal{D})} = 2\exp(-2\rho)\left\{(\partial_a v^a)\partial_b S^b - (\partial_a S^b)\partial_b v^a\right\} + \exp(-\rho)\delta_{ab}(\mathrm{curl}\,\Omega)^a S^b. \tag{2.26b}$$

Moreover, $\mathfrak{Q}_{(v)}^i$, $\mathfrak{Q}_{(\pm)}$, $\mathfrak{Q}_{(\rho)}$, $\mathfrak{Q}_{(\mathcal{C})}^i$, and $\mathfrak{Q}_{(\mathcal{D})}$ are[46] *the null forms relative to* $\mathbf{g}$ *defined by:*

$$\mathfrak{Q}_{(v)}^i \overset{def}{=} -\left\{1 + c^{-1}c_{;\rho}\right\}(\mathbf{g}^{-1})^{\alpha\beta}(\partial_\alpha\rho)\partial_\beta v^i, \tag{2.27a}$$

$$\mathfrak{Q}_{(\pm)} \overset{def}{=} \mathfrak{Q}_{(v)}^1 \mp 2c_{;\rho}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\rho\partial_\beta\rho \pm c\left\{(\partial_a v^a)(\partial_b v^b) - (\partial_a v^b)\partial_b v^a\right\}, \tag{2.27b}$$

$$\mathfrak{Q}_{(\rho)} \overset{def}{=} -3c^{-1}c_{;\rho}(\mathbf{g}^{-1})^{\alpha\beta}(\partial_\alpha\rho)\partial_\beta\rho + \left\{(\partial_a v^a)\partial_b v^b - (\partial_a v^b)\partial_b v^a\right\}, \tag{2.27c}$$

$$\begin{aligned}\mathfrak{Q}_{(\mathcal{C})}^i &\overset{def}{=} \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^i\left\{(\partial_a v^b)\partial_b v^a - (\partial_a v^a)\partial_b v^b\right\} \\ &\quad + \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^b\left\{(\partial_a v^a)\partial_b v^i - (\partial_a v^i)\partial_b v^a\right\} \\ &\quad + 2\exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^a\left\{(\partial_a\rho)\mathbf{B}v^i - (\partial_a v^i)\mathbf{B}\rho\right\} \\ &\quad + 2\exp(-3\rho)c^{-3}c_{;\rho}\frac{p_{;s}}{\bar{\varrho}}S^a\left\{(\partial_a\rho)\mathbf{B}v^i - (\partial_a v^i)\mathbf{B}\rho\right\} \\ &\quad + \exp(-3\rho)c^{-2}\frac{p_{;s;\rho}}{\bar{\varrho}}S^a\left\{(\partial_a v^i)\mathbf{B}\rho - (\partial_a\rho)\mathbf{B}v^i\right\} \\ &\quad + \exp(-3\rho)c^{-2}\frac{p_{;s;\rho}}{\bar{\varrho}}S^i\{(\mathbf{B}v^a)\partial_a\rho - (\mathbf{B}\rho)\partial_a v^a\} \\ &\quad + 2\exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^i\{(\mathbf{B}\rho)\partial_a v^a - (\mathbf{B}v^a)\partial_a\rho\} \\ &\quad + 2\exp(-3\rho)c^{-3}c_{;\rho}\frac{p_{;s}}{\bar{\varrho}}S^i\{(\mathbf{B}\rho)\partial_a v^a - (\mathbf{B}v^a)\partial_a\rho\},\end{aligned} \tag{2.27d}$$

$$\mathfrak{Q}_{(\mathcal{D})} \overset{def}{=} 2\exp(-2\rho)S^a\left\{(\partial_a v^b)\partial_b\rho - (\partial_a\rho)\partial_b v^b\right\}. \tag{2.27e}$$

[45] Actually, the last the last term on RHS (2.26b) is not a null form, but rather a simpler harmless error term.

[46] The term $\mathfrak{M}_{(\mathcal{C})}^i$ on RHS (2.24b) and the term $\mathfrak{M}_{(\mathcal{D})}$ on RHS (2.25a) are also null forms relative to $\mathbf{g}$. We have isolated these two null forms with different notation because they are more difficult to treat than $\mathfrak{Q}_{(v)}^i$, $\mathfrak{Q}_{(\pm)}$ $\mathfrak{Q}_{(\rho)}$, $\mathfrak{Q}_{(\mathcal{C})}^i$, and $\mathfrak{Q}_{(\mathcal{D})}$; to bound the top-order derivatives of the "$\mathfrak{M}$" terms, we rely on the delicate "elliptic-hyperbolic" identities that we derive in Sect. 21.

In addition, the terms $\mathfrak{L}_{(v)}^i$, $\mathfrak{L}_{(\pm)}$, $\mathfrak{L}_{(\rho)}$, $\mathfrak{L}_{(s)}$, $\mathfrak{L}_{(\Omega)}^i$, $\mathfrak{L}_{(S)}^i$, $\mathfrak{L}_{(\mathrm{div}\,\Omega)}$, and $\mathfrak{L}_{(C)}^i$, which are at most linear in the derivatives of the unknowns, are defined as follows:

$$\begin{aligned}\mathfrak{L}_{(v)}^i &\stackrel{def}{=} 2\exp(\rho)\epsilon_{iab}(\mathbf{B}v^a)\Omega^b - \frac{p_{;s}}{\bar{\varrho}}\epsilon_{iab}\Omega^a S^b \\ &\quad - \frac{1}{2}\exp(-\rho)\frac{p_{;\rho;s}}{\bar{\varrho}} S^a \partial_a v^i \\ &\quad - 2\exp(-\rho)c^{-1}c_{;\rho}\frac{p_{;s}}{\bar{\varrho}}(\mathbf{B}\rho)S^i + \exp(-\rho)\frac{p_{;s;\rho}}{\bar{\varrho}}(\mathbf{B}\rho)S^i,\end{aligned} \tag{2.28a}$$

$$\begin{aligned}\mathfrak{L}_{(\pm)} &\stackrel{def}{=} \mathfrak{L}_{(v)}^1 \mp \frac{5}{2}c\exp(-\rho)\frac{p_{;S^b\rho}}{\bar{\varrho}} S^a\partial_a\rho \pm 2c^2 c_{;s} S^a\partial_a\rho \\ &\quad \mp c\exp(-\rho)\frac{p_{;s;s}}{\bar{\varrho}}\delta_{ab}S^a S^b \pm F_{;s}\mathfrak{L}_{(s)},\end{aligned} \tag{2.28b}$$

$$\mathfrak{L}_{(\rho)} \stackrel{def}{=} -\frac{5}{2}\exp(-\rho)\frac{p_{;s;\rho}}{\bar{\varrho}} S^a\partial_a\rho - \exp(-\rho)\frac{p_{;s;s}}{\bar{\varrho}}\delta_{ab}S^a S^b, \tag{2.28c}$$

$$\mathfrak{L}_{(s)} \stackrel{def}{=} c^2 S^a\partial_a\rho - cc_{;\rho}S^a\partial_a\rho - cc_{;s}\delta_{ab}S^a S^b, \tag{2.28d}$$

$$\mathfrak{L}_{(\Omega)}^i \stackrel{def}{=} \Omega^a\partial_a v^i - \exp(-2\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\epsilon_{iab}(\mathbf{B}v^a)S^b, \tag{2.28e}$$

$$\mathfrak{L}_{(S)}^i \stackrel{def}{=} -S^a\partial_a v^i + \epsilon_{iab}\exp(\rho)\Omega^a S^b, \tag{2.28f}$$

$$\mathfrak{L}_{(\mathrm{div}\,\Omega)} \stackrel{def}{=} -\Omega^a\partial_a\rho, \tag{2.28g}$$

$$\begin{aligned}\mathfrak{L}_{(C)}^i &\stackrel{def}{=} 2\exp(-3\rho)c^{-3}c_{;s}\frac{p_{;s}}{\bar{\varrho}}(\mathbf{B}v^i)\delta_{ab}S^a S^b \\ &\quad - 2\exp(-3\rho)c^{-3}c_{;s}\frac{p_{;s}}{\bar{\varrho}}\delta_{ab}S^a(\mathbf{B}v^b)S^i \\ &\quad + \exp(-3\rho)c^{-2}\frac{p_{;s;s}}{\bar{\varrho}}\delta_{ab}(\mathbf{B}v^a)S^b S^i \\ &\quad - \exp(-3\rho)c^{-2}\frac{p_{;s;s}}{\bar{\varrho}}(\mathbf{B}v^i)\delta_{ab}S^a S^b.\end{aligned} \tag{2.28h}$$

Discussion of the proof. Theorem 2.15 was essentially proved as [72, Theorem 1], except that the wave equations (2.22b) for $\mathcal{R}_{(\pm)}$ were not derived there. In [52, Theorem 5.1], those wave equations were derived as a straightforward consequence of [72, Theorem 1].

□

3. The acoustic geometry and the arrays γ and $\underline{\gamma}$

In this section, we construct the acoustic geometry, reveal its basic properties, and provide the evolution equations satisfied by various geometric tensors. Our approach is based on the one pioneered by Christodoulou [24] in his study of irrotational and isentropic solutions. The fundamental object behind all the constructions is an acoustic eikonal function, that is, a solution u to the acoustic eikonal equation. The eikonal function is fundamental for our approach because our proof shows that the fluid solution remains rather smooth relative to the geometric coordinates (t, u, x^2, x^3). In particular, later on, we will use u to construct suitable commutation and multiplier vectorfields out of the geometric coordinates to control the solution and acoustic geometry up to top-order. This allows us, at least in some ways, to treat the problem of shock formation as a long-time existence problem relative to the geometric coordinates. We also introduce the solution variable arrays γ and $\underline{\gamma}$, which contain the wave-variables and various components of the acoustic geometry. We use these arrays throughout the paper to simplify the notation and to allow for convenient, schematic expressions. We sometimes refer to γ and $\underline{\gamma}$ as the "controlling quantities" because all of the quantities that we analyze can, in principle, be constructed out of them.

3.1. **The eikonal function and inverse foliation density.** In the following definition, we introduce the *eikonal function* u and the inverse foliation density μ. The level-sets of u are the characteristic for the wave operator $\square_{\mathbf{g}}$, while the inverse (i.e., reciprocal) of the inverse foliation density measures the density of these characteristics. In particular, the vanishing of $\upmu$ corresponds to the infinite density of the characteristics. Our main results show that in the regime under study, the vanishing of $\upmu$ coincides with the blowup of $|\partial_1 \mathcal{R}_{(+)}|$.

Definition 3.1 (Eikonal function and inverse foliation density)**.** The *eikonal function* u is the solution of the following fully nonlinear hyperbolic initial value problem, where $\mathbf{g}$ is the acoustical metric defined in (2.15a) and the PDE is known as the *acoustic eikonal equation*:

$$\begin{cases} (\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \\ \partial_t u > 0, \\ u|_{\Sigma_0} = -x^1. \end{cases} \tag{3.1}$$

We define the *inverse foliation density* $\upmu$ by:

$$\upmu \stackrel{\text{def}}{=} -\frac{1}{(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha t \partial_\beta u} > 0. \tag{3.2}$$

3.2. **Acoustical subsets of spacetime.**

Definition 3.2 (Acoustical subsets of spacetime)**.** We define the following "acoustical subsets" of spacetime:

$$\Sigma_{t'} \stackrel{\text{def}}{=} \{(t,x^1,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid t = t'\}, \tag{3.3a}$$

$$\mathcal{P}_{u'} \stackrel{\text{def}}{=} \{(t,x^1,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid u(t,x^1,x^2,x^3) = u'\}, \tag{3.3b}$$

$$\ell_{t',u'} \stackrel{\text{def}}{=} \Sigma_{t'} \cap \mathcal{P}_{u'} = \{(t,x^1,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid t = t',\ u(t,x^1,x^2,x^3) = u'\}. \tag{3.3c}$$

Given real numbers $u_1 \leq u_2$ and $t_1 \leq t_2$, we define the following "truncated" subsets of spacetime:

$$\Sigma_{t'}^{[u_1,u_2]} \stackrel{\text{def}}{=} \Sigma_{t'} \cap \{(t,x^1,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid u_1 \leq u(t,x^1,x^2,x^3) \leq u_2\}, \tag{3.4a}$$

$$\mathcal{P}_{u'}^{[t_1,t_2]} \stackrel{\text{def}}{=} \mathcal{P}_{u'} \cap \{(t,x^1,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid t_1 \leq t \leq t_2\}. \tag{3.4b}$$

We refer to the Σ_t as "constant Cartesian-time hypersurfaces," the $\mathcal{P}_u$ as "null hypersurfaces," "acoustic characteristics," or "characteristics," and the $\ell_{t,u}$ as "acoustic tori." We emphasize that with the exception of the appendices, in this paper, we will *not* derive estimates on Σ_t or the $\ell_{t',u'}$. Instead, we will control the solution on the rough hypersurfaces and the rough tori of Def. 4.11.

3.3. **Projection tensorfields and related differential operators.**

Definition 3.3 (Projection tensorfields and tangency to hypersurfaces)**.**

1. We define the type $\binom{1}{1}$ Σ_t-projection tensorfield Π and the type $\binom{1}{1}$ $\ell_{t,u}$-projection tensorfield ${\Pi\!\!\!/}$ as follows, where δ_β^α denotes the Kronecker delta:

$$\Pi_\beta{}^\alpha \stackrel{\text{def}}{=} \delta_\beta^\alpha + \mathbf{B}^\alpha \mathbf{B}_\beta, \tag{3.5a}$$

$${\Pi\!\!\!/}_\beta{}^\alpha \stackrel{\text{def}}{=} \delta_\beta^\alpha + \mathbf{B}^\alpha \mathbf{B}_\beta - X^\alpha X_\beta = \delta_\beta^\alpha - L^\alpha \delta_\beta^0 + X^\alpha L_\beta. \tag{3.5b}$$

2. Given any type $\binom{m}{n}$ spacetime tensorfield ξ, we respectively define its $\mathbf{g}$-orthogonal projection onto Σ_t, denoted by $\Pi\xi$, and its $\mathbf{g}$-orthogonal projection onto $\ell_{t,u}$, denoted by ${\Pi\!\!\!/}\xi$, as follows:

$$(\Pi\xi)^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n} \stackrel{\text{def}}{=} \Pi_{\widetilde{\alpha}_1}^{\ \alpha_1}\cdots\Pi_{\widetilde{\alpha}_m}^{\ \alpha_m}\Pi_{\beta_1}^{\ \widetilde{\beta}_1}\cdots\Pi_{\beta_n}^{\ \widetilde{\beta}_n}\xi^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}, \tag{3.6a}$$

$$({\Pi\!\!\!/}\xi)^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n} \stackrel{\text{def}}{=} {\Pi\!\!\!/}_{\widetilde{\alpha}_1}^{\ \alpha_1}\cdots{\Pi\!\!\!/}_{\widetilde{\alpha}_m}^{\ \alpha_m}{\Pi\!\!\!/}_{\beta_1}^{\ \widetilde{\beta}_1}\cdots{\Pi\!\!\!/}_{\beta_n}^{\ \widetilde{\beta}_n}\xi^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}. \tag{3.6b}$$

3. We say that a spacetime tensorfield ξ is Σ_t-tangent if $\Pi\xi = \xi$. We say that a spacetime tensorfield ξ is $\ell_{t,u}$-tangent if ${\Pi\!\!\!/}\xi = \xi$.
4. If ξ is a symmetric type $\binom{0}{2}$-spacetime tensor and V is a vectorfield, then we define ${\xi\!\!\!/}_V \stackrel{\text{def}}{=} {\Pi\!\!\!/}(\xi \cdot V)$, where $\xi \cdot V$ is the one-form with components $(\xi\cdot V)_\alpha \stackrel{\text{def}}{=} \xi_{\alpha\beta}V^\beta$.
5. If ξ is a spacetime tensor, then we define ${\xi\!\!\!/} = {\Pi\!\!\!/}\xi$. From 3 above, it follows that ξ is $\ell_{t,u}$-tangent if and only if ${\xi\!\!\!/} = \xi$.

It is straightforward to check that $\Pi\mathbf{B} = 0$, while if V is Σ_t-tangent, then $\Pi V = V$, i.e, in view of the properties of $\mathbf{B}$ from Lemma 3.9, we see that Π *is* the $\mathbf{g}$-orthogonal projection onto Σ_t. Similarly, $\not{\Pi} L = \not{\Pi} X = \not{\Pi}\mathbf{B} = 0$, while if Y is $\ell_{t,u}$-tangent, then $\not{\Pi} Y = Y$.

3.4. **First fundamental forms.**

Definition 3.4 (First fundamental forms)**.**

1. We define g, the first fundamental form of Σ_t relative to $\mathbf{g}$, to be the symmetric type $\binom{0}{2}$ tensorfield $\Pi\mathbf{g}$. Note that $g(Y,Z) = \mathbf{g}(Y,Z)$ for all pairs (Y,Z) of Σ_t-tangent vectorfields. We define the corresponding inverse first fundamental form g^{-1} to be the symmetric type $\binom{2}{0}$ tensorfield that is $\mathbf{g}$-dual to g i.e., $(g^{-1})^{\alpha\beta} \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\widetilde{\alpha}}(\mathbf{g}^{-1})^{\beta\widetilde{\beta}} g_{\widetilde{\alpha}\widetilde{\beta}}$. Note that the restriction of g to Σ_t-tangent tensorfields is the Riemannian[47] metric on Σ_t induced by $\mathbf{g}$. In particular, relative to the Cartesian spatial coordinates, we have:
$$g_{ij} = \mathbf{g}_{ij} = c^{-2}\delta_{ij}, \tag{3.7}$$
where δ_{ij} denotes the Kronecker delta, and to obtain the last equality in (3.7), we have used (2.15a).
2. We define $\not{g}$, the first fundamental form of the acoustic tori $\ell_{t,u}$ relative to $\mathbf{g}$, to be the symmetric type $\binom{0}{2}$ tensorfield $\not{\Pi}\mathbf{g}$. Note that $\not{g}(Y,Z) = \mathbf{g}(Y,Z)$ for all pairs (Y,Z) of $\ell_{t,u}$-tangent vectorfields. Note that the restriction of $\not{g}$ to $\ell_{t,u}$-tangent tensorfields is the Riemannian[48] metric on $\ell_{t,u}$ induced by $\mathbf{g}$. We define the corresponding inverse first fundamental form $\not{g}^{-1}$ to be the symmetric type $\binom{2}{0}$ tensorfield that is $\mathbf{g}$-dual to $\not{g}$ i.e., $(\not{g}^{-1})^{\alpha\beta} \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\widetilde{\alpha}}(\mathbf{g}^{-1})^{\beta\widetilde{\beta}} \not{g}_{\widetilde{\alpha}\widetilde{\beta}}$.

3.5. **Geometric coordinates, metric duality, and related vectorfields.**

3.5.1. *Geometric coordinates.*

Definition 3.5 (The geometric coordinates and their corresponding partial derivative vectorfields)**.** We define the *geometric coordinate system* to be (t,u,x^2,x^3). We define $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ to be the coordinate partial derivative vectorfields in the geometric coordinate system.

Remark 3.6 (Coordinate systems on $\ell_{t,u}$ and $\mathcal{P}_u$)**.** Note that (x^2,x^3) form a coordinate system on the acoustic tori $\ell_{t,u}$ and that $\left\{\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ span the tangent space of $\ell_{t,u}$. Similarly, (t,x^2,x^3) form a coordinate system on the null hypersurfaces $\mathcal{P}_u$ and $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ span the tangent space of $\mathcal{P}_u$. We will silently use these basic facts throughout the rest of the article.

Notation 3.1 (Conventions used with (x^2,x^3) and $\{\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}$)**.**

1. If V is a vectorfield, then for $A = 2,3$, $V^A \stackrel{\text{def}}{=} Vx^A = V^\alpha\partial_\alpha x^A$. In particular, if V is $\ell_{t,u}$-tangent, then $V = V^A\frac{\partial}{\partial x^A}$, and V^A are the components of V with respect to geometric coordinates (x^2,x^3) on $\ell_{t,u}$.
2. If ξ is a one-form, then we denote its contraction with $\frac{\partial}{\partial x^A}$ by using the abbreviated notation $\xi_A \stackrel{\text{def}}{=} \xi\left(\frac{\partial}{\partial x^A}\right) = \xi_\alpha\left(\frac{\partial}{\partial x^A}\right)^\alpha$ for $A = 2,3$.
3. We adopt a similar convention for contractions involving higher order tensorfields, e.g., $\not{g}_{AB} = \not{g}(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B})$ for $A,B = 2,3$.
4. We sum repeated uppercase Latin indices over $A = 2,3$, e.g., $\xi_{AA} \stackrel{\text{def}}{=} \xi_{22} + \xi_{33}$.

3.5.2. *Metric duality and musical notation.*

Definition 3.7 (Metric duality and musical notation)**.** If V is a spacetime vectorfield, then $V_\flat$ denotes the one-form that is $\mathbf{g}$-dual to V, i.e., $(V_\flat)_\alpha = \mathbf{g}_{\alpha\beta}V^\beta$. Consistent with the conventions of Sect. 2.1, we typically write V_α instead of $(V_\flat)_\alpha$.

If $Y = Y^B\frac{\partial}{\partial x^B}$ is an $\ell_{t,u}$-tangent vectorfield, then $Y_\flat$ denotes the $\ell_{t,u}$-tangent one-form that is $\not{g}$-dual to Y, i.e., for $A = 2,3$, $(Y_\flat)_A = \not{g}_{AB}Y^B$. Since Y is $\ell_{t,u}$-tangent, $Y_\flat$ can also be viewed as the dual of Y with respect to $\mathbf{g}$, i.e., $(Y_\flat)_\alpha \stackrel{\text{def}}{=} Y_\flat\cdot\partial_\alpha = \mathbf{g}(Y,\partial_\alpha) = \mathbf{g}_{\alpha\beta}Y^\beta$. Similarly, if ξ is an $\ell_{t,u}$-tangent one-form, then $\xi^\sharp$ denotes the $\ell_{t,u}$-tangent vectorfield that is $\not{g}$-dual to ξ, i.e., for $A = 2,3$, $(\xi^\#)^A = (\not{g}^{-1})^{AB}\xi_B$, where $\xi_B = \xi\cdot\frac{\partial}{\partial x^B}$. Similarly, if ξ is a symmetric

[47]It is Riemannian is because the $\mathbf{g}$-normal to Σ_t is the $\mathbf{g}$-timelike vectorfield $\mathbf{B}$; see Lemma 3.9.
[48]It is Riemannian is because $\ell_{t,u}$ is a sub-manifold of the spacelike hypersurface Σ_t.

type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, then $\xi^\#$ denotes the type $\binom{1}{1}$ $\ell_{t,u}$-tangent tensorfield obtained by raising one index of ξ, with $\not{g}^{-1}$, while $\xi^{\#\#}$ denotes the type $\binom{2}{0}$ $\ell_{t,u}$-tangent tensorfield obtained by raising both indices of ξ, with $\not{g}^{-1}$.

3.5.3. *The important acoustic vectorfields.* The vectorfields in the next definition are fundamental for the rest of the paper. We will use them to control the solution up to top-order.

Definition 3.8 (The important acoustic vectorfields)**.**

1. We define the *geodesic null vectorfield* by:

$$L^\alpha_{(\mathrm{Geo})} \stackrel{\mathrm{def}}{=} -(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta u, \tag{3.8}$$

and the *rescaled null vectorfield* as follows, where $\upmu$ is the inverse foliation density defined in (3.2):

$$L \stackrel{\mathrm{def}}{=} \upmu L_{(\mathrm{Geo})}. \tag{3.9}$$

For $i = 1, 2, 3$, we define the scalar functions $L^i_{(\mathrm{Small})}$ as follows, where throughout the paper, L^i denotes the Cartesian component Lx^i:

$$L^1_{(\mathrm{Small})} \stackrel{\mathrm{def}}{=} L^1 - 1, \qquad L^2_{(\mathrm{Small})} \stackrel{\mathrm{def}}{=} L^2, \qquad L^3_{(\mathrm{Small})} \stackrel{\mathrm{def}}{=} L^3. \tag{3.10}$$

2. We define X to be the unique vectorfield that is Σ_t-tangent and $\mathbf{g}$-orthogonal to the acoustic tori $\ell_{t,u}$, and normalized by:

$$\mathbf{g}(L, X) = -1, \tag{3.11}$$

and we define the rescaled vectorfield $\breve{X}$ by:

$$\breve{X} \stackrel{\mathrm{def}}{=} \upmu X. \tag{3.12}$$

For $i = 1, 2, 3$, we define the scalar functions $X^i_{(\mathrm{Small})}$ as follows, where throughout the paper, X^i denotes the Cartesian component Xx^i:[49]

$$X^1_{(\mathrm{Small})} \stackrel{\mathrm{def}}{=} X^1 + 1, \qquad X^2_{(\mathrm{Small})} \stackrel{\mathrm{def}}{=} X^2, \qquad X^3_{(\mathrm{Small})} \stackrel{\mathrm{def}}{=} X^3. \tag{3.13}$$

3. We define $Y_{(2)}$, $Y_{(3)}$, to be the following $\ell_{t,u}$-tangent vectorfields:

$$Y_{(2)} \stackrel{\mathrm{def}}{=} \partial_2 - \mathbf{g}(\partial_2, X)X, \qquad Y_{(3)} \stackrel{\mathrm{def}}{=} \partial_3 - \mathbf{g}(\partial_3, X)X. \tag{3.14}$$

We also define $Y_{(2;\mathrm{Small})}$, $Y_{(3;\mathrm{Small})}$, to be the following vectorfields (which are not generally $\ell_{t,u}$-tangent):

$$Y_{(2;\mathrm{Small})} \stackrel{\mathrm{def}}{=} Y_{(2)} - \partial_2, \qquad Y_{(3;\mathrm{Small})} \stackrel{\mathrm{def}}{=} Y_{(3)} - \partial_2. \tag{3.15}$$

We similarly define the Cartesian component functions $Y^i_{(2)}, Y^i_{(3;\mathrm{Small})}$, analogously to (3.13).

4. We define the *commutation vectorfields* $\mathscr{Z}$, the $\mathcal{P}_u$-tangential subset $\mathscr{P}$, and the $\ell_{t,u}$-tangential subset $\mathscr{Y}$ as follows:

$$\mathscr{Z} \stackrel{\mathrm{def}}{=} \{L, \breve{X}, Y_{(2)}, Y_{(3)}\}, \qquad \mathscr{P} \stackrel{\mathrm{def}}{=} \{L, Y_{(2)}, Y_{(3)}\}, \qquad \mathscr{Y} \stackrel{\mathrm{def}}{=} \{Y_{(2)}, Y_{(3)}\}. \tag{3.16}$$

Lemma 5.5 shows that $\mathscr{Z}$ spans the tangent spaces of spacetime equipped with the differential structure corresponding to the geometric coordinates (t, u, x^2, x^3). We sometimes refer to $\mathscr{Z}$ as the *rescaled frame* because the vectorfield $\breve{X} = \upmu X$ degenerates with respect to the Cartesian differential structure as $\upmu \downarrow 0$, i.e., $\breve{X}^i = \upmu X^i$ tends to 0. Similarly, the lemma shows that $\mathscr{P}$ spans the tangent spaces of the characteristics $\mathcal{P}_u$ and that $\mathscr{Y}$ spans the tangent spaces of the $\ell_{t,u}$. To derive L^∞ and Hölder estimates, we commute various PDEs with elements of $\mathscr{Z}$. To derive energy estimates, we will commute various PDEs with the elements of $\mathscr{P}$. For a handful of key estimates, we will refer to the set $\mathscr{Y}$. We also note that from definitions (3.6b) and (3.14), it is straightforward to check that $Y_{(A)} = \not{\Pi}\partial_A$, i.e., $Y_{(A)}$ is the $\mathbf{g}$-orthogonal projection of the Cartesian partial derivative vectorfield ∂_A onto $\ell_{t,u}$; see also the first equality in (3.34a).

Throughout the paper, we will often silently use the identities featured in the following lemma.

Lemma 3.9 (Basic properties of the vectorfields)**.** *The follow results hold.*

[49]For the solutions covered by our main results, the functions $L^i_{(\mathrm{Small})}$ and $X^i_{(\mathrm{Small})}$ will have magnitudes that are $\ll 1$.

1. *The vectorfield $L_{(\mathrm{Geo})}$ is geodesic and $\mathbf{g}$-null, i.e., with $\mathbf{D}$ denoting the Levi-Civita connection of $\mathbf{g}$, we have:*

$$\mathbf{g}(L_{(\mathrm{Geo})}, L_{(\mathrm{Geo})}) = 0, \qquad \mathbf{D}_{L_{(\mathrm{Geo})}} L_{(\mathrm{Geo})} = 0. \tag{3.17}$$

The rescaled vectorfield L is also $\mathbf{g}$-null:

$$\mathbf{g}(L, L) = 0, \tag{3.18}$$

and it satisfies the following identity, where μ is the inverse foliation density defined in (3.2):

$$\mathbf{D}_L L = \frac{(L\mu)}{\mu} L. \tag{3.19}$$

2. *L is $\mathbf{g}$-orthogonal to the characteristics $\mathcal{P}_u$, that is, for any vectorfield P tangent to $\mathcal{P}_u$, we have:*

$$\mathbf{g}(L, P) = 0. \tag{3.20}$$

3. *The following identities hold:*

$$Lu = 0, \qquad Lt = L^0 = 1, \qquad \breve{X}u = 1, \qquad \breve{X}t = \breve{X}^0 = 0, \tag{3.21}$$

$$\mathbf{g}(X, X) = 1, \qquad \mathbf{g}(\breve{X}, \breve{X}) = \mu^2, \qquad \mathbf{g}(L, X) = -1, \qquad \mathbf{g}(L, \breve{X}) = -\mu. \tag{3.22}$$

4. *The material vectorfield $\mathbf{B}$ is future directed, $\mathbf{g}$-orthogonal to Σ_t (and hence also to $\ell_{t,u}$), and it is of $\mathbf{g}$-unit size:*

$$\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1. \tag{3.23}$$

Moreover, we have:

$$\mathbf{B} = L + X, \tag{3.24}$$

and relative to the Cartesian coordinates, we have:

$$\mathbf{B}_\alpha = -\delta_\alpha^0, \tag{3.25}$$

where δ_α^β is the Kronecker delta.

5. *Finally, the following identities hold for $i = 1, 2, 3$ and $A = 2, 3$:*

$$X^i_{(\mathrm{Small})} = -L^i_{(\mathrm{Small})} + v^i, \tag{3.26a}$$

$$Y^i_{(A;\mathrm{Small})} = -c^{-2} X^A_{(\mathrm{Small})} X^i = -c^{-2}(-L^A_{(\mathrm{Small})} + v^A)(-L^i + v^i). \tag{3.26b}$$

Proof. All aspects of the lemma except for (3.19) and (3.26b) follow from minor modifications of the proofs of [73, (2.12), (2.13) and Lemma 2.1]. The identity (3.19) follows from definition (3.9), (3.17), and the Leibniz rule for the connection $\mathbf{D}$. The identity (3.26b) follows from definitions (3.13)–(3.14), the form (2.15a) of $\mathbf{g}_{\alpha\beta}$, and (3.26a). ☐

3.6. **Differential operators associated with the projections and metrics.**

Definition 3.10 ($\ell_{t,u}$-differential)**.** If φ is a scalar function, then we define $d\!\!\!/\varphi$ to be the following $\ell_{t,u}$-tangent one-form:

$$d\!\!\!/\varphi \overset{\mathrm{def}}{=} {/\!\!\!\Pi}\, d\varphi. \tag{3.27}$$

Note that $d\!\!\!/_A\varphi = d\!\!\!/\varphi \cdot \frac{\partial}{\partial x^A} = \frac{\partial}{\partial x^A}\varphi$ for $A = 2, 3$ and that $d\!\!\!/_\alpha\varphi = d\!\!\!/\varphi \cdot \partial_\alpha = {/\!\!\!\Pi}_\alpha^{\ \beta}\partial_\beta\varphi$ for $\alpha = 0, 1, 2, 3$.

Definition 3.11 (Levi-Civita connections and associated differential operators)**.**

1. We denote the Levi-Civita connection of $\mathbf{g}$ by $\mathbf{D}$.
2. We denote the Levi-Civita connection of $g\!\!\!/$ by $\nabla\!\!\!\!/$. In particular, for $\ell_{t,u}$-tangent tensorfields ξ, we have $\nabla\!\!\!\!/\xi = {/\!\!\!\Pi}\mathbf{D}\xi$.
3. If ξ is an $\ell_{t,u}$-tangent one-form, then we define its $\ell_{t,u}$-divergence to be the scalar function $\mathrm{d}{/\!\!\!\mathrm{iv}}\,\xi \overset{\mathrm{def}}{=} g\!\!\!/^{-1} \cdot \nabla\!\!\!\!/\xi$. Similarly, if Y is an $\ell_{t,u}$-tangent vectorfield, then we define its $\ell_{t,u}$-divergence to be the scalar function $\mathrm{d}{/\!\!\!\mathrm{iv}}\, Y \overset{\mathrm{def}}{=} g\!\!\!/^{-1} \cdot \nabla\!\!\!\!/ Y_\flat$. where $Y_\flat$ is the $\ell_{t,u}$-tangent one-form that is $\mathbf{g}$-dual to Y.
4. If ξ is a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, then we define its $\ell_{t,u}$-divergence to be the $\ell_{t,u}$-tangent one-form with the following $\ell_{t,u}$-components for $A = 2, 3$: $(\mathrm{d}{/\!\!\!\mathrm{iv}}\,\xi)_A \overset{\mathrm{def}}{=} (g\!\!\!/^{-1})^{BC} \cdot \nabla\!\!\!\!/_B \xi_{CA}$.
5. We denote the covariant wave operator of $\mathbf{g}$ by $\square_{\mathbf{g}} \overset{\mathrm{def}}{=} \mathbf{g}^{-1} \cdot \mathbf{D}^2 = (\mathbf{g}^{-1})^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta$.
6. We denote the $\ell_{t,u}$-Laplacian associated to $g\!\!\!/$ by $\Delta\!\!\!\!/ \overset{\mathrm{def}}{=} g\!\!\!/^{-1} \cdot \nabla\!\!\!\!/^2 = (g\!\!\!/^{-1})^{AB}\nabla\!\!\!\!/_A\nabla\!\!\!\!/_B$.

Definition 3.12 (Projected Lie derivatives)**.** Given a spacetime tensorfield ξ and a vectorfield Z, we define $\underline{\mathcal{L}}_Z \xi$ and $\not{\mathcal{L}}_Z \xi$ to respectively be the following Σ_t-tangent and $\ell_{t,u}$-tangent tensorfields:

$$\underline{\mathcal{L}}_Z \xi \stackrel{\text{def}}{=} \Pi \mathcal{L}_Z \xi, \qquad \not{\mathcal{L}}_Z \xi \stackrel{\text{def}}{=} \not{\Pi} \mathcal{L}_Z \xi. \tag{3.28}$$

We will use the following simple commutation lemma when deriving various equations.

Lemma 3.13 (Angular differential $\not{d}$ commutes with $\not{\mathcal{L}}$)**.** *Let f be a scalar function and let $Z \in \mathscr{Z}$ (see definition* (3.16)*). Then the following identity holds:*

$$\not{\mathcal{L}}_Z \not{d} f = \not{d} Z f. \tag{3.29}$$

Proof. The same proof of [73, Lemma 2.10] holds. □

3.7. **Controlling quantities γ and $\underline{\gamma}$.** In the next definition, we introduce the solution variable arrays γ and $\underline{\gamma}$. They allow us to provide simple, schematic formulas in contexts where precise structure is not important for the PDE analysis.

Definition 3.14 (The controlling quantities)**.** We define γ and $\underline{\gamma}$ to be the following arrays of scalar functions:

$$\gamma \stackrel{\text{def}}{=} (\vec{\Psi}, L^1_{(\text{Small})}, L^2_{(\text{Small})}, L^3_{(\text{Small})}), \tag{3.30a}$$

$$\underline{\gamma} \stackrel{\text{def}}{=} (\vec{\Psi}, \mu - 1, L^1_{(\text{Small})}, L^2_{(\text{Small})}, L^3_{(\text{Small})}). \tag{3.30b}$$

For the solutions that we study in our main results, along the data-hypersurface Σ_0, γ and $\underline{\gamma}$ are small in L^∞.

3.8. **Identities for the $\ell_{t,u}$-projection tensorfield and the first fundamental forms.** The following lemma provides useful identities for the $\ell_{t,u}$-projection tensorfield $\not{\Pi}$, the first fundamental form $\not{g}$ of $\ell_{t,u}$, and the first fundamental form g of Σ_t.

Lemma 3.15 (Useful identities for the first fundamental forms)**.** *Recall that $\not{\Pi}$ is the $\ell_{t,u}$-projection tensorfield from Def. 3.3, that $\not{g}$ is the first fundamental form of $\ell_{t,u}$ from Def.3.4, and that g is the first fundamental form of Σ_t from Def.3.4. Let X be the vectorfield defined in Def. 3.8. Then the following identities hold relative to the geometric coordinates ($A, B = 2, 3$):*

$$\not{g}_{AB} = c^{-2}\delta_{AB} + c^{-2}\frac{X^A X^B}{(X^1)^2}, \tag{3.31a}$$

$$(\not{g}^{-1})^{AB} = c^2 \delta^{AB} - X^A X^B, \tag{3.31b}$$

$$\det \not{g} = \frac{1}{c^2 (X^1)^2}. \tag{3.31c}$$

Moreover, the following identities hold relative to arbitrary coordinates:

$$g_{\alpha\beta} = \mathbf{g}_{\alpha\beta} + \mathbf{B}_\alpha \mathbf{B}_\beta, \tag{3.32a}$$

$$(g^{-1})^{\alpha\beta} = (\mathbf{g}^{-1})^{\alpha\beta} + \mathbf{B}^\alpha \mathbf{B}^\beta. \tag{3.32b}$$

Furthermore, the following identities hold relative to the Cartesian coordinates:

$$g = c^{-2} \sum_{a=1}^{3} (\mathrm{d}x^a - v^a \mathrm{d}t) \otimes (\mathrm{d}x^a - v^a \mathrm{d}t), \tag{3.33a}$$

$$g^{-1} = c^2 \sum_{a=1}^{3} \partial_a \otimes \partial_a. \tag{3.33b}$$

In addition, the following identities hold relative to arbitrary coordinates:

$$\not{g}_{\alpha\beta} = g_{\alpha\beta} - X_\alpha X_\beta = \mathbf{g}_{\alpha\beta} + \mathbf{B}_\alpha \mathbf{B}_\beta - X_\alpha X_\beta = \mathbf{g}_{\alpha\beta} + L_\alpha L_\beta + L_\alpha X_\beta + X_\alpha L_\beta, \tag{3.34a}$$

$$(\not{g}^{-1})^{\alpha\beta} = (g^{-1})^{\alpha\beta} - X^\alpha X^\beta = (\mathbf{g}^{-1})^{\alpha\beta} + \mathbf{B}^\alpha \mathbf{B}^\beta - X^\alpha X^\beta = (\mathbf{g}^{-1})^{\alpha\beta} + L^\alpha L^\beta + L^\alpha X^\beta + X^\alpha L^\beta, \tag{3.34b}$$

$$\not{\Pi}_\beta{}^\alpha = \mathbf{g}_{\beta\gamma} (\not{g}^{-1})^{\alpha\gamma} = \not{g}_{\beta\gamma} (\not{g}^{-1})^{\alpha\gamma}. \tag{3.34c}$$

Finally, relative to the Cartesian coordinates, the following identities hold for $\alpha, \beta = 0, 1, 2, 3$:

$$(\not{g}^{-1})^{0\alpha} = (\not{g}^{-1})^{\alpha 0} = 0, \tag{3.35a}$$

$$(g^{-1})^{0\alpha} = (g^{-1})^{\alpha 0} = 0, \tag{3.35b}$$

$$\not{\Pi}_{\beta}^{\ 0} = 0. \tag{3.35c}$$

Proof. The identities (3.31a)–(3.31b) were proved in [52, Lemma 2.31]. The identity (3.31c) follows from (3.31a) and the following identity:

$$c^2 = \sum_{i=1}^{3} (X^i)^2, \tag{3.36}$$

which follows from (2.15a) and $\mathbf{g}(X, X) = 1$ (see (3.22)).

(3.32a) follows from definition (3.5a) and the fact that $g = \Pi \mathbf{g}$. (3.32b) follows from raising the indices in (3.32a) with $\mathbf{g}^{-1}$.

Since $\ell_{t,u} \subset \Sigma_t$, the first equality in (3.34a) follows from the fact that X is Σ_t-tangent, $\mathbf{g}$-orthogonal to $\ell_{t,u}$, and normalized by $\mathbf{g}(X, X) = g(X, X) = 1$. The second equality in (3.34a) follows from (3.32a). The last equality in (3.34a) follows from (3.24). (3.34b) follows from raising the indices in (3.34a) with $\mathbf{g}^{-1}$.

The first equality in (3.34c) follows from definition (3.5b) and the second equality in (3.34b). The second equality in (3.34c) follows from the second equality in (3.34a) and the fact that $\not{g}^{-1}$ vanishes when contracted against $\mathbf{B}$ or X.

(3.33b) follows from (2.15b) and (3.32b). (3.33a) follows from (3.33b) and (2.15a), which in particular implies that in Cartesian coordinates, the $\mathbf{g}$-dual of ∂_a is $c^{-2}\{\mathrm{d}x^a - v^a \mathrm{d}t\}$.

(3.35a) follows from (3.33b), the first equality in (3.34b), and the fact that $X^0 = 0$, i.e., X is Σ_t-tangent. (3.35b) follows from (3.33b). (3.35c) follows from (3.34c) and (3.35a). □

3.9. **Traces of tensorfields.** In our analysis, we will encounter various traces of tensorfields.

Definition 3.16 (Traces of tensorfields)**.**

1. If ξ is a type $\binom{0}{2}$ spacetime tensorfield, then we define its $\mathbf{g}$-trace as follows:

$$\mathrm{tr}_{\mathbf{g}} \xi \overset{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\beta} \xi_{\alpha\beta}. \tag{3.37a}$$

2. If ξ is a type $\binom{0}{2}$ spacetime tensorfield, then we define its $\not{g}$-trace as follows:

$$\mathrm{tr}_{\not{g}} \xi \overset{\text{def}}{=} (\not{g}^{-1})^{\alpha\beta} \xi_{\alpha\beta}. \tag{3.37b}$$

3.10. **Pointwise norms and semi-norms of tensorfields.** In the next definition, we define various pointwise norms and semi-norms that we will use to measure the size of tensorfields.

Definition 3.17 (Pointwise norms)**.**

1. If ξ is a type $\binom{m}{n}$ spacetime tensorfield such that $\mathbf{g}_{\alpha_1\widetilde{\alpha}_1} \cdots \mathbf{g}_{\alpha_m\widetilde{\alpha}_m} (\mathbf{g}^{-1})^{\beta_1\widetilde{\beta}_1} \cdots (\mathbf{g}^{-1})^{\beta_n\widetilde{\beta}_n} \xi_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_n} \xi_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m} \geq 0$, then we define $|\xi|_{\mathbf{g}} \geq 0$ by:

$$|\xi|_{\mathbf{g}}^2 \overset{\text{def}}{=} \mathbf{g}_{\alpha_1\widetilde{\alpha}_1} \cdots \mathbf{g}_{\alpha_m\widetilde{\alpha}_m} (\mathbf{g}^{-1})^{\beta_1\widetilde{\beta}_1} \cdots (\mathbf{g}^{-1})^{\beta_n\widetilde{\beta}_n} \xi_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_n} \xi_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}. \tag{3.38a}$$

2. If ξ is a type $\binom{m}{n}$ tensorfield, then we define $|\xi|_{\not{g}} \geq 0$ by:

$$|\xi|_{\not{g}}^2 \overset{\text{def}}{=} \not{g}_{\alpha_1\widetilde{\alpha}_1} \cdots \not{g}_{\alpha_m\widetilde{\alpha}_m} (\not{g}^{-1})^{\beta_1\widetilde{\beta}_1} \cdots (\not{g}^{-1})^{\beta_n\widetilde{\beta}_n} \xi_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_n} \xi_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}. \tag{3.38b}$$

3. If ξ is a type $\binom{m}{n}$ tensorfield, then we define $|\xi|_g \geq 0$ by:

$$|\xi|_g^2 \overset{\text{def}}{=} g_{\alpha_1\widetilde{\alpha}_1} \cdots g_{\alpha_m\widetilde{\alpha}_m} (g^{-1})^{\beta_1\widetilde{\beta}_1} \cdots (g^{-1})^{\beta_n\widetilde{\beta}_n} \xi_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_n} \xi_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}. \tag{3.38c}$$

Remark 3.18 (Norms vs. semi-norms)**.** $|\cdot|_{\mathbf{g}}$ is a pointwise norm on the space of $\mathbf{g}$-spacelike tensorfields. $|\cdot|_g$ is a pointwise norm on the space of Σ_t-tangent tensorfields and a pointwise semi-norm on the space of all tensorfields. $|\cdot|_{\not{g}}$ is a pointwise norm on the space of $\ell_{t,u}$-tangent tensorfields and a pointwise semi-norm on the space of all tensorfields.

Similarly, the function $|\cdot|_{\widetilde{\not{g}}}$ from Def. 6.9 below is a pointwise norm on the space of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfields a pointwise semi-norm on the space of all tensorfields.

Remark 3.19 (Omitting the 0 component in Cartesian coordinates)**.** In view of (3.25), Def. 3.3, definitions (3.38b)–(3.38c), and (3.35a)–(3.35b), we see that if ξ is a type $\binom{m}{n}$ Σ_t-tangent tensorfield, then relative to the Cartesian coordinates, we have:

$$|\xi|^2_{\not g} = \not g_{a_1\widetilde{a}_1}\cdots\not g_{a_m\widetilde{a}_m}(\not g^{-1})^{b_1\widetilde{b}_1}\cdots(\not g^{-1})^{b_n\widetilde{b}_n}\xi^{a_1\cdots a_n}_{b_1\cdots b_n}\xi^{\widetilde{a}_1\cdots\widetilde{a}_m}_{\widetilde{b}_1\cdots\widetilde{b}_n}, \tag{3.39}$$

$$|\xi|^2_{g} = g_{a_1\widetilde{a}_1}\cdots g_{a_m\widetilde{a}_m}(g^{-1})^{b_1\widetilde{b}_1}\cdots(g^{-1})^{b_n\widetilde{b}_n}\xi^{a_1\cdots a_n}_{b_1\cdots b_n}\xi^{\widetilde{a}_1\cdots\widetilde{a}_m}_{\widetilde{b}_1\cdots\widetilde{b}_n}, \tag{3.40}$$

i.e., we can omit all "0" components on RHSs (3.39)–(3.40).

Similarly, taking into account definition (3.6b) and (3.35c), we see that if V is a Σ_t-tangent vectorfield, then relative to the Cartesian coordinates, we have $\not\Pi_\beta^{\ \alpha}\partial_\alpha V^\beta = \not\Pi_b^{\ a}\partial_a V^b$.

In the rest of the paper, we will use these basic facts without always explicitly mentioning them.

3.11. **Second fundamental forms and the torsion.** In this section, we provide the standard definitions of the second fundamental form k of Σ_t, the null second fundamental form χ of $\ell_{t,u}$, and the one-form ζ. These quantities will appear in various PDEs throughout the article. It is well-known that there are many technical difficulties that have to be overcome to obtain top-order energy estimates for $\mathrm{tr}_{\not g}\chi$ and χ. To achieve control, we will use the modified quantities defined in Sect. 19 and elliptic estimates on the rough tori, which we derive in Sect. 28.

Definition 3.20 (The second fundamental forms k and χ, and the one-form ζ)**.**

1. We define the *second fundamental form* k of Σ_t as follows:

$$k \stackrel{\text{def}}{=} \frac{1}{2}\underline{\mathcal{L}}_{\mathbf{B}} g. \tag{3.41}$$

2. We define the *null second fundamental form* of $\ell_{t,u}$ as follows:

$$\chi \stackrel{\text{def}}{=} \frac{1}{2}\not{\mathcal{L}}_L \not g. \tag{3.42}$$

3. We define ζ to be the $\ell_{t,u}$-tangent one-form with the following components:

$$\zeta_A \stackrel{\text{def}}{=} \mathbf{g}(\mathbf{D}_A L, X). \tag{3.43}$$

3.12. **Transport equations for the eikonal function quantities.** To control the eikonal function quantities μ and L^i, we will use the following transport equations.

Lemma 3.21 (Transport equations satisfied by μ and L^i)**.** *The scalar functions μ and L^i satisfy the following transport equations (see Def. 2.12 regarding the notation):*

$$L\mu = \frac{1}{2}\vec{G}_{LL}\diamond \breve{X}\vec{\Psi} - \frac{1}{2}\mu\vec{G}_{LL}\diamond L\vec{\Psi} - \mu\vec{G}_{LX}\diamond L\vec{\Psi}, \tag{3.44}$$

$$LL^i_{(\text{Small})} = \frac{1}{2}(\vec{G}_{LL}\diamond L\vec{\Psi})X^i - (\vec{\not G}{}^{\#}_L \diamond L\vec{\Psi})\cdot \not d x^i + \frac{1}{2}(\vec{G}_{LL}\diamond \not d^{\#}\vec{\Psi})\cdot \not d x^i. \tag{3.45}$$

Proof. The same proof of [73, Lemma 2.12] holds. □

3.13. **The factor driving the shock formation and formulas involving G_{LL}.** In the following lemma, we compute an expression for the product $\frac{1}{2}\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}$ on the RHS of the evolution equation (3.44) for μ. For every smooth equation of state besides that of a Chaplygin gas, there exist open sets of background densities $\overline{\varrho} > 0$ such that the non-degeneracy condition (2.4) holds. The identity (3.46) then shows that for solutions that are close to the trivial solution $\vec{\Psi} \equiv 0$, the expansion of $\frac{1}{2}\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}$ features a non-zero term proportional to $\breve{X}\mathcal{R}_{(+)}$; the presence of this term is crucial for our main results, as it drives the formation of the shock, i.e., it drives μ to 0. In contrast, for the equation of state $p = C_0 - C_1\exp(-\rho)$ of a Chaplygin gas, one can compute that $c^{-1}c_{;\rho}+1 \equiv 0$, and the non-degeneracy condition (2.4) is therefore impossible. In this case, equation (3.46) shows that the product $\frac{1}{2}\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}$ does not depend on the solution's $\breve{X}$ derivative, and hence our main results do not apply. We note that one can show that for irrotational and isentropic solutions, the equation $c^{-1}c_{;\rho}+1 = 0$ is equivalent to the statement that the quasilinear wave equation for a potential function satisfies Klainerman's null condition [44].

Lemma 3.22 (Identity for the factor driving the shock formation)**.** *For solutions to the compressible Euler equations* (2.6a)–(2.6c)*, the following identity holds, where $F(\rho,s)$ is the scalar function from* (2.7)*:*

$$\begin{aligned}\frac{1}{2}\vec{G}_{LL}\diamond \breve{X}\vec{\Psi} = &-\frac{1}{2}c^{-1}(c^{-1}c_{;\rho}+1)\left\{\breve{X}\mathcal{R}_{(+)}-\breve{X}\mathcal{R}_{(-)}\right\}\\ &-\frac{1}{2}\mu c^{-1}X^1\left\{L\mathcal{R}_{(+)}+L\mathcal{R}_{(-)}\right\}-\mu c^{-2}\{X^2Lv^2+X^3Lv^3\}\\ &-\mu c^{-1}c_{;s}X^aS^a+\mu c^{-1}(c^{-1}c_{;\rho}+1)F_{;s}X^aS^a.\end{aligned} \tag{3.46}$$

Proof. This is the same as [50, Lemmas 2.45, 2.46], except for minor modifications incorporating the third dimension and the entropy (via the $c_{;s}$-dependent and $F_{;s}$-dependent products). □

In the next lemma, we derive expressions for G^2_{LL} and G^3_{LL}. When deriving estimates, we will use the expressions to track smallness.

Lemma 3.23 (Formulas for G^A_{LL})**.** *The following identities hold for $A=2,3$:*

$$G^A_{LL}=2c^{-2}(v^A-L^A)=2c^{-2}X^A=2c^{-2}X^A_{(\mathrm{Small})}. \tag{3.47}$$

Proof. The identities follow from the expression (2.15a) for the Cartesian component $\mathbf{g}_{\alpha\beta}$ viewed as a function of $(\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s)$, the identity $G^A_{LL}=(\frac{\partial}{\partial v^A}\mathbf{g}_{\alpha\beta})L^\alpha L^\beta$, and the identities $L^0=1$ and $L^A+X^A=v^A$, which follow from Lemma 3.9 and (1.2). □

3.14. **Useful geometric decompositions.** In this section, we provide some geometric decompositions that we will use throughout the article.

We start with the following alternate expressions for χ and $\not{k}$, which are useful for computations.

Lemma 3.24 (Alternate expressions for χ and $\not{k}$)**.** *The second fundamental forms from Def. 3.20 satisfy the following identities:*

$$\chi_{AB}=\mathbf{g}\left(\mathbf{D}_AL,\frac{\partial}{\partial x^B}\right), \qquad \not{k}_{AB}=\mathbf{g}\left(\mathbf{D}_A\mathbf{B},\frac{\partial}{\partial x^B}\right). \tag{3.48}$$

Proof. The same proof of [69, Lemma 3.61] holds in the present setting. □

We will use the following identities and decompositions when deriving estimates for χ, $\not{k}$, and ζ.

Lemma 3.25 (Useful identities and decompositions for χ, $\not{k}$, and ζ)**.** *Let χ, k, and ζ be the tensorfields from Def. 3.20. Then the following*[50] *identities hold:*

$$\chi=\mathbf{g}_{ab}\not{d}L^a\otimes\not{d}x^b+\frac{1}{2}\vec{\not{G}}\diamond L\vec{\Psi}+\frac{1}{2}\not{d}\vec{\Psi}\overset{\diamond}{\otimes}\vec{\not{G}}_L-\frac{1}{2}\vec{\not{G}}_L\overset{\diamond}{\otimes}\not{d}\vec{\Psi}, \tag{3.49a}$$

$$\mathrm{tr}_{\not{g}}\chi=\mathbf{g}_{ab}\not{g}^{-1}\cdot\left\{\not{d}L^a\otimes\not{d}x^b\right\}+\frac{1}{2}\not{g}^{-1}\cdot\vec{\not{G}}\diamond L\vec{\Psi}. \tag{3.49b}$$

Moreover, we can decompose $\not{k}$ and ζ into μ^{-1}-singular and μ^{-1}-regular pieces as follows:

$$\zeta=\zeta^{(\mathrm{Tan}-\vec{\Psi})}+\mu^{-1}\zeta^{(\mathrm{Trans}-\vec{\Psi})}, \qquad \not{k}=\not{k}^{(\mathrm{Tan}-\vec{\Psi})}+\mu^{-1}\not{k}^{(\mathrm{Trans}-\vec{\Psi})}, \tag{3.50a}$$

where:

$$\not{k}^{(\mathrm{Tan}-\vec{\Psi})}\overset{def}{=}\frac{1}{2}\vec{\not{G}}\diamond L\vec{\Psi}-\frac{1}{2}\vec{\not{G}}_L\overset{\diamond}{\otimes}\not{d}\vec{\Psi}-\frac{1}{2}\not{d}\vec{\Psi}\overset{\diamond}{\otimes}\vec{\not{G}}_L-\frac{1}{2}\vec{\not{G}}_X\overset{\diamond}{\otimes}\not{d}\vec{\Psi}-\frac{1}{2}\not{d}\vec{\Psi}\overset{\diamond}{\otimes}\vec{\not{G}}_X, \tag{3.50b}$$

$$\not{k}^{(\mathrm{Trans}-\vec{\Psi})}\overset{def}{=}\frac{1}{2}\mu^{-1}\vec{\not{G}}\diamond\breve{X}\vec{\Psi}, \tag{3.50c}$$

$$\zeta^{(\mathrm{Tan}-\vec{\Psi})}\overset{def}{=}\frac{1}{2}\vec{\not{G}}_X\diamond L\vec{\Psi}-\frac{1}{2}\vec{G}_{LX}\diamond\not{d}\vec{\Psi}-\frac{1}{2}\vec{G}_{XX}\diamond\not{d}\vec{\Psi}, \tag{3.50d}$$

$$\zeta^{(\mathrm{Trans}-\vec{\Psi})}\overset{def}{=}-\frac{1}{2}\mu^{-1}\vec{\not{G}}_L\diamond\breve{X}\vec{\Psi}. \tag{3.50e}$$

Proof. The same proofs of [50, Lemmas 2.13, 2.15] holds with minor modifications accounting for the third spatial dimension. □

[50]Here, $\overset{\diamond}{\otimes}$ is defined by $\vec{\not{G}}_L\overset{\diamond}{\otimes}\not{d}\vec{\Psi}\overset{\mathrm{def}}{=}\sum_{\iota=0}^4\vec{\not{G}}^\iota_L\otimes\not{d}\Psi_\iota$, and similarly for $\not{d}\Psi\overset{\diamond}{\otimes}\vec{\not{G}}_L$, $\vec{\not{G}}_X\overset{\diamond}{\otimes}\not{d}\vec{\Psi}$, etc.

In the next lemma, we decompose the μ-weighted covariant wave operator relative to the rescaled frame $\{L, \breve{X}, Y_{(2)}, Y_{(3)}\}$ and the second fundamental forms.

Lemma 3.26 (Frame decomposition of $\mu\Box_{\mathbf{g}(\vec{\Psi})}f$)**.** *Let f be a scalar function, and let $\Box_{\mathbf{g}(\vec{\Psi})}$ be the covariant wave operator of the acoustical metric, as in Def. 2.13. Then the following identities hold:*

$$\begin{aligned}\mu\Box_{\mathbf{g}(\vec{\Psi})}f &= -L(\mu Lf + 2\breve{X}f) + \mu\not{\Delta}f - (\mathrm{tr}_{\not{g}}\chi)\breve{X}f \\ &\quad - \mu\mathrm{tr}_{\not{g}}\not{k}Lf - 2\mu\zeta^{\#}\cdot\not{d}f,\end{aligned} \tag{3.51a}$$

$$\begin{aligned}\mu\Box_{\mathbf{g}(\vec{\Psi})}f &= -(\mu L + 2\breve{X})(Lf) + \mu\not{\Delta}f - (\mathrm{tr}_{\not{g}}\chi)\breve{X}f - (L\mu)Lf \\ &\quad - \mu\mathrm{tr}_{\not{g}}\not{k}Lf + 2\mu\zeta^{\#}\cdot\not{d}f + 2(\not{d}^{\#}\mu)\cdot\not{d}f.\end{aligned} \tag{3.51b}$$

Proof. The same proof of [69, Proposition 5.4] holds with $\breve{X}$ in the role of the vectorfield denoted by "$\breve{R}$" there. □

4. Rough time functions, adapted rough coordinates, and rough subsets

The geometric coordinates (t, u, x^2, x^3) from Sect. 3 are fundamental for our construction of commutation and multiplier vectorfields. However, these coordinates, in particular the Cartesian time function t, are not adapted to the shape of the singular boundary. For this reason, in this section, we construct a one-parameter family of rough time functions $\{^{(\mathfrak{n})}\tau\}_{\mathfrak{n}\in[0,\mathfrak{n}_0]}$ that *are* adapted to structure of the singular boundary, where $\mathfrak{n}_0 > 0$ is a constant depending on the initial data on Σ_0. We refer to $(^{(\mathfrak{n})}\tau, u, x^2, x^3)$ as *adapted rough coordinates*.

In our forthcoming PDE analysis, we will derive estimates on the level-sets of the $^{(\mathfrak{n})}\tau$, which we will prove are $\mathbf{g}$-spacelike (see (6.20c)). We construct $^{(\mathfrak{n})}\tau$ by solving a well-chosen transport equation (see Def. 4.5) with data equal to $-\mu$ on the "initial hypersurface" $\{\breve{X}\mu = -\mathfrak{n}\}$. Standard well-posedness and Cauchy stability results (see Appendices A and B) imply that for perturbations of simple isentropic plane-symmetric solutions, our construction is well-defined on short "rough time" intervals of the form $[\tau_0, \tau_{\mathrm{Boot}})$, where $\tau_0 < 0$ is a data-dependent constant (independent of $\mathfrak{n} \in \mathfrak{n}_0$) and $\tau_0 < \tau_{\mathrm{Boot}} < 0$ is a "bootstrap parameter." Our main results will show that each $^{(\mathfrak{n})}\tau$ exists in a neighborhood of the singular boundary and has range $[\tau_0, 0]$,

There are several subtleties tied to the analysis and regularity of $^{(\mathfrak{n})}\tau$. In particular, our main results crucially rely on our proofs that $^{(\mathfrak{n})}\tau$ is *one degree more differentiable* than the initial hypersurface $\{\breve{X}\mu = -\mathfrak{n}\}$ and that near the singular boundary, for $\mathfrak{m}$ sufficiently small and positive, $\{\breve{X}\mu = -\mathfrak{n}\} \cap \{\mu = \mathfrak{m}\}$ is an embedded two-dimensional, spacelike torus with sufficient regularity. The proofs of these results and many supporting ones are located in Sects. 14–18.

> Starting now, we consider a fixed $\mathfrak{n} \in [0, \mathfrak{n}_0]$; see Sect. 10 for discussion of how $\mathfrak{n}_0$ is tied to the initial data. $\mathfrak{n}$ will remain fixed until Sects. 32–34, where we prove our main results by exploiting all the time functions $^{(\mathfrak{n})}\tau$ for $\mathfrak{n} \in [0, \mathfrak{n}_0]$. All of our estimates will involve constants that can be chosen to be uniform with respect to $\mathfrak{n}$ over the interval $[0, \mathfrak{n}_0]$.

4.1. **Basic constructions.** We now introduce some basic ingredients that we will use to construct $^{(\mathfrak{n})}\tau$.

4.1.1. *The constant $U_{☢} > 0$, the cut-off function ϕ, and the vectorfield $^{(\mathfrak{n})}\breve{W}$.* Our constructions rely on the vectorfield $^{(\mathfrak{n})}\breve{W}$ featured in the next definition, which is crucial for the rest of the paper. In what follows, $U_{☢} > 0$ is a positive constant; see Sect. 10 for further discussion of how the specific choice of $U_{☢}$ that we make in our main theorem is tied to the initial data.

Definition 4.1 (The cut-off ϕ and the vectorfield $^{(\mathfrak{n})}\breve{W}$)**.** Let $\psi: \mathbb{R} \to [0,1]$ be a fixed C^∞ cut-off function such that $\psi(u) = 1$ when $|u| \leq \frac{3}{4}$ and $\psi(u) = 0$ when $|u| \geq 1$. Let ϕ be the cut-off function defined by $\phi(u) \stackrel{\mathrm{def}}{=} \psi\left(\frac{u}{U_{☢}}\right)$. In particular:

$$\begin{cases}\phi(u) = 1, & \text{when } |u| \leq \frac{3}{4}U_{☢}, \\ 0 \leq \phi(u) \leq 1, & \text{when } \frac{3}{4}U_{☢} \leq |u| \leq U_{☢}, \\ \phi(u) = 0, & \text{when } |u| \geq U_{☢}.\end{cases} \tag{4.1}$$

For fixed $\mathfrak{n} \geq 0$, we define the **rough transversal vectorfield** $^{(\mathfrak{n})}\breve{W}$ as follows:

$$^{(\mathfrak{n})}\breve{W} \stackrel{\mathrm{def}}{=} \breve{X} + \phi\frac{\mathfrak{n}}{L\mu}L. \tag{4.2}$$

In our main results, we will have $L\mu < 0$ on the support of ϕ; see (18.8a).

4.1.2. $\upmu$-*adapted subsets.* To follow the solution up to the singular boundary, we will analyze it on the following subsets (and others as well), which are adapted to the shape of the singular boundary.

Definition 4.2 (Level-sets of $\upmu$ and $\breve{X}\upmu$ and the $\upmu$-adapted tori $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$)**.** Recall that $\breve{X}$ is defined in (3.12). Given real numbers $\mathfrak{m},\mathfrak{n} \geq 0$, we define:

$$\breve{\mathbb{M}}_{\mathfrak{m}} \overset{\text{def}}{=} \left\{(t,u,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid \upmu(t,u,x^2,x^3) = \mathfrak{m}\right\} \cap \{|u| \leq U_{\maltese}\}, \tag{4.3a}$$

$$\breve{\mathbb{X}}_{-\mathfrak{n}} \overset{\text{def}}{=} \left\{(t,u,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid \breve{X}\upmu(t,u,x^2,x^3) = -\mathfrak{n}\right\} \cap \{|u| \leq U_{\maltese}\}, \tag{4.3b}$$

$$\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} \overset{\text{def}}{=} \breve{\mathbb{M}}_{\mathfrak{m}} \cap \breve{\mathbb{X}}_{-\mathfrak{n}}. \tag{4.3c}$$

Remark 4.3 (The values of $\mathfrak{m}$ and $\mathfrak{n}$ featured in our main results)**.** Our main results concern solutions and values of $\mathfrak{m} \in [0,\mathfrak{m}_0]$ and $\mathfrak{n} \in [0,\mathfrak{n}_0]$ such that: **i)** ${}^{(\mathfrak{n})}\breve{W}$ is transversal to $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$; **ii)** $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ is a torus, specifically, a $C^{1,1}$ graph over $\mathbb{T}^2$ in geometric coordinates.

Remark 4.4 (Differential structure with respect to the geometric coordinates)**.** In the rest of the paper, we usually implicitly consider $\breve{\mathbb{M}}_{\mathfrak{m}}$, $\breve{\mathbb{X}}_{-\mathfrak{n}}$, and $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ to be subsets of spacetime with the differential structure *induced by the geometric coordinates* (t,u,x^2,x^3); this is already apparent from Def. 4.2. Similar remarks apply to the sets ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{I}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, ${}^{(\mathfrak{n})}\mathcal{P}_{u}^{I}$, ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$, $\breve{\mathbb{M}}_{\mathfrak{m}}^{I}$, and $\breve{\mathbb{X}}_{-\mathfrak{n}}^{I}$ defined below.

4.2. **Rough time functions, the parameter τ_0, and adapted rough coordinates.**

We now provide the transport equation initial value problem whose solution is the rough time function.

4.2.1. *Rough time functions and the parameter* τ_0.

Definition 4.5 (The rough time function ${}^{(\mathfrak{n})}\tau$)**.** Let $\mathfrak{m}_0 > 0$ be a real number and let $\mathfrak{m}_{\text{Boot}} \in [0,\mathfrak{m}_0]$ (see Sect. 10 for further discussion of how the specific choice of $\mathfrak{m}_0$ that we make in our main theorem is tied to the initial data). Let ${}^{(\mathfrak{n})}\breve{W} = \breve{X} + \phi\frac{\mathfrak{n}}{L\upmu}L$ be the vectorfield defined in (4.2), and let $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ be the $\upmu$-adapted torus defined by (4.3c). For $\mathfrak{m} \in [\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]$, we define the **rough time function** ${}^{(\mathfrak{n})}\tau$ to be the solution to the following transport equation initial value problem:[51]

$${}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\tau = 0, \tag{4.4a}$$

$${}^{(\mathfrak{n})}\tau|_{\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}} = -\mathfrak{m} = -\upmu|_{\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}}. \tag{4.4b}$$

We sometimes refer to $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ as a *primal torus* for ${}^{(\mathfrak{n})}\tau$ because ${}^{(\mathfrak{n})}\tau$ is "flowed out" from it.

Remark 4.6 (${}^{(\mathfrak{n})}\breve{W}$ is tangent to the level-sets of ${}^{(\mathfrak{n})}\tau$)**.** Note that equation (4.4a) implies that ${}^{(\mathfrak{n})}\breve{W}$ is tangent to the level-sets of ${}^{(\mathfrak{n})}\tau$.

Remark 4.7 (The regularity of ${}^{(\mathfrak{n})}\tau$)**.** Since ${}^{(\mathfrak{n})}\breve{W}$ depends on $\upmu$, and since $\upmu$ is one degree less differentiable than the fluid wave-variables $\vec{\Psi}$ (because it solves the transport equation (3.44)), it follows that ${}^{(\mathfrak{n})}\tau$ is also less regular than $\vec{\Psi}$; that is why we refer to ${}^{(\mathfrak{n})}\tau$ as the "rough time function." See Sect. 1.3 for further discussion.

Definition 4.8 (The parameter τ_0)**.** We define the parameter $\tau_0 < 0$ as follows:

$$\tau_0 \overset{\text{def}}{=} -\mathfrak{m}_0. \tag{4.5}$$

In the bulk of the paper, portions of the rough hypersurface $\{{}^{(\mathfrak{n})}\tau = \tau_0\}$ will play the role of an "initial data" hypersurface near the singularity. Note that by construction, we have $\tau_0 \leq {}^{(\mathfrak{n})}\tau \leq -\mathfrak{m}_{\text{Boot}} \leq 0$.

4.2.2. *Adapted rough coordinates.* Having constructed the eikonal function u and the rough time function ${}^{(\mathfrak{n})}\tau$, we now define a system of coordinates adapted to them.

Definition 4.9 (The adapted rough coordinates and their partial derivative vectorfields)**.** We call $({}^{(\mathfrak{n})}\tau,u,x^2,x^3)$ the *adapted rough coordinates*. We denote the corresponding adapted rough coordinate partial derivative vectorfields by $\left\{\frac{\widetilde{\partial}}{\widetilde{\partial}{}^{(\mathfrak{n})}\tau},\frac{\widetilde{\partial}}{\widetilde{\partial}u},\frac{\widetilde{\partial}}{\widetilde{\partial}x^2},\frac{\widetilde{\partial}}{\widetilde{\partial}x^3}\right\}$.

[51] See Lemmas 14.2 and 15.1 for the well-posedness theory of this Cauchy problem.

Our analysis will show that the map $(t, x^1, x^2, x^3) \to \left({}^{(\mathfrak{n})}\tau, u, x^2, x^3\right)$ is a homeomorphism – all the way up to the singular boundary – and that it is a diffeomorphism away from the singular boundary. Moreover, the map $(t, u, x^2, x^3) \to \left({}^{(\mathfrak{n})}\tau, u, x^2, x^3\right)$ is a diffeomorphism all the way up to the singular boundary; see Theorems 31.1 and 34.1.

Remark 4.10 (Suppressing the value of $\mathfrak{n}$)**.** The notation $\left\{\frac{\widetilde{\partial}}{\partial u}, \frac{\widetilde{\partial}}{\partial x^2}, \frac{\widetilde{\partial}}{\partial x^3}\right\}$ suppresses the dependence of these operators on $\mathfrak{n}$. Moreover, we often write τ in place of ${}^{(\mathfrak{n})}\tau$ or $\frac{\widetilde{\partial}}{\partial \tau}$ in place of $\frac{\widetilde{\partial}}{\partial^{(\mathfrak{n})}\tau}$ when there is no danger of confusion about the value of $\mathfrak{n}$.

4.3. **Rough subsets.** In this section, we define various subsets of spacetime that are tied to u and ${}^{(\mathfrak{n})}\tau$. Most of the delicate PDE analysis in the bulk of paper will take place on these subsets. Our analysis will show that these subsets are well-adapted to the structure of the singular boundary.

4.3.1. *Truncated* ${}^{(\mathfrak{n})}\tau$*-adapted subsets.*

Definition 4.11 (Truncated ${}^{(\mathfrak{n})}\tau$-adapted subsets)**.** Given intervals $I, J \subset \mathbb{R}$ and real numbers $\tau, u \in \mathbb{R}$, we define:

$$ {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{J} \overset{\text{def}}{=} \{(t, u, x^2, x^3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}^2 \mid {}^{(\mathfrak{n})}\tau(t, u, x^2, x^3) = \tau, \ u \in J\}, \tag{4.6a} $$

$$ {}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u} \overset{\text{def}}{=} \{(t, u, x^2, x^3) \mid (t, x^2, x^3) \in \mathbb{R} \times \mathbb{T}^2, {}^{(\mathfrak{n})}\tau(t, u, x^2, x^3) = \tau\}, \tag{4.6b} $$

$$ {}^{(\mathfrak{n})}\mathcal{P}_{u}^{I} \overset{\text{def}}{=} \bigcup_{\tau' \in I} {}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u}, \tag{4.6c} $$

$$ {}^{(\mathfrak{n})}\mathcal{M}_{I,J} \overset{\text{def}}{=} \bigcup_{\tau' \in I} {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{J} = \bigcup_{u' \in J} {}^{(\mathfrak{n})}\mathcal{P}_{u'}^{I} = \bigcup_{(\tau', u') \in I \times J} {}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u'}. \tag{4.6d} $$

We refer to the ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{J}$ as *rough hypersurfaces*. We sometimes refer to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{J}$ as the *initial rough hypersurface*, where $\tau_0 < 0$ is the parameter from Sect. 4.2.1. We refer to the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ as *rough tori*. We also note that ${}^{(\mathfrak{n})}\mathcal{P}_{u}^{I}$ is a portion of the $\mathbf{g}$-null surface $\mathcal{P}_u$.

From Defs. 4.9 and 4.11, it follows that $\left\{\frac{\widetilde{\partial}}{\partial u}, \frac{\widetilde{\partial}}{\partial x^2}, \frac{\widetilde{\partial}}{\partial x^3}\right\}$ spans the tangent space of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{I}$, that $\left\{\frac{\widetilde{\partial}}{\partial x^2}, \frac{\widetilde{\partial}}{\partial x^3}\right\}$ spans the tangent space of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, that $\left\{\frac{\widetilde{\partial}}{\partial t}, \frac{\widetilde{\partial}}{\partial x^2}, \frac{\widetilde{\partial}}{\partial x^3}\right\}$ spans the tangent space of ${}^{(\mathfrak{n})}\mathcal{P}_{u}^{I}$, and that $\left\{\frac{\widetilde{\partial}}{\partial \tau}, \frac{\widetilde{\partial}}{\partial u}, \frac{\widetilde{\partial}}{\partial x^2}, \frac{\widetilde{\partial}}{\partial x^3}\right\}$ spans the tangent space of ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$.

4.3.2. *Truncated* μ*-adapted subsets.* In our analysis, we will often derive estimates on truncated versions of the level-sets of various functions, which we now define.

Definition 4.12 (Truncated level-sets of μ and $\breve{X}\mu$)**.** Let $I \subset [\tau_0, 0]$ be an interval, let $\mathfrak{m} \in [0, \mathfrak{m}_0]$, and let $\breve{\mathbb{M}}_{\mathfrak{m}}$ and $\breve{\mathbb{X}}_{-\mathfrak{n}}$ be the sets from Def. 4.2. We define:

$$ \breve{\mathbb{M}}_{\mathfrak{m}}^{I} \overset{\text{def}}{=} \breve{\mathbb{M}}_{\mathfrak{m}} \cap \left\{(t, u, x^2, x^3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}^2 \mid {}^{(\mathfrak{n})}\tau(t, u, x^2, x^3) \in I\right\}, \tag{4.7a} $$

$$ \breve{\mathbb{X}}_{-\mathfrak{n}}^{I} \overset{\text{def}}{=} \breve{\mathbb{X}}_{-\mathfrak{n}} \cap \left\{(t, u, x^2, x^3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}^2 \mid {}^{(\mathfrak{n})}\tau(t, u, x^2, x^3) \in I\right\}. \tag{4.7b} $$

Just as in Remark 4.4, we view ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{J}$, ${}^{(\mathfrak{n})}\mathcal{P}_{u}^{I}$, ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$, $\breve{\mathbb{M}}_{\mathfrak{m}}^{I}$, and $\breve{\mathbb{X}}_{-\mathfrak{n}}^{I}$ as sub-manifolds of spacetime equipped with the differential structure induced by the geometric coordinates (t, u, x^2, x^3).

5. Coordinate transformations

To prove our main results, we will have to control the transformations between the Cartesian coordinates (t, x^1, x^2, x^3), the geometric coordinates (t, u, x^2, x^3), the adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$, and a few other coordinate systems whose role will become clear later in the paper. In this section, we define the relevant change of variables maps and derive some basic relationships between the partial derivative vectorfields in the different coordinate systems.

5.1. **Change of variables maps.** In this short section, we define various change of variables maps that we use to prove our main results.

Definition 5.1 (Change of variables maps)**.** We define the change of variables map from geometric to Cartesian coordinates as follows:

$$\Upsilon(t,u,x^2,x^3) \stackrel{\text{def}}{=} (t,x^1,x^2,x^3). \tag{5.1}$$

We define the change of variables map from geometric coordinates to adapted rough coordinates as follows:

$$^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3) \stackrel{\text{def}}{=} (^{(\mathfrak{n})}\tau,u,x^2,x^3). \tag{5.2}$$

We define the map $\breve{\mathscr{M}}$ from geometric coordinates to "$(\mu,\breve{X}\mu,x^2,x^3)$-space" and its Jacobian $^{(\breve{\mathscr{M}})}\mathbf{J}$ as follows:

$$\breve{\mathscr{M}}(t,u,x^2,x^3) \stackrel{\text{def}}{=} (\mu,\breve{X}\mu,x^2,x^3), \tag{5.3a}$$

$$^{(\breve{\mathscr{M}})}\mathbf{J}(t,u,x^2,x^3) \stackrel{\text{def}}{=} \frac{\partial\breve{\mathscr{M}}(t,u,x^2,x^3)}{\partial(t,u,x^2,x^3)} = \frac{\partial(\mu,\breve{X}\mu,x^2,x^3)}{\partial(t,u,x^2,x^3)}. \tag{5.3b}$$

We define the map $^{(\mathfrak{n})}\Phi$ from adapted rough coordinates to "$(\mu,\breve{X}\mu,x^2,x^3)$-space" and its Jacobian $^{(^{(\mathfrak{n})}\Phi)}\mathbf{J}$ as follows:

$$^{(\mathfrak{n})}\Phi(^{(\mathfrak{n})}\tau,u,x^2,x^3) \stackrel{\text{def}}{=} \left(\mu(^{(\mathfrak{n})}\tau,u,x^2,x^3),\breve{X}\mu(^{(\mathfrak{n})}\tau,u,x^2,x^3),x^2,x^3\right), \tag{5.4a}$$

$$^{(^{(\mathfrak{n})}\Phi)}\mathbf{J}(^{(\mathfrak{n})}\tau,u,x^2,x^3) \stackrel{\text{def}}{=} \frac{\partial^{(\mathfrak{n})}\Phi(^{(\mathfrak{n})}\tau,u,x^2,x^3)}{\partial(^{(\mathfrak{n})}\tau,u,x^2,x^3)} = \frac{\partial(\mu,\breve{X}\mu,x^2,x^3)}{\partial(^{(\mathfrak{n})}\tau,u,x^2,x^3)}. \tag{5.4b}$$

We note the following identity:

$$\breve{\mathscr{M}} = {}^{(\mathfrak{n})}\Phi \circ {}^{(\mathfrak{n})}\mathscr{T}. \tag{5.5}$$

Remark 5.2 (Invertibility of the change of variables maps)**.** We justify the invertibility of Υ in Prop. 18.4 and the invertibility of $^{(\mathfrak{n})}\mathscr{T}$ in Lemma 15.5. We justify the *local* invertibility of $^{(\mathfrak{n})}\Phi$ in Lemma 15.7.

Remark 5.3 (Implicit functional dependence)**.** In most of the paper, our convention is that functions and tensorfields should be viewed as depending on the geometric coordinates (t,u,x^2,x^3), *unless we explicitly indicate otherwise*. For example, it is understood that in (5.1)–(5.3b), we are viewing the quantities on the RHSs as functions of (t,u,x^2,x^3). Whenever we make statements relative to the Cartesian coordinates and there is the possibility of confusion, we explicitly indicate the presence of Υ; see, for example, Prop. 33.1.

We now highlight some occasions when we abuse notation that involves functional dependence on the adapted rough coordinates $(^{(\mathfrak{n})}\tau,u,x^2,x^3)$, such as on RHSs (5.4a)–(5.4b).

- Whenever the wave-variables $\vec{\Psi}$, the acoustic geometry variables, etc. are shown to depend on the adapted rough coordinates (e.g., when we write $\mu(^{(\mathfrak{n})}\tau,u,x^2,x^3)$), it should be understood that we are implicitly composing with $^{(\mathfrak{n})}\mathscr{T}^{-1}$, e.g., by writing $\mu(^{(\mathfrak{n})}\tau,u,x^2,x^3)$, we mean $\mu\circ{}^{(\mathfrak{n})}\mathscr{T}^{-1}(^{(\mathfrak{n})}\tau,u,x^2,x^3)$. Put differently, to avoid cluttering the notation, we often avoid explicitly writing the composition with $^{(\mathfrak{n})}\mathscr{T}^{-1}$.
- In view of the previous bullet point, in the language of differential geometry, one can view Lemma 5.8 as describing the pushforward of $\{\frac{\widetilde{\partial}}{\partial\tau},\frac{\widetilde{\partial}}{\partial u},\frac{\widetilde{\partial}}{\partial x^2},\frac{\widetilde{\partial}}{\partial x^3}\}$ by $^{(\mathfrak{n})}\mathscr{T}^{-1}$ in terms of $\{\frac{\partial}{\partial t},\frac{\partial}{\partial u},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\}$. To avoid cluttering the discussion, we do not use the language of pushforwards in this article, except in Prop. 33.2, where we use the notion of a pushforward to carefully address some degeneracies that occur along the singular boundary. Similar remarks apply to Lemma 5.4 (which could be described as identities involving pushforward by Υ).
- We highlight that we define our area and volume forms on the $^{(\mathfrak{n})}\tau$-adapted regions $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, $^{(\mathfrak{n})}\mathcal{P}_u^I$, $^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^I$, and $^{(\mathfrak{n})}\mathcal{M}_{I,J}$ in terms of the adapted rough coordinates (see Def. 8.3), and that we define our L^2-type norms and energies in terms of the adapted rough coordinates (see, for example (8.12a)–(8.12d)). Moreover, in Sects. 24–29, we derive our L^2 estimates in terms of the adapted rough coordinates.
- In Lemmas 22.6 and 22.9, we derive estimates for modified quantities $^{(\mathcal{P}^N)}\mathscr{X}$ and $^{(\mathcal{P}^N)}\widetilde{\mathscr{X}}$ along integral curves of $^{(\mathfrak{n})}\widetilde{L}$ (see (6.3)) in adapted rough coordinates. When stating and deriving these estimates, as is explained in the previous bullet points, it should be understood that we are implicitly composing with $^{(\mathfrak{n})}\mathscr{T}^{-1}$. Similar remarks apply during parts of the proof of Prop. 17.1.

5.2. Coordinate partial derivative transformations.

Lemma 5.4 (Geometric coordinate vectorfields in terms of the Cartesian ones)**.** *The following identities hold, where* $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ *are the geometric coordinate partial derivative vectorfields and* $\{\partial_t, \partial_1, \partial_2, \partial_3\}$ *are the Cartesian coordinate partial derivative vectorfields:*

$$\frac{\partial}{\partial t} = \partial_t + \left\{\frac{L^1X^1 + L^2X^2 + L^3X^3}{X^1}\right\}\partial_1, \qquad \frac{\partial}{\partial u} = \frac{\mu c^2}{X^1}\partial_1, \qquad \frac{\partial}{\partial x^2} = \partial_2 - \frac{X^2}{X^1}\partial_1, \qquad \frac{\partial}{\partial x^3} = \partial_3 - \frac{X^3}{X^1}\partial_1. \tag{5.6}$$

Proof. The identities for $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial x^2}$, and $\frac{\partial}{\partial x^3}$ were proved in [52, Lemma 2.24]. To derive the identity (5.6) for $\frac{\partial}{\partial t}$, we first equate the following two expressions for L: $L = \partial_t + L^1\partial_1 + L^A\partial_A = \frac{\partial}{\partial t} + L^A\frac{\partial}{\partial x^A}$. We then solve for $\frac{\partial}{\partial t}$ to deduce that $\frac{\partial}{\partial t} = \partial_t + L^1\partial_1 + L^A\partial_A - L^A\frac{\partial}{\partial x^A}$. Finally, we use the identities for $\left\{\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ in (5.6) to substitute for the factors of $\frac{\partial}{\partial x^A}$ in the expression $L^A\frac{\partial}{\partial x^A}$. □

The following lemma reveals the relationship between the geometric coordinate partial derivative vectorfields and the commutation vectorfields from Def. 3.8.

Lemma 5.5 (Relationship between $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ and $\{L, X, Y_{(2)}, Y_{(3)}\}$)**.** *The following identities hold:*

$$L = \frac{\partial}{\partial t} + L^A\frac{\partial}{\partial x^A}, \tag{5.7a}$$

$$\breve{X} = \frac{\partial}{\partial u} + \mu X^A\frac{\partial}{\partial x^A}, \tag{5.7b}$$

$$Y_{(2)} = \left\{1 - c^{-2}(X^2)^2\right\}\frac{\partial}{\partial x^2} - c^{-2}X^2X^3\frac{\partial}{\partial x^3}, \tag{5.7c}$$

$$Y_{(3)} = \left\{1 - c^{-2}(X^3)^2\right\}\frac{\partial}{\partial x^3} - c^{-2}X^2X^3\frac{\partial}{\partial x^2}. \tag{5.7d}$$

Moreover, the following identities hold:

$$\frac{\partial}{\partial t} = L - L^AY_{(A)} - \left\{\frac{L^2X^2 + L^3X^3}{(X^1)^2}\right\}X^AY_{(A)}, \tag{5.8a}$$

$$\frac{\partial}{\partial u} = \breve{X} - \frac{1}{(X^1)^2}\mu c^2X^AY_{(A)}, \tag{5.8b}$$

$$\frac{\partial}{\partial x^2} = \left\{\frac{(X^1)^2 + (X^2)^2}{(X^1)^2}\right\}Y_{(2)} + \frac{X^2X^3}{(X^1)^2}Y_{(3)}, \tag{5.8c}$$

$$\frac{\partial}{\partial x^3} = \left\{\frac{(X^1)^2 + (X^3)^2}{(X^1)^2}\right\}Y_{(3)} + \frac{X^2X^3}{(X^1)^2}Y_{(2)}. \tag{5.8d}$$

Proof. The identities (5.7a)–(5.7d) were proved in [52, Lemma 2.23]. The identities (5.8c)–(5.8d) then follow from solving for $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial x^3}$ in (5.7c)–(5.7d) and using (3.36). To derive (5.8b), we solve for $\frac{\partial}{\partial u}$ in (5.7b) and use the expressions (5.8c)–(5.8d) as well as (3.36). To derive (5.8a), we solve for $\frac{\partial}{\partial t}$ in (5.7a) and use the expressions (5.8c)–(5.8d). □

The following lemma is an analog of Lemma 5.5 with the Cartesian coordinate partial derivative vectorfields in place of the geometric ones.

Lemma 5.6 (Relationship between $\{\partial_t, \partial_1, \partial_2, \partial_3\}$ and $\{L, X, Y_{(2)}, Y_{(3)}\}$)**.** *The following identities hold:*

$$\partial_t = L - \frac{L^1X^1 + L^2X^2 + L^3X^3}{c^2}X + \frac{L^1}{X^1}X^AY_{(A)} - L^AY_{(A)}, \tag{5.9a}$$

$$\partial_1 = \frac{X^1}{c^2}X - \frac{1}{X^1}X^AY_{(A)}, \tag{5.9b}$$

$$\partial_2 = \frac{X^2}{c^2}X + Y_{(2)}, \tag{5.9c}$$

$$\partial_3 = \frac{X^3}{c^2}X + Y_{(3)}. \tag{5.9d}$$

Proof. To derive the identities (5.9c)–(5.9d), we use the definition (3.14) of the vectorfields $Y_{(A)}$ and the expression (2.15a) for the acoustical metric in Cartesian coordinates. To prove (5.9b), we first use (5.6) to deduce $\partial_1 = \frac{X^1}{\mu c^2}\frac{\partial}{\partial u}$. We then use (5.8b) to substitute for $\frac{\partial}{\partial u}$ in the RHS of the previous expression, thereby obtaining (5.9b). To prove (5.9a), we first use (5.6) and the already proven (5.9b) to express ∂_t in terms of $\frac{\partial}{\partial t}$ and the commutation vectorfields. To handle the $\frac{\partial}{\partial t}$ term, we first use (5.7a) to express $\frac{\partial}{\partial t} = L - L^A \frac{\partial}{\partial x^A}$ and then use (5.8c)–(5.8d) to substitute for the factors of $\frac{\partial}{\partial x^A}$. □

Corollary 5.7 (Expressions for ${\Pi\mkern-12mu/}_\beta^{\ \alpha}\partial_\alpha$ in terms of $\{Y_{(2)}, Y_{(3)}\}$). *The following identities hold relative to the Cartesian coordinates:*

$$
{\Pi\mkern-12mu/}_0^{\ \alpha}\partial_\alpha = \frac{L^1}{X^1} X^A Y_{(A)} - L^A Y_{(A)}, \tag{5.10}
$$

$$
{\Pi\mkern-12mu/}_1^{\ \alpha}\partial_\alpha = -\frac{1}{X^1} X^A Y_{(A)}, \tag{5.11}
$$

$$
{\Pi\mkern-12mu/}_2^{\ \alpha}\partial_\alpha = Y_{(2)}, \qquad {\Pi\mkern-12mu/}_3^{\ \alpha}\partial_\alpha = Y_{(3)}. \tag{5.12}
$$

Proof. To prove (5.10), we first note that by (3.6b), relative to the Cartesian coordinates, we have $({\Pi\mkern-12mu/}\partial_t)^\alpha = {\Pi\mkern-12mu/}_\beta^{\ \alpha}(\partial_t)^\beta = {\Pi\mkern-12mu/}_0^{\ \alpha}$. Hence, the vectorfield on LHS (5.10) is the $\mathbf{g}$-orthogonal projection of ∂_t onto $\ell_{t,u}$. The identity (5.10) therefore follows from (5.9a). The identities (5.11)–(5.12) follow from similar arguments based on (5.9b)–(5.9d). □

The next lemma reveals the relationship between the geometric coordinate partial derivative vectorfields and the adapted rough coordinate partial derivative vectorfields.

Lemma 5.8 (Relationship between $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}$ and $\{\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}, \frac{\widetilde{\partial}}{\widetilde{\partial}u}, \frac{\widetilde{\partial}}{\widetilde{\partial}x^2}, \frac{\widetilde{\partial}}{\widetilde{\partial}x^3}\}$). *The following identities hold for $A = 2, 3$:*

$$
\frac{\widetilde{\partial}}{\widetilde{\partial}\tau} = \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\frac{\partial}{\partial t}, \tag{5.13a}
$$

$$
\frac{\widetilde{\partial}}{\widetilde{\partial}u} = \frac{\partial}{\partial u} - \frac{\frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\frac{\partial}{\partial t}, \tag{5.13b}
$$

$$
\frac{\widetilde{\partial}}{\widetilde{\partial}x^A} = \frac{\partial}{\partial x^A} - \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\frac{\partial}{\partial t} = \frac{\partial}{\partial x^A} + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} L^B \frac{\partial}{\partial x^B} - \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} L. \tag{5.13c}
$$

Moreover, the following identity holds:

$$
\frac{\partial}{\partial x^A} = \frac{\widetilde{\partial}}{\widetilde{\partial}x^A} - \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{L^{(\mathfrak{n})}\tau} L^B \frac{\widetilde{\partial}}{\widetilde{\partial}x^B} + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{L^{(\mathfrak{n})}\tau} L. \tag{5.14}
$$

Proof. The identity (5.13a) follows from the chain rule identity $\frac{\partial}{\partial t} = (\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau)\frac{\widetilde{\partial}}{\widetilde{\partial}\tau} + (\frac{\partial}{\partial t}u)\frac{\widetilde{\partial}}{\widetilde{\partial}u} + (\frac{\partial}{\partial t}x^A)\frac{\widetilde{\partial}}{\widetilde{\partial}x^A}$ and the fact that $\frac{\partial}{\partial t}u = \frac{\partial}{\partial t}x^A = 0$. The identities (5.13b)–(5.13c) follow from similar arguments and (5.7a).

Finally, with the help of (3.21), it is straightforward to confirm the identity (5.14) by checking that both sides evaluate to the same values when acting on the adapted rough coordinate functions τ, u, x^2, x^3. □

5.3. **An identity for $\frac{\widetilde{\partial}}{\widetilde{\partial}u}$.** We will use the following simple identity in Sect. 18.2, when we study the homeomorphism and diffeomorphism properties of the change of variables map Υ.

Lemma 5.9 (An identity for $\frac{\widetilde{\partial}}{\widetilde{\partial}u}$). *The following identity holds:*

$$
\frac{\widetilde{\partial}}{\widetilde{\partial}u} = \breve{X} + \phi\frac{\mathfrak{n}}{L\mu}L - \left\{\mu X^A + \phi\frac{\mathfrak{n}}{L\mu}L^A\right\}\left\{\frac{\partial}{\partial x^A} + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} L^B\frac{\partial}{\partial x^B} - \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}L\right\}. \tag{5.15}
$$

Proof. First, we use (4.2), (4.4a), and Lemma 3.9 to deduce that ${}^{(\mathfrak{n})}\breve{W}u = 1$ and ${}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\tau = 0$. It follows that $\frac{\widetilde{\partial}}{\widetilde{\partial}u} = {}^{(\mathfrak{n})}\breve{W} - {}^{(\mathfrak{n})}\breve{W}^A\frac{\widetilde{\partial}}{\widetilde{\partial}x^A}$, where as usual, ${}^{(\mathfrak{n})}\breve{W}^A = {}^{(\mathfrak{n})}\breve{W}x^A$. From this identity, (4.2), (5.7a), and (5.13c), we conclude (5.15). □

6. The rough acoustical geometry and curvature tensors

In this section, we set up the acoustical geometry of the rough foliations, that is, the geometry associated to the rough hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$, the characteristics $\mathcal{P}_u$, and their intersection ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. In particular, we define the first fundamental forms induced by the acoustical metric $\mathbf{g}$ on the sub-manifolds ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$, we define various geometric vectorfields tied to these sub-manifolds, we exhibit various geometric decompositions, and we introduce the Riemann and Ricci curvature of the acoustical metric $\mathbf{g}$ and various curvature tensors of the first fundamental form of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$.

Remark 6.1 (Sometimes suppressing dependence on $\mathfrak{n}$)**.** In view of (4.2) and Def. 4.5 for ${}^{(\mathfrak{n})}\tau$, it follows that all our constructions in this section depend on the choice of $\mathfrak{n}$. On the other hand, the constants "C" in our forthcoming estimates can be chosen to independent of $\mathfrak{n}$ for $\mathfrak{n} \in [0, \mathfrak{n}_0]$. This is important because in Sects. 32–34, we will vary $\mathfrak{n} \in [0, \mathfrak{n}_0]$ to obtain a continuum of time functions ${}^{(\mathfrak{n})}\tau$ and their respective foliations, and our results depend on the fact that the "C" can be chosen uniformly with respect to $\mathfrak{n}$. However, to simplify the notation, there are many geometric objects for which we often suppress their dependence on $\mathfrak{n}$ (for example, the tensorfield $\widetilde{g\!\!/}$ introduced below depends on $\mathfrak{n}$). Nonetheless, in some objects, such as ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$, we will retain the explicit $\mathfrak{n}$-dependence to provide the reader mental reminders regarding our constructions.

6.1. First fundamental forms of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$.

We now introduce the first fundamental forms of the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and the rough hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ induced by the acoustical metric $\mathbf{g}$.

Definition 6.2 (The first fundamental forms of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$)**.**

First fundamental form of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$:

- We define the *first fundamental form of* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ relative to $\mathbf{g}$ to be the symmetric type $\binom{0}{2}$ tensorfield $\widetilde{g\!\!/}$ such that $\widetilde{g\!\!/}(Y,Z) = \mathbf{g}(Y,Z)$ for all pairs (Y,Z) of vectorfields tangent to ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and such that $\widetilde{g\!\!/}(V,\cdot) = \widetilde{g\!\!/}(\cdot,V) = 0$ if V is $\mathbf{g}$-orthogonal to ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$.
- We define the *inverse first fundamental form* $\widetilde{g\!\!/}^{-1}$ to be the dual of $\widetilde{g\!\!/}$ relative to $\mathbf{g}$, i.e., relative to arbitrary coordinates, it is the symmetric type $\binom{2}{0}$ tensorfield with the following components:

$$(\widetilde{g\!\!/}^{-1})^{\alpha\beta} \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\gamma}(\mathbf{g}^{-1})^{\beta\delta}\widetilde{g\!\!/}_{\gamma\delta}. \tag{6.1}$$

First fundamental form of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$:

- We define the *first fundamental form of* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ relative to $\mathbf{g}$ to be the symmetric type $\binom{0}{2}$ tensorfield $\widetilde{g}$ such that $\widetilde{g}(Y,Z) = \mathbf{g}(Y,Z)$ for all pairs (Y,Z) of vectorfields tangent to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ and such that $\widetilde{g}(V,\cdot) = \widetilde{g}(\cdot,V) = 0$ if V is $\mathbf{g}$-orthogonal to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$.
- We define the *inverse first fundamental form* $\widetilde{g}^{-1}$ to be the dual of $\widetilde{g}$ relative to $\mathbf{g}$, i.e., relative to arbitrary coordinates, it is the symmetric type $\binom{2}{0}$ tensorfield with the following components:

$$(\widetilde{g}^{-1})^{\alpha\beta} \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\gamma}(\mathbf{g}^{-1})^{\beta\delta}\widetilde{g}_{\gamma\delta}. \tag{6.2}$$

6.2. Geometric vectorfields associated to the rough foliations.

In this section, we define several vectorfields that play a crucial role in our analysis of the rough geometry, and we reveal their basic properties.

We start by introducing the rough null vectorfield ${}^{(\mathfrak{n})}\widetilde{L}$, which is the unique $\mathbf{g}$-null, $\mathcal{P}_u$-tangent vectorfield normalized by ${}^{(\mathfrak{n})}\widetilde{L}\,{}^{(\mathfrak{n})}\tau = 1$, i.e., ${}^{(\mathfrak{n})}\widetilde{L}$ is normalized relative to the rough time function. Roughly speaking, ${}^{(\mathfrak{n})}\widetilde{L}$ plays a similar role that L played in the works [24, 50, 69, 73], which relied on foliations by level-sets of the Cartesian time function. However, ${}^{(\mathfrak{n})}\widetilde{L}$ enjoys less regularity than L, so to obtain top-order estimates, we must commute the equations with L (rather than ${}^{(\mathfrak{n})}\widetilde{L}$).

Definition 6.3 (The rough null vectorfield)**.** We define the *rough null vectorfield* vectorfield ${}^{(\mathfrak{n})}\widetilde{L}$ as follows:

$${}^{(\mathfrak{n})}\widetilde{L} \stackrel{\text{def}}{=} \frac{1}{L\,{}^{(\mathfrak{n})}\tau} L. \tag{6.3}$$

We will often use the following basic property of ${}^{(\mathfrak{n})}\widetilde{L}$, which follows immediately from (6.3):

$${}^{(\mathfrak{n})}\widetilde{L}\,{}^{(\mathfrak{n})}\tau = 1. \tag{6.4}$$

Definition 6.4 (The vectorfields ${}^{(\mathfrak{n})}U$, ${}^{(\mathfrak{n})}\breve{R}$, ${}^{(\mathfrak{n})}\hat{R}$, ${}^{(\mathfrak{n})}\widetilde{N}$, and ${}^{(\mathfrak{n})}\hat{N}$)**.**

- We define ${}^{(\mathfrak{n})}U$ to be the following ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent vectorfield (see Remark 4.6):

$$ {}^{(\mathfrak{n})}U \stackrel{\text{def}}{=} \widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right) \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}. \tag{6.5} $$

- We define ${}^{(\mathfrak{n})}\breve{R}$ to be the following ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$-tangent vectorfield:

$$ {}^{(\mathfrak{n})}\breve{R} \stackrel{\text{def}}{=} {}^{(\mathfrak{n})}\breve{W} - \mu {}^{(\mathfrak{n})}U = \breve{X} + \phi \frac{\mathfrak{n}}{L\mu} L - \mu {}^{(\mathfrak{n})}U, \tag{6.6} $$

where $\phi = \phi(u)$ is the cut-off function introduced in Def. 4.1.

For the solutions featured in our main results, ${}^{(\mathfrak{n})}\breve{R}$ will be $\mathbf{g}$-spacelike, i.e., $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}^2 \stackrel{\text{def}}{=} \mathbf{g}({}^{(\mathfrak{n})}\breve{R}, {}^{(\mathfrak{n})}\breve{R}) > 0$. Moreover, we define ${}^{(\mathfrak{n})}\hat{R}$ to be the $\mathbf{g}$-unit-length rescaling of ${}^{(\mathfrak{n})}\breve{R}$:

$$ {}^{(\mathfrak{n})}\hat{R} \stackrel{\text{def}}{=} \frac{1}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}} {}^{(\mathfrak{n})}\breve{R}. \tag{6.7} $$

- We define ${}^{(\mathfrak{n})}\widetilde{N}$ to be the following vectorfield:

$$ {}^{(\mathfrak{n})}\widetilde{N} \stackrel{\text{def}}{=} L + \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}^2} {}^{(\mathfrak{n})}\breve{R}. \tag{6.8} $$

For the solutions featured in our main results, ${}^{(\mathfrak{n})}\widetilde{N}$ will be $\mathbf{g}$-timelike, i.e., $\mathbf{g}({}^{(\mathfrak{n})}\widetilde{N}, {}^{(\mathfrak{n})}\widetilde{N}) < 0$. Moreover, we define ${}^{(\mathfrak{n})}\hat{N}$ to be the $\mathbf{g}$-unit-length rescaling of ${}^{(\mathfrak{n})}\widetilde{N}$:

$$ {}^{(\mathfrak{n})}\hat{N} \stackrel{\text{def}}{=} \frac{1}{\sqrt{-\mathbf{g}({}^{(\mathfrak{n})}\widetilde{N}, {}^{(\mathfrak{n})}\widetilde{N})}} {}^{(\mathfrak{n})}\widetilde{N}. \tag{6.9} $$

6.3. **Identities involving the first fundamental forms and geometric vectorfields.**

6.3.1. *Identities involving* $\widetilde{g\!\!\!/}$, $\widetilde{g\!\!\!/}^{-1}$, *and* $g\!\!\!/$.

Lemma 6.5 (Identities for $\widetilde{g\!\!\!/}$ and $\widetilde{g\!\!\!/}^{-1}$)**.** *When restricted to the tangent space of* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, *we have the following identity for* $\widetilde{g\!\!\!/}$ *relative to the adapted rough coordinates:*

$$ \widetilde{g\!\!\!/} = \widetilde{g\!\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right) \mathrm{d}x^A \otimes \mathrm{d}x^B, \tag{6.10} $$

where with $g\!\!\!/_{AB}$ *as in* (3.31a), *we have:*

$$ \widetilde{g\!\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right) = g\!\!\!/_{AB} + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} g\!\!\!/_{BC} L^C + \frac{\frac{\partial}{\partial x^B}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} g\!\!\!/_{AC} L^C + \frac{(\frac{\partial}{\partial x^A}\tau)\frac{\partial}{\partial x^B}\tau}{(\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau)^2} g\!\!\!/_{CD} L^C L^D. \tag{6.11} $$

Moreover, relative to the adapted rough coordinates, the following identity holds:

$$ \widetilde{g\!\!\!/}^{-1} = \widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right) \frac{\widetilde{\partial}}{\widetilde{\partial} x^A} \otimes \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}, \tag{6.12} $$

where:

$$ \widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^C\right) \widetilde{g\!\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^C}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right) = \delta_B^A, \tag{6.13} $$

and δ_B^A *is the Kronecker delta.*

Proof. (6.10) is a simple consequence of the fact that (x^2, x^3) are coordinates on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. The identity (6.11) follows from the fact that $\widetilde{g\!\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right) = \mathbf{g}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right)$, (5.13c), and the fact that $\mathbf{g}(L, L) = \mathbf{g}(L, \frac{\partial}{\partial x^A}) = 0$. The identity (6.12) is a simple consequence of the fact that the rough coordinate vectorfields $\left\{\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\right\}_{A=2,3}$ span the tangent space of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. Next, we note that it is straightforward to check, using (6.1), that the type $\binom{1}{1}$ tensorfield with components $(\widetilde{g\!\!\!/}^{-1})^{\alpha\gamma}\widetilde{g\!\!\!/}_{\gamma\beta}$ is the

$\mathbf{g}$-orthogonal projection tensorfield onto ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ (see also (6.28)). From this fact and the fact that (x^2, x^3) are coordinates on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, the identity (6.13) readily follows. □

Lemma 6.6 (Relationship between components $\not{g}$ and $\widetilde{\not{g}}$)**.** *We define* $\mathbf{A}$ *to be the* 2×2 *matrix with the following entries:*

$$\mathbf{A}_A^B \overset{def}{=} \delta_A^B - \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{L^{(\mathfrak{n})}\tau} L^B, \tag{6.14}$$

where δ_A^B *is the Kronecker delta. Then the following identity holds:*

$$\frac{\partial}{\partial x^A} = \mathbf{A}_A^B \frac{\widetilde{\partial}}{\widetilde{\partial} x^B} + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{L^{(\mathfrak{n})}\tau} L. \tag{6.15}$$

Moreover, recalling that $\not{g}$ *is the first fundamental form of* $\ell_{t,u}$*, we have the following relationship between* $\not{g}_{AB} \overset{def}{=} \not{g}(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B})$ *and* $\widetilde{\not{g}}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right)$*:*

$$\not{g}_{AB} = \mathbf{A}_A^C \mathbf{A}_B^D \widetilde{\not{g}}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^C}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^D}\right), \tag{6.16}$$

$$(\not{g}^{-1})^{AB} = (\mathbf{A}^{-1})_C^A (\mathbf{A}^{-1})_D^B \widetilde{\not{g}}^{-1}\left(\mathrm{d}x^C, \mathrm{d}x^D\right). \tag{6.17}$$

In addition, the inverse $(\mathbf{A}^{-1})_A^B$ *of* $\mathbf{A}_B^A$*, defined by* $(\mathbf{A}^{-1})_B^C \mathbf{A}_C^A = \delta_B^A$*, can be expressed as follows:*

$$(\mathbf{A}^{-1})_A^B = \delta_A^B + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} L^B. \tag{6.18}$$

Finally, we have the following identity:

$$\widetilde{\not{g}}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right) = (\mathbf{A}^{-1})_A^C (\mathbf{A}^{-1})_B^D \not{g}_{CD}. \tag{6.19}$$

Proof. The identity (6.15) is a restatement of (5.14). To derive (6.18), we note that (6.15) implies that $\frac{\widetilde{\partial}}{\widetilde{\partial} x^A} = (\mathbf{A}^{-1})_A^B \frac{\partial}{\partial x^B} + fL$ for some scalar function f. We then note that the coefficients $(\mathbf{A}^{-1})_A^B$ are given by (5.13c). Next, using (6.18), we see that (6.19) follows from (6.11). (6.16) follows from applying two factors of $\mathbf{A}$ to each side of (6.19). (6.17) follows from taking the inverse of (6.16) and using (6.13). □

6.3.2. *Properties of the geometric vectorfields associated to the rough foliations.*

Proposition 6.7 (Properties of the geometric vectorfields associated to the rough foliations)**.** *The vectorfield* ${}^{(\mathfrak{n})}\breve{R}$ *defined in* (6.6) *is* $\mathbf{g}$*-orthogonal to* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$*, i.e.,* $\mathbf{g}$*-orthogonal to the elements of* $\left\{\frac{\partial}{\partial x^C}\right\}_{C=2,3}$*. Moreover, its square norm as measured by* $\mathbf{g}$ *(which is the same as its square norm as measured by* $\widetilde{g}$*) satisfies:*

$$|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}^2 = |{}^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}^2 = \mu^2(1 - {}^{(\mathfrak{n})}r) - 2\phi \frac{\mathfrak{n}\mu}{L\mu}, \tag{6.20a}$$

where $\phi = \phi(u)$ *is the cut-off function from Def. 4.1, and* ${}^{(\mathfrak{n})}r \geq 0$ *satisfies:*

$${}^{(\mathfrak{n})}r \overset{def}{=} \frac{\widetilde{\not{g}}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right)(\frac{\partial}{\partial x^A}\tau)\frac{\partial}{\partial x^B}\tau}{(\frac{\partial}{\partial t}\tau)^2} = |{}^{(\mathfrak{n})}U|_{\mathbf{g}}^2. \tag{6.20b}$$

In particular, if $\mu > 0$*,* ${}^{(\mathfrak{n})}r < 1$*, and* $L\mu < 0$ *on the support of* ϕ *(all of which are satisfied for the solutions featured in our main results), then* ${}^{(\mathfrak{n})}\breve{R}$ *is* $\mathbf{g}$*-spacelike, and the vectorfield* ${}^{(\mathfrak{n})}\hat{R}$ *defined in* (6.7) *is the* $\mathbf{g}$*-unit normal to* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ *in* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$*.*

In addition, the vectorfield ${}^{(\mathfrak{n})}\widetilde{N}$ *defined in* (6.8) *is* $\mathbf{g}$*-normal to* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$*, i.e.* $\mathbf{g}$*-orthogonal to* $\left\{{}^{(\mathfrak{n})}\breve{R}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^2}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^3}\right\}$*. Moreover, its square size as measured by* $\mathbf{g}$ *satisfies:*

$$\mathbf{g}({}^{(\mathfrak{n})}\widetilde{N}, {}^{(\mathfrak{n})}\widetilde{N}) = -\frac{\mu^2}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}^2}. \tag{6.20c}$$

In particular, in the solution regime under study, in which $\upmu > 0$ *and* $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}} > 0$, ${}^{(\mathfrak{n})}\widetilde{N}$ *is* $\mathbf{g}$*-timelike, and the vectorfield* ${}^{(\mathfrak{n})}\hat{N}$ *defined in* (6.9) *is the future directed* $\mathbf{g}$*-unit normal to* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$*, which is consequently* $\mathbf{g}$***-spacelike****.*

Finally, ${}^{(\mathfrak{n})}\hat{N}$ *admits the following decomposition:*

$$
{}^{(\mathfrak{n})}\hat{N} = \frac{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}{\upmu} L + {}^{(\mathfrak{n})}\hat{R}. \tag{6.20d}
$$

Proof. We first prove the statements regarding ${}^{(\mathfrak{n})}\breve{R}$. To derive the orthogonality properties of ${}^{(\mathfrak{n})}\breve{R}$, we first use definitions (6.5)–(6.6), the second identity in (5.13c), and the fact that $\mathbf{g}\left(L, \frac{\widetilde{\partial}}{\partial x^C}\right) = 0$ to obtain:

$$
\mathbf{g}\left({}^{(\mathfrak{n})}\breve{R}, \frac{\widetilde{\partial}}{\partial x^C}\right) = \mathbf{g}\left(\breve{X}, \frac{\partial}{\partial x^C} - \frac{\frac{\partial}{\partial x^C}\tau}{\frac{\partial}{\partial t}\tau} L + \frac{\frac{\partial}{\partial x^C}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} L^D \frac{\partial}{\partial x^D}\right) - \mathbf{g}\left(\upmu \widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right) \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}\tau} \frac{\widetilde{\partial}}{\partial x^B}, \frac{\widetilde{\partial}}{\partial x^C}\right). \tag{6.21}
$$

Since $\breve{X}$ is $\mathbf{g}$-orthogonal to the elements of $\left\{\frac{\partial}{\partial x^C}\right\}_{C=2,3}$, since $\mathbf{g}(\breve{X}, L) = -\upmu$, and since (6.13) implies that:

$$
\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right)\mathbf{g}\left(\frac{\widetilde{\partial}}{\partial x^B}, \frac{\widetilde{\partial}}{\partial x^C}\right) = \widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right)\widetilde{g\!\!\!/}\left(\frac{\widetilde{\partial}}{\partial x^B}, \frac{\widetilde{\partial}}{\partial x^C}\right) = \delta_C^A, \tag{6.22}
$$

we conclude that RHS (6.21) = 0, i.e., that ${}^{(\mathfrak{n})}\breve{R}$ is $\mathbf{g}$-orthogonal to ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$.

To prove (6.20a)–(6.20b), we first use (6.6), the $\mathbf{g}$-orthogonality of ${}^{(\mathfrak{n})}\breve{R}$ to ${}^{(\mathfrak{n})}U$, and the relations $\mathbf{g}(\breve{X}, \breve{X}) = \upmu^2$, $\mathbf{g}(\breve{X}, L) = -\upmu$, and $\mathbf{g}(L, L) = 0$ to compute that $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}^2 + |{}^{(\mathfrak{n})}U|_{\mathbf{g}}^2 = \upmu^2 - 2\phi \frac{\mathfrak{n}\upmu}{L\upmu}$. Moreover, using definition (6.5) and (6.22), we compute that $|{}^{(\mathfrak{n})}U|_{\mathbf{g}}^2 = \frac{\widetilde{g\!\!\!/}^{-1}(\mathrm{d}x^A, \mathrm{d}x^B)(\frac{\partial}{\partial x^A}\tau)\frac{\partial}{\partial x^B}\tau}{(\frac{\partial}{\partial t}\tau)^2}$. Combining these calculations, we conclude (6.20a)–(6.20b).

Next, in view of definition (6.8), we note that the vectorfield ${}^{(\mathfrak{n})}\widetilde{N}$ is $\mathbf{g}$-orthogonal to $\left\{\frac{\widetilde{\partial}}{\partial x^C}\right\}_{C=2,3}$ because both L and ${}^{(\mathfrak{n})}\breve{R}$ are. Furthermore, we note that the relation $\mathbf{g}({}^{(\mathfrak{n})}\widetilde{N}, {}^{(\mathfrak{n})}\breve{R}) = 0$ follows easily from definition (6.8) and the relations $\mathbf{g}(L, L) = 0$ and $\mathbf{g}(L, {}^{(\mathfrak{n})}\breve{R}) = -\upmu$. Hence, ${}^{(\mathfrak{n})}\widetilde{N}$ is $\mathbf{g}$-orthogonal to the set $\left\{{}^{(\mathfrak{n})}\breve{R}, \frac{\widetilde{\partial}}{\partial x^2}, \frac{\widetilde{\partial}}{\partial x^3}\right\}$, which spans the tangent space of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ (see Remark 4.6). Therefore, ${}^{(\mathfrak{n})}\widetilde{N}$ is $\mathbf{g}$-orthogonal to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$.

Similarly, (6.20c) follows from definition (6.8) and the relations $\mathbf{g}(L, L) = 0$ and $\mathbf{g}(L, {}^{(\mathfrak{n})}\breve{R}) = -\upmu$.

Finally, (6.20d) follows from definitions (6.7) and (6.8)–(6.9) and (6.20c). □

6.3.3. *Decompositions of* $\mathbf{g}^{-1}$ *and* $\widetilde{g}^{-1}$.

Corollary 6.8 (Decompositions of $\mathbf{g}^{-1}$ and $\widetilde{g}^{-1}$)**.** *The inverse acoustical metric* $\mathbf{g}^{-1}$ *and the inverse first fundamental form* $\widetilde{g}^{-1}$ *of* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ *from Def. 6.2 can be expressed as follows relative to the vectorfields* ${}^{(\mathfrak{n})}\hat{N}$ *and* ${}^{(\mathfrak{n})}\hat{R}$ *from Def. 6.4 and the inverse first fundamental form* $\widetilde{g\!\!\!/}^{-1}$ *of* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ *from Def. 6.2:*

$$
\mathbf{g}^{-1} = -{}^{(\mathfrak{n})}\hat{N} \otimes {}^{(\mathfrak{n})}\hat{N} + {}^{(\mathfrak{n})}\hat{R} \otimes {}^{(\mathfrak{n})}\hat{R} + \widetilde{g\!\!\!/}^{-1}, \tag{6.23a}
$$

$$
\widetilde{g}^{-1} = {}^{(\mathfrak{n})}\hat{R} \otimes {}^{(\mathfrak{n})}\hat{R} + \widetilde{g\!\!\!/}^{-1}. \tag{6.23b}
$$

Proof. (6.23a)–(6.23b) are straightforward consequences of Prop. 6.7. □

6.4. **The pointwise semi-norms of tensors with respect to $\widetilde{g\!\!\!/}$ and the $\widetilde{g\!\!\!/}$-trace.**

Definition 6.9 (Pointwise norms)**.** If ξ is a type $\binom{m}{n}$ tensorfield, then we define $|\xi|_{\widetilde{g\!\!\!/}} \geq 0$ by:

$$
|\xi|_{\widetilde{g\!\!\!/}}^2 \stackrel{\text{def}}{=} \widetilde{g\!\!\!/}_{\alpha_1\widetilde{\alpha}_1} \cdots \widetilde{g\!\!\!/}_{\alpha_m\widetilde{\alpha}_m} (\widetilde{g\!\!\!/}^{-1})^{\beta_1\widetilde{\beta}_1} \cdots (\widetilde{g\!\!\!/}^{-1})^{\beta_n\widetilde{\beta}_n} \xi_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_m} \xi_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}. \tag{6.24}
$$

Definition 6.10 ($\widetilde{g\!\!\!/}$-trace)**.** If ξ is a type $\binom{0}{2}$ tensorfield, then we define its $\widetilde{g\!\!\!/}$-trace $\mathrm{tr}_{\widetilde{g\!\!\!/}}\xi$ as follows:

$$
\mathrm{tr}_{\widetilde{g\!\!\!/}}\xi \stackrel{\text{def}}{=} (\widetilde{g\!\!\!/}^{-1})^{\alpha\beta} \xi_{\alpha\beta}. \tag{6.25}
$$

6.5. **g-orthogonal projection onto the rough tori and $\widetilde{\not{d}}$.**

Definition 6.11 ($\mathbf{g}$-orthogonal projection onto the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangency)**.**

1. We define the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-projection tensorfield $\widetilde{\not{\Pi}}$ as follows, where ${}^{(\mathfrak{n})}\hat{N}$ and ${}^{(\mathfrak{n})}\hat{R}$ are the vectorfields from Def. 6.4 and δ_β^α denotes the Kronecker delta:

$$\widetilde{\not{\Pi}}_\beta^{\ \alpha} \stackrel{\text{def}}{=} \delta_\beta^\alpha + {}^{(\mathfrak{n})}\hat{N}_\beta {}^{(\mathfrak{n})}\hat{N}^\alpha - {}^{(\mathfrak{n})}\hat{R}_\alpha {}^{(\mathfrak{n})}\hat{R}^\beta. \tag{6.26}$$

2. Given any type $\binom{m}{n}$ spacetime tensorfield ξ, we define its $\mathbf{g}$-orthogonal projection onto ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, denoted by $\widetilde{\not{\Pi}}\xi$, as follows:

$$(\widetilde{\not{\Pi}}\xi)_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_m} \stackrel{\text{def}}{=} \widetilde{\not{\Pi}}_{\widetilde{\alpha}_1}^{\ \alpha_1}\cdots\widetilde{\not{\Pi}}_{\widetilde{\alpha}_m}^{\ \alpha_m}\widetilde{\not{\Pi}}_{\beta_1}^{\ \widetilde{\beta}_1}\cdots\widetilde{\not{\Pi}}_{\beta_n}^{\ \widetilde{\beta}_n}\xi_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}. \tag{6.27}$$

3. We say that a spacetime tensorfield ξ is ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent if $\widetilde{\not{\Pi}}\xi = \xi$.

With the help of Lemma 3.9 and Prop. 6.7, it is straightforward to check that $\widetilde{\not{\Pi}}{}^{(\mathfrak{n})}\hat{N} = \widetilde{\not{\Pi}}{}^{(\mathfrak{n})}\hat{R} = \widetilde{\not{\Pi}}L = 0$, while if Z is ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent, then $\widetilde{\not{\Pi}}Z = Z$. That is, $\widetilde{\not{\Pi}}$ *is* the $\mathbf{g}$-orthogonal projection onto ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. Moreover, with the help of (6.23a), it is straightforward to check that the first fundamental form $\widetilde{\not{g}}$ of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ from Def. 6.2 satisfies:

$$\widetilde{\not{g}} = \widetilde{\not{\Pi}}\mathbf{g}. \tag{6.28}$$

Definition 6.12 (${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-differential)**.** Let φ be a scalar function. We define $\widetilde{\not{d}}\varphi$ to be the following ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent one-form:

$$\widetilde{\not{d}}\varphi \stackrel{\text{def}}{=} \widetilde{\not{\Pi}}\mathrm{d}\varphi. \tag{6.29}$$

Note that $[\widetilde{\not{d}}\varphi]\left(\frac{\widetilde{\partial}}{\partial x^A}\right) = \frac{\widetilde{\partial}}{\partial x^A}\varphi$ for $A = 2, 3$.

6.6. **The Levi-Civita connection $\widetilde{\not{\nabla}}$ of $\widetilde{\not{g}}$ and related differential operators.**

Definition 6.13 (The Levi-Civita connection $\widetilde{\not{\nabla}}$ of $\widetilde{\not{g}}$ and related differential operators)**.**

1. We denote the Levi-Civita connection of $\widetilde{\not{g}}$ by $\widetilde{\not{\nabla}}$. In particular, for ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfields ξ, we have $\widetilde{\not{\nabla}}\xi = \widetilde{\not{\Pi}}\mathbf{D}\xi$.
2. If ξ is an ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent one-form, then we define its ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-divergence to be the scalar function $\widetilde{\mathrm{d}\not{i}\mathrm{v}}\,\xi \stackrel{\text{def}}{=} \widetilde{\not{g}}^{-1}\cdot\widetilde{\not{\nabla}}\xi$. Similarly, if V is an ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent vectorfield, then we define its ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-divergence to be the scalar function $\widetilde{\mathrm{d}\not{i}\mathrm{v}}\,V \stackrel{\text{def}}{=} \not{g}^{-1}\cdot\widetilde{\not{\nabla}}V_\flat$, where $V_\flat$ is the one-form that is $\mathbf{g}$-dual to V.
3. If ξ is a symmetric type $\binom{0}{2}$ ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfield, then we define its ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-divergence $\widetilde{\mathrm{d}\not{i}\mathrm{v}}\,\xi$ to be the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent one-form with the following ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-components for $A = 2, 3$:

$$[\widetilde{\mathrm{d}\not{i}\mathrm{v}}\,\xi]\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\right) \stackrel{\text{def}}{=} (\widetilde{\not{g}}^{-1})(\mathrm{d}x^B, \mathrm{d}x^C)\left[\widetilde{\not{\nabla}}_{\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}}\xi\right]\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^C}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\right). \tag{6.30}$$

6.7. **Curvature tensors of $\mathbf{g}$ and $\widetilde{\not{g}}$.** The Riemann curvature tensors of $\mathbf{g}$ and $\widetilde{\not{g}}$ play a central role in the geometric analysis of the acoustical geometry.

Definition 6.14 (Curvature tensors of $\mathbf{g}$ and $\widetilde{\not{g}}$)**.** The *Riemann curvature tensor* $\mathbf{Riem}$ of the acoustical metric $\mathbf{g}$ is the type $\binom{0}{4}$ spacetime tensorfield defined by:

$$\mathbf{Riem}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) \stackrel{\text{def}}{=} \mathbf{g}(-\mathbf{D}_{\mathbf{XY}}^2\mathbf{Z} + \mathbf{D}_{\mathbf{YX}}^2\mathbf{Z}, \mathbf{W}), \tag{6.31}$$

where $\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}$ are arbitrary spacetime vectorfields, and $\mathbf{D}_{\mathbf{XY}}^2\mathbf{Z} \stackrel{\text{def}}{=} \mathbf{X}^\alpha\mathbf{Y}^\beta\mathbf{D}_\alpha\mathbf{D}_\beta\mathbf{Z}$.

The *Ricci curvature tensor* $\mathbf{Ric}$ of the acoustical metric $\mathbf{g}$ is the type $\binom{0}{2}$ spacetime tensor defined relative to arbitrary coordinates as follows:

$$\mathbf{Ric}_{\alpha\beta} \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\kappa\lambda}\mathbf{Riem}_{\alpha\kappa\beta\lambda}. \tag{6.32}$$

Similarly, the Riemann curvature tensor $\widetilde{\mathfrak{Riem}}$ of the Riemannian metric $\widetilde{g\!\!/}$ on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ is the type $\binom{0}{4}$ ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ tensorfield defined as follows:

$$\widetilde{\mathfrak{Riem}}(X,Y,Z,W) \stackrel{\text{def}}{=} \widetilde{g\!\!/}(-\widetilde{\nabla\!\!\!/}^2_{XY}Z + \widetilde{\nabla\!\!\!/}^2_{YX}Z, W), \tag{6.33}$$

where X,Y,Z,W are arbitrary ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent vectorfields and $\widetilde{\nabla\!\!\!/}$ is the Levi-Civita connection of $\widetilde{g\!\!/}$.

The *Ricci curvature tensor* $\widetilde{\mathfrak{Ric}}$ of $\widetilde{g\!\!/}$ is the type $\binom{0}{2}$ spacetime tensor defined relative to arbitrary coordinates as follows:

$$\widetilde{\mathfrak{Ric}}_{\alpha\beta} \stackrel{\text{def}}{=} (\widetilde{g\!\!/}^{-1})^{\kappa\lambda}\widetilde{\mathfrak{Riem}}_{\alpha\kappa\beta\lambda}. \tag{6.34}$$

The *scalar curvature* $\widetilde{\mathfrak{R}}$ of $\widetilde{g\!\!/}$ is the scalar function defined relative to arbitrary coordinates as follows:

$$\widetilde{\mathfrak{R}} \stackrel{\text{def}}{=} (\widetilde{g\!\!/}^{-1})^{\alpha\beta}\widetilde{\mathfrak{Ric}}_{\alpha\beta}. \tag{6.35}$$

It is well-known that because ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ is two-dimensional, the *Gauss curvature* $\widetilde{\mathfrak{K}}$ of $\widetilde{g\!\!/}$ can be expressed as follows in terms of it scalar curvature:

$$\widetilde{\mathfrak{K}} = \frac{1}{2}\widetilde{\mathfrak{R}}. \tag{6.36}$$

7. The acoustic double-null frame and its relationship with the rough acoustical geometry

To control the top-order derivatives of Ω and S, we will rely on a family of "elliptic-hyperbolic" integral identities that we derive in Sect. 21. In this section, we construct the acoustic double-null frame that we use to derive the elliptic-hyperbolic identities. Moreover, we provide various identities that relate the acoustic double-null frame to the rough acoustical geometry constructed in Sect. 6.

7.1. **The acoustic double-null frame.** The new ingredient in the acoustic double-null frame is the vectorfield $\underline{L}$, which we now define. In Lemma 7.3, we will show that $\underline{L}$ is $\mathbf{g}$-null and transversal to the characteristics $\mathcal{P}_u$.

Definition 7.1 (The vectorfield $\underline{L}$)**.** We define $\underline{L}$ to be the vectorfield whose Cartesian components are:

$$\underline{L}^{\alpha} \stackrel{\text{def}}{=} L^{\alpha} + 2X^{\alpha}. \tag{7.1}$$

Remark 7.2 (The role of $\underline{L}$)**.** In this paper, we use $\underline{L}$ only to construct a sufficiently smooth projection operator $\overline{\Pi}_{\beta}^{\ \alpha}$ onto the characteristics $\mathcal{P}_u$; see Def. 21.1. In particular, we do not derive estimates for the solution's $\underline{L}$ derivatives, and we never need to integrate along the integral curves of $\underline{L}$ to derive our estimates. We use $\overline{\Pi}_{\beta}^{\ \alpha}$ as a tool in deriving the elliptic-hyperbolic integral identities that we use to control the top-order derivatives of the specific vorticity and entropy gradient; see Sect. 21.

Lemma 7.3 (Basic properties of $\underline{L}$)**.** *The vectorfields $\underline{L}$ and $\mathbf{B}$ satisfy the following identities:*

$$\underline{L} = \mathbf{B} + X, \qquad \mathbf{B} = \frac{1}{2}(L + \underline{L}) \tag{7.2}$$

Moreover, $\underline{L}$ is $\mathbf{g}$-null and transversal to the characteristics $\mathcal{P}_u$:

$$\mathbf{g}(\underline{L},\underline{L}) = 0, \tag{7.3}$$

$$\mathbf{g}(\underline{L},L) = -2. \tag{7.4}$$

In addition, the acoustical metric $\mathbf{g}$ and its inverse $\mathbf{g}^{-1}$ satisfy the following identities, where $g\!\!/$ is the first fundamental form of $\ell_{t,u}$ from Def. 3.4:

$$\mathbf{g}_{\alpha\beta} = -\frac{1}{2}L_{\alpha}\underline{L}_{\beta} - \frac{1}{2}\underline{L}_{\alpha}L_{\beta} + g\!\!/_{\alpha\beta}, \qquad (\mathbf{g}^{-1})^{\alpha\beta} = -\frac{1}{2}L^{\alpha}\underline{L}^{\beta} - \frac{1}{2}\underline{L}^{\alpha}L^{\beta} + (g\!\!/^{-1})^{\alpha\beta}. \tag{7.5}$$

Finally, we have the following identity, where δ_{β}^{α} is the Kronecker delta and $\Pi\!\!\!/$ is the $\ell_{t,u}$-projection tensorfield from Def. 3.3:

$$\delta_{\beta}^{\alpha} = -\frac{1}{2}L^{\alpha}\underline{L}_{\beta} - \frac{1}{2}\underline{L}^{\alpha}L_{\beta}\beta + \Pi\!\!\!/_{\beta}^{\ \alpha}. \tag{7.6}$$

Proof. Equations (7.2)–(7.4) are straightforward consequences of the definition of $\underline{L}$ in (7.1), the identity $\mathbf{B} = L + X$ (see (3.24)), and the identities $\mathbf{g}(L, L) = 0$, $\mathbf{g}(L, X) = -1$, and $\mathbf{g}(X, X) = 1$ from Lemma 3.9. The statement that $\underline{L}$ is transversal to $\mathcal{P}_u$ follows from (7.4) and the fact that L is $\mathbf{g}$-orthogonal to $\mathcal{P}_u$. The identity for $\mathbf{g}_{\alpha\beta}$ in (7.5) follows from contracting both sides of the identities against the frame $\left\{L, \underline{L}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ and computing that both sides are equal. The identity for $(\mathbf{g}^{-1})^{\alpha\beta}$ in (7.5) follows from raising the indices in the first identity with $\mathbf{g}^{-1}$. The identity (7.6) follows from using $\mathbf{g}$ to lower the β index in the last identity stated in (7.5) and using (3.34c). □

Definition 7.4 (The acoustic double-null frame)**.** We refer to $\left\{L, \underline{L}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ as the *acoustic double-null frame.*

7.2. **Identities involving the acoustic double-null frame and the rough geometry.** The following lemma provides several identities involving the acoustic double-null frame and the rough acoustical geometry.

Lemma 7.5 (Identities involving the acoustic double-null frame and the rough acoustic geometry)**.** *Let* ${}^{(\mathfrak{n})}r$ *be as in* (6.20b), *and recall that* $\nabla\!\!\!/$ *is the Levi-Civita connection of* $g\!\!\!/$, *the first fundamental form of* $\ell_{t,u} = \Sigma_t \cap \mathcal{P}_u$ *relative to* $\mathbf{g}$. *Then the following identity holds:*

$$
{}^{(\mathfrak{n})}r = \frac{|\nabla\!\!\!/ {}^{(\mathfrak{n})}\tau|^2_{g\!\!\!/}}{(L{}^{(\mathfrak{n})}\tau)^2}. \tag{7.7}
$$

Moreover, the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$*-tangent vectorfield* ${}^{(\mathfrak{n})}U$ *defined in* (6.5) *admits the following decomposition, where we recall that the* $\ell_{t,u}$*-tangent vectorfield* $\nabla\!\!\!/^{\#}{}^{(\mathfrak{n})}\tau$ *is the dual of* $\nabla\!\!\!/ {}^{(\mathfrak{n})}\tau$ *with respect to* $\mathbf{g}$.*:*

$$
{}^{(\mathfrak{n})}U = -{}^{(\mathfrak{n})}rL + \frac{1}{L{}^{(\mathfrak{n})}\tau}\nabla\!\!\!/^{\#}{}^{(\mathfrak{n})}\tau. \tag{7.8}
$$

In addition, with ${}^{(\mathfrak{n})}\breve{R}$ *the vectorfield defined in* (6.6), *the vectorfield* $\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}$ *admits the following decomposition:*

$$
\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R} = X + \left(\phi\frac{\mathfrak{n}}{\mu L\mu} + {}^{(\mathfrak{n})}r\right)L - \frac{1}{L{}^{(\mathfrak{n})}\tau}\nabla\!\!\!/^{\#}{}^{(\mathfrak{n})}\tau. \tag{7.9}
$$

Furthermore, the following differentiation identities hold:

$$
{}^{(\mathfrak{n})}\breve{R}\left[\frac{1}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}\right] = -\frac{{}^{(\mathfrak{n})}\breve{R}\mu}{(\mu - \phi\frac{\mathfrak{n}}{L\mu})^2} + \frac{\mathfrak{n}\frac{\phi'}{L\mu}}{(\mu - \phi\frac{\mathfrak{n}}{L\mu})^2} - \frac{\phi\mathfrak{n}\frac{{}^{(\mathfrak{n})}\breve{R}L\mu}{(L\mu)^2}}{(\mu - \phi\frac{\mathfrak{n}}{L\mu})^2}, \tag{7.10}
$$

$$
{}^{(\mathfrak{n})}U\left[\frac{\mu}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}\right] = -\frac{({}^{(\mathfrak{n})}U\mu)\phi\frac{\mathfrak{n}}{L\mu}}{(\mu - \phi\frac{\mathfrak{n}}{L\mu})^2} - \frac{\mu\phi\mathfrak{n}\frac{{}^{(\mathfrak{n})}UL\mu}{(L\mu)^2}}{(\mu - \phi\frac{\mathfrak{n}}{L\mu})^2}. \tag{7.11}
$$

In addition, we have the following decompositions for L *and* $\underline{L}$*:*

$$
L = \frac{\mu}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B} - \frac{1}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}{}^{(\mathfrak{n})}\breve{R} - \frac{\mu}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}{}^{(\mathfrak{n})}U, \tag{7.12a}
$$

$$
\underline{L} = \frac{\mu - \phi\frac{2\mathfrak{n}}{L\mu}}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B} + \frac{1}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}{}^{(\mathfrak{n})}\breve{R} + \frac{\mu}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}{}^{(\mathfrak{n})}U. \tag{7.12b}
$$

Moreover, the following identities hold:

$$
{}^{(\mathfrak{n})}\breve{R}^{\alpha}L_{\alpha} = -\mu, \qquad {}^{(\mathfrak{n})}\breve{R}^{\alpha}\underline{L}_{\alpha} = \mu(1 - 2{}^{(\mathfrak{n})}r) - 2\phi\frac{\mathfrak{n}}{L\mu}, \tag{7.13a}
$$

$$
{}^{(\mathfrak{n})}\breve{R}^{\alpha}\mathbf{B}_{\alpha} = -\left(\mu{}^{(\mathfrak{n})}r + \phi\frac{\mathfrak{n}}{L\mu}\right). \tag{7.13b}
$$

Finally, the following identities hold for any Σ_t-tangent vectorfield V, where g is the first fundamental form of Σ_t relative to $\mathbf{g}$:

$$
{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha = \left\{ \mu(1 - {}^{(\mathfrak{n})}r) - \phi \frac{\mathfrak{n}}{L\mu} \right\} X_a V^a - \frac{\mu}{L^{(\mathfrak{n})}\tau} \not{V}^\alpha \not{\nabla}_\alpha {}^{(\mathfrak{n})}\tau, \tag{7.14a}
$$

$$
\begin{aligned}
-\frac{1}{2} {}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4} {}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|_g^2 &= -\frac{1}{4}\mu (V^a X_a)^2 + \frac{1}{4}\left\{ \mu(1 - 2^{(\mathfrak{n})}r) - 2\phi \frac{\mathfrak{n}}{L\mu} \right\} |\not{V}|_{\not{g}}^2 \\
&\quad + \frac{1}{2} \frac{\mu}{L^{(\mathfrak{n})}\tau} X_a V^a \not{V}^\alpha \not{\nabla}_\alpha {}^{(\mathfrak{n})}\tau.
\end{aligned} \tag{7.14b}
$$

Proof. To prove (7.7), we expand $\frac{|\not{\nabla}^{(\mathfrak{n})}\tau|_{\not{g}}^2}{(L^{(\mathfrak{n})}\tau)^2} = \frac{1}{(L^{(\mathfrak{n})}\tau)^2} (\not{g}^{-1})^{AB} \left(\frac{\partial}{\partial x^A} {}^{(\mathfrak{n})}\tau \right) \frac{\partial}{\partial x^B} {}^{(\mathfrak{n})}\tau$ and use the identities (6.17)–(6.18) and $L = \frac{\partial}{\partial t} + L^A \frac{\partial}{\partial x^A}$, thereby confirming that $\frac{|\not{\nabla}^{(\mathfrak{n})}\tau|_{\not{g}}^2}{(L^{(\mathfrak{n})}\tau)^2}$ agrees with the expression for ${}^{(\mathfrak{n})}r$ given in (6.20b), as desired. (7.8) then follows from a similar argument based on expanding $-{}^{(\mathfrak{n})}rL + \frac{1}{L^{(\mathfrak{n})}\tau} \not{\nabla}^{\#(\mathfrak{n})}\tau = -\frac{|\not{\nabla}^{(\mathfrak{n})}\tau|^2}{(L^{(\mathfrak{n})}\tau)^2} L + \frac{1}{L^{(\mathfrak{n})}\tau} (\not{g}^{-1})^{AB} \left(\frac{\partial}{\partial x^A} {}^{(\mathfrak{n})}\tau \right) \frac{\partial}{\partial x^B}$ and using (6.14)–(6.15) and (6.17)–(6.18).

The expression (7.9) for $\frac{1}{\mu} {}^{(\mathfrak{n})}\breve{R}$ follows from (6.6) and the already proved identity (7.8).

The identities (7.10)–(7.11) are straightforward consequences of the chain and Leibniz rules and the fact that ${}^{(\mathfrak{n})}\breve{R}u = 1$, which follows from Lemma 3.9, (6.6), and (7.8).

The decompositions of L and $\underline{L}$ stated in (7.12a)–(7.12b) follow from (6.6) and the identities $\mathbf{B} = L + X$, $\underline{L} = \mathbf{B} + X$, and $\mathbf{B} = \frac{1}{2}(L + \underline{L})$.

Next, we note that if V is Σ_t-tangent, then since $\mathbf{B} = L + X$, it follows from (3.25) that $L_\alpha V^\alpha = -X_a V^a$. From this identity and (7.9), we deduce (7.14a).

The identities in (7.13a) follow from Lemma 3.9, (6.6), definition (7.1), and (7.8). Moreover, (7.13b) follows from the same arguments.

Finally, the identity (7.14b) follows from (7.13a), (7.14a), and the decomposition $g_{ab} = \not{g}_{ab} + X_a X_b$, which follows from (3.34a). □

8. Norms, area and volume forms, and strings of commutation vectorfields

In this section, we define various norms on regions of spacetime that are tied to the rough geometry. We also introduce the area and volume forms that we use in our L^2 analysis. Finally, we introduce notation for repeated differentiation with respect to the commutation vectorfields.

8.1. L^∞-type Sobolev norms and Hölder norms.

8.1.1. *Multi-index notation in various coordinate systems.*

Definition 8.1 (Multi-index notation in various coordinate systems)**.** Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}$, and let $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the corresponding multi-index of order $|\vec{\alpha}| \stackrel{\text{def}}{=} \sum_{i=1}^4 \alpha_i$. We define the following order $|\vec{\alpha}|$ differential operator with respect to the geometric coordinates:

$$
\frac{\partial^{\vec{\alpha}}}{\partial(t, u, x^2, x^3)} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial t} \right)^{\alpha_1} \left(\frac{\partial}{\partial u} \right)^{\alpha_2} \left(\frac{\partial}{\partial x^2} \right)^{\alpha_3} \left(\frac{\partial}{\partial x^3} \right)^{\alpha_4}. \tag{8.1}
$$

Similarly, we define the following order $|\vec{\alpha}|$ differential operator with respect to the adapted rough coordinates:

$$
\frac{\widetilde{\partial}^{\vec{\alpha}}}{\widetilde{\partial}({}^{(\mathfrak{n})}\tau, u, x^2, x^3)} \stackrel{\text{def}}{=} \left(\frac{\widetilde{\partial}}{\widetilde{\partial}{}^{(\mathfrak{n})}\tau} \right)^{\alpha_1} \left(\frac{\widetilde{\partial}}{\widetilde{\partial} u} \right)^{\alpha_2} \left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^2} \right)^{\alpha_3} \left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^3} \right)^{\alpha_4}. \tag{8.2}
$$

8.1.2. *Definitions of essential sup-norm-type Sobolev norms and Hölder norms.*

Definition 8.2 (L^∞-type Sobolev norms and Hölder norms)**.** Let f be a scalar function, let $m \geq 0$ be an integer, and let $\beta \in (0, 1]$ be a real number. On the spacetime regions ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$ defined in (4.6d), we define the following L^∞-type Sobolev norms and Hölder norms of f relative to the geometric coordinates:

$$\|f\|_{W^{m,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{I,J})} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \operatorname*{ess\,sup}_{p\in {}^{(\mathfrak{n})}\mathcal{M}_{I,J}} \left| \frac{\partial^{\vec{\alpha}} f(p)}{\partial(t,u,x^2,x^3)} \right|, \tag{8.3a}$$

$$\|f\|_{C^{m,\beta}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{I,J})} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \max_{p\in {}^{(\mathfrak{n})}\mathcal{M}_{I,J}} \left| \frac{\partial^{\vec{\alpha}} f(p)}{\partial(t,u,x^2,x^3)} \right| + \sum_{|\vec{\alpha}|=m} \sup_{\substack{p_1,p_2\in {}^{(\mathfrak{n})}\mathcal{M}_{I,J}\\ p_1\neq p_2}} \frac{\left| \frac{\partial^{\vec{\alpha}} f(p_1)}{\partial(t,u,x^2,x^3)} - \frac{\partial^{\vec{\alpha}} f(p_2)}{\partial(t,u,x^2,x^3)} \right|}{\left(\text{dist}_{\text{geo}}(p_1,p_2)\right)^{\beta}}, \tag{8.3b}$$

where $\text{dist}_{\text{geo}}(p_1,p_2)$ is the standard Euclidean distance between p_1 and p_2 in the flat geometric coordinate space $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$, i.e., if $p_i \overset{\text{def}}{=} (t_i,u_i,x_i^2,x_i^3)$, $\Delta t \overset{\text{def}}{=} t_2 - t_1$, and $\Delta u \overset{\text{def}}{=} u_2 - u_1$, then $\text{dist}_{\text{geo}}(p_1,p_2) \overset{\text{def}}{=} \sqrt{|\Delta t|^2 + |\Delta u|^2 + |\Delta x^2|^2_{\mathbb{T}} + |\Delta x^3|^2_{\mathbb{T}}}$, where for $j=2,3$, $|\Delta x^j|_{\mathbb{T}}$ is the Euclidean distance between x_2^j and x_1^j in the torus.

Similarly, for intervals $I,J \in \mathbb{R}$, we define the following norms in the adapted rough coordinate spacetime region $I\times J\times\mathbb{T}^2 \subset \mathbb{R}_\tau\times\mathbb{R}_u\times\mathbb{T}^2$:

$$\|f\|_{W^{m,\infty}_{\text{rough}}(I\times J\times\mathbb{T}^2)} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \operatorname*{ess\,sup}_{q\in I\times J\times\mathbb{T}^2} \left| \frac{\widetilde{\partial}^{\vec{\alpha}} f(q)}{\widetilde{\partial}({}^{(\mathfrak{n})}\tau,u,x^2,x^3)} \right|, \tag{8.4}$$

$$\|f\|_{C^{m,\beta}_{\text{rough}}(I\times J\times\mathbb{T}^2)} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \max_{q\in I\times J\times\mathbb{T}^2} \left| \frac{\widetilde{\partial}^{\vec{\alpha}} f(q)}{\widetilde{\partial}({}^{(\mathfrak{n})}\tau,u,x^2,x^3)} \right| + \sum_{|\vec{\alpha}|=m} \sup_{\substack{q_1,q_2\in I\times J\times\mathbb{T}^2\\ q_1\neq q_2}} \frac{\left| \frac{\widetilde{\partial}^{\vec{\alpha}} f(q_1)}{\widetilde{\partial}({}^{(\mathfrak{n})}\tau,u,x^2,x^3)} - \frac{\widetilde{\partial}^{\vec{\alpha}} f(q_2)}{\widetilde{\partial}({}^{(\mathfrak{n})}\tau,u,x^2,x^3)} \right|}{\left(\text{dist}_{\text{rough}}(q_1,q_2)\right)^{\beta}}, \tag{8.5}$$

where on RHS (8.5), $\text{dist}_{\text{rough}}(q_1,q_2)$ is the standard Euclidean distance between q_1 and q_2 in the flat adapted rough coordinate space $\mathbb{R}_\tau\times\mathbb{R}_u\times\mathbb{T}^2$, i.e., if $q_i \overset{\text{def}}{=} (\tau_i,u_i,x_i^2,x_i^3)$, $\Delta\tau \overset{\text{def}}{=} \tau_2-\tau_1$, and $\Delta u \overset{\text{def}}{=} u_2-u_1$, then $\text{dist}_{\text{rough}}(q_1,q_2) \overset{\text{def}}{=} \sqrt{|\Delta\tau|^2+|\Delta u|^2+|\Delta x^2|^2_{\mathbb{T}}+|\Delta x^3|^2_{\mathbb{T}}}$.

Moreover, on the torus $\mathbb{T}^2$ equipped with the Cartesian coordinates (x^2,x^3), we define the following norms:

$$\|f\|_{L^\infty(\mathbb{T}^2)} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{p\in\mathbb{T}^2} |f(p)|, \tag{8.6a}$$

$$\|f\|_{W^{m,\infty}_{\text{geo}}(\mathbb{T}^2)} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \operatorname*{ess\,sup}_{p\in\mathbb{T}^2} \left| \frac{\partial^{\vec{\alpha}} f(p)}{\partial(x^2,x^3)} \right|, \tag{8.6b}$$

$$\|f\|_{C^{m,\beta}_{\text{geo}}(\mathbb{T}^2)} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \max_{p\in\mathbb{T}^2} \left| \frac{\partial^{\vec{\alpha}} f(p)}{\partial(x^2,x^3)} \right| + \sum_{|\vec{\alpha}|=m} \sup_{\substack{p_1,p_2\in\mathbb{T}^2\\ p_1\neq p_2}} \frac{\left| \frac{\partial^{\vec{\alpha}} f(p_1)}{\partial(x^2,x^3)} - \frac{\partial^{\vec{\alpha}} f(p_2)}{\partial(x^2,x^3)} \right|}{(\text{dist}_{\text{flat}}(p_1,p_2))^{\beta}}, \tag{8.6c}$$

where on RHSs (8.6b)–(8.6c), the multi-indices correspond to repeated differentiation with respect to $\{\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\}$ and $\text{dist}_{\text{flat}}(p_1,p_2)$ is the standard Euclidean distance between p_1 and p_2 in $\mathbb{T}^2$.

Similarly, on subsets $\mathcal{S}$ of $\mathbb{R}\times\mathbb{T}^2$ or $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$, we define the following norms:

$$\|f\|_{L^\infty(\mathcal{S})} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{p\in\mathcal{S}} |f(p)|, \tag{8.7a}$$

$$\|f\|_{W^{m,\infty}_{\text{geo}}(\mathcal{S})} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \operatorname*{ess\,sup}_{p\in\mathcal{S}} \left| \partial^{\vec{\alpha}} f(p) \right|, \tag{8.7b}$$

$$\|f\|_{C^{m,\beta}_{\text{geo}}(\mathcal{S})} \overset{\text{def}}{=} \sum_{|\vec{\alpha}|\leq m} \max_{p\in\mathcal{S}} \left| \partial^{\vec{\alpha}} f(p) \right| + \sum_{|\vec{\alpha}|=m} \sup_{\substack{p_1,p_2\in\mathcal{S}\\ p_1\neq p_2}} \frac{\left| \partial^{\vec{\alpha}} f(p_1) - \partial^{\vec{\alpha}} f(p_2) \right|}{(\text{dist}_{\text{flat}}(p_1,p_2))}, \tag{8.7c}$$

where on RHSs (8.7b)–(8.7c), $\partial^{\vec{\alpha}}$ represents repeated differentiation with respect to the coordinate partial derivative vectorfields on $\mathcal{S}$ (the coordinates on $\mathcal{S}$ will always be clear from context, and the coordinates on the factor $\mathbb{T}^2$ will always be the Cartesian coordinates (x^2, x^3)) and $\mathrm{dist}_{\mathrm{flat}}(p_1, p_2)$ is the standard Euclidean distance between p_1 and p_2 in the flat ambient space $\mathbb{R}\times\mathbb{T}^2$ or $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$ (which will always be clear from context).

If $\vec{\varphi} = \{\varphi_i\}_{i=1,\cdots,M}$ is an array- or matrix-valued function with M scalar function entries, then we extend the definitions of the above norms to $\vec{\varphi}$ by summing the scalar norms over i, e.g., $\|\vec{\varphi}\|_{C^{m,\beta}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{I,J})} \stackrel{\mathrm{def}}{=} \sum_{i=1}^{M} \|\varphi_i\|_{C^{m,\beta}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{I,J})}$.

8.2. **Area forms, volume forms, and corresponding L^2 norms.** We now define the area and volume forms on the rough subsets (see Def. 4.11) ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, ${}^{(\mathfrak{n})}\widetilde{\Sigma}^I_\tau$, and ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$ that we will use in our analysis. We also define corresponding L^2-type norms. Our definitions are in terms of the adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$ because those are the coordinates that we use in our energy identities, where the forms arise.

8.2.1. *Geometric forms and related integrals.*

Definition 8.3 (Geometric forms and related integrals)**.**

- Recall that $\widetilde{g\!\!\!/}$ is the first fundamental form of the rough torus ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. We define the canonical area form of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ induced by $\widetilde{g\!\!\!/}$ in the adapted rough coordinates (τ, u, x^2, x^3) by:

$$d\varpi_{\widetilde{g\!\!\!/}} = d\varpi_{\widetilde{g\!\!\!/}}(\tau, u', x^2, x^3) \stackrel{\mathrm{def}}{=} \sqrt{\det\widetilde{g\!\!\!/}(\tau, u', x^2, x^3)}\, dx^2 dx^3, \tag{8.8}$$

  where $\det\widetilde{g\!\!\!/}(\tau, u', x^2, x^3)$ is the determinant of the 2×2 matrix $\left(\widetilde{g\!\!\!/}(\tau, u', x^2, x^3)\left(\frac{\widetilde{\partial}}{\partial x^A}, \frac{\widetilde{\partial}}{\partial x^B}\right)\right)_{A,B=2,3}$ (see RHS (6.10)).
- We define the (non-canonical) area form $d\underline{\varpi}$ of ${}^{(\mathfrak{n})}\widetilde{\Sigma}^I_\tau$ in the adapted rough coordinates (τ, u, x^2, x^3) by:

$$d\underline{\varpi} = d\underline{\varpi}(\tau, u', x^2, x^3) = d\varpi_{\widetilde{g\!\!\!/}}(\tau, u', x^2, x^3)\, du'. \tag{8.9}$$

- We define the (non-canonical) area form $d\overline{\varpi}$ on ${}^{(\mathfrak{n})}\mathcal{P}^J_u$ in the adapted rough coordinates (τ, u, x^2, x^3) by:

$$d\overline{\varpi} = d\overline{\varpi}(\tau', u, x^2, x^3) \stackrel{\mathrm{def}}{=} d\varpi_{\widetilde{g\!\!\!/}}(\tau', u, x^2, x^3)\, d\tau'. \tag{8.10}$$

- We define the (non-canonical) volume form $d\boldsymbol{\varpi}$ of ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$ in the adapted rough coordinates (τ, u, x^2, x^3) by:

$$d\boldsymbol{\varpi} = d\boldsymbol{\varpi}(\tau', u', x^2, x^3) \stackrel{\mathrm{def}}{=} d\varpi_{\widetilde{g\!\!\!/}}(\tau', u', x^2, x^3)\, du'\, d\tau'. \tag{8.11}$$

Unless we explicitly indicate otherwise, all integrals along ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, ${}^{(\mathfrak{n})}\widetilde{\Sigma}^I_\tau$, and ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$ are defined with respect to the above forms. Moreover, we will often suppress the variables with respect to which we integrate, e.g., we write:

$$\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} f\, d\varpi_{\widetilde{g\!\!\!/}} \stackrel{\mathrm{def}}{=} \int_{(x^2,x^3)\in\mathbb{T}^2} f(\tau, u, x^2, x^3)\, d\varpi_{\widetilde{g\!\!\!/}}(\tau, u, x^2, x^3), \tag{8.12a}$$

$$\int_{{}^{(\mathfrak{n})}\mathcal{P}^J_u} f\, d\overline{\varpi} \stackrel{\mathrm{def}}{=} \int_{\tau'\in J}\int_{(x^2,x^3)\in\mathbb{T}^2} f(\tau', u, x^2, x^3)\, d\varpi_{\widetilde{g\!\!\!/}}(\tau', u, x^2, x^3)\, d\tau', \tag{8.12b}$$

$$\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}^I_\tau} f\, d\underline{\varpi} \stackrel{\mathrm{def}}{=} \int_{u'\in I}\int_{(x^2,x^3)\in\mathbb{T}^2} f(\tau, u', x^2, x^3)\, d\varpi_{\widetilde{g\!\!\!/}}(\tau, u', x^2, x^3)\, du', \tag{8.12c}$$

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{I,J}} f\, d\boldsymbol{\varpi} \stackrel{\mathrm{def}}{=} \int_{\tau'\in J}\int_{u'\in I}\int_{(x^2,x^3)\in\mathbb{T}^2} f(\tau', u', x^2, x^3)\, d\varpi_{\widetilde{g\!\!\!/}}(\tau', u', x^2, x^3)\, du'\, d\tau'. \tag{8.12d}$$

Remark 8.4 (Abuse of notation)**.** Strictly speaking, we have abused notation in (8.12a)–(8.12d) because the RHSs are with respect to the adapted rough coordinates while the sets ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, ${}^{(\mathfrak{n})}\mathcal{P}^J_u$, ${}^{(\mathfrak{n})}\widetilde{\Sigma}^I_\tau$, and ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$ on the LHSs are subsets of geometric coordinate space (see Remark 4.4). Thus, for example, it would be more accurate to write $\int_{\{\tau\}\times\{u\}\times\mathbb{T}^2}\cdots$ on LHS (8.12a) instead of $\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}}\cdots$ because $\{\tau\}\times\{u\}\times\mathbb{T}^2$ is the image of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ in adapted rough coordinates under the map ${}^{(\mathfrak{n})}\mathscr{T}$ (see (5.2)).

In a few of our calculations, we will also refer to the canonical volume forms of $\widetilde{g}$ and $\mathbf{g}$ relative to the adapted rough coordinates, which we provide in the following definition.

Definition 8.5 (Canonical volume forms relative to the adapted rough coordinates)**.**

- Recall that $\widetilde{g}$ is the first fundamental form of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$, as in Def. 6.2. We define $\mathrm{dvol}_{\widetilde{g}} \stackrel{\mathrm{def}}{=} \sqrt{\det \widetilde{g}}\, \mathrm{d}x^2\, \mathrm{d}x^3\, \mathrm{d}u'$ to be the canonical area form on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}$ induced by $\widetilde{g}$, where $\det \widetilde{g}$ is evaluated relative to the adapted rough coordinates (u', x^2, x^3) on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$.
- Recall that $\mathbf{g}$ denotes the acoustical metric, defined relative to the Cartesian coordinates in (2.15a). We define $\mathrm{dvol}_{\mathbf{g}} \stackrel{\mathrm{def}}{=} \sqrt{|\det \mathbf{g}|}\, \mathrm{d}x^2\, \mathrm{d}x^3\, \mathrm{d}u'\, \mathrm{d}\tau'$ to be the canonical volume form on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$ induced by the acoustical metric $\mathbf{g}$, where $\det \mathbf{g}$ is evaluated relative to the adapted rough coordinates (τ', u', x^2, x^3) on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$.

8.2.2. *Identities involving the forms.* For future use, in the following lemma, we establish several identities involving $\mathrm{dvol}_{\widetilde{g}}, \mathrm{dvol}_{\mathbf{g}}, \mathrm{d}\underline{\varpi}$, and $\mathrm{d}\boldsymbol{\varpi}$ relative to the adapted rough coordinates.

Lemma 8.6 (Identities involving $\mathrm{dvol}_{\widetilde{g}}, \mathrm{dvol}_{\mathbf{g}}, \mathrm{d}\underline{\varpi}$, and $\mathrm{d}\boldsymbol{\varpi}$)**.** *Let $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}} \geq 0$ be as in (6.20a). Then the following identities hold relative to the adapted rough coordinates (τ, u, x^2, x^3), e.g., $\det \widetilde{g\!\!\!/}$ is the determinant of the 2×2 matrix $\left(\widetilde{g\!\!\!/}\left(\frac{\widetilde{\partial}}{\partial x^A}, \frac{\widetilde{\partial}}{\partial x^B}\right)\right)_{A,B=2,3}$:*

$$\det \widetilde{g} = |{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}^2 \det \widetilde{g\!\!\!/}, \tag{8.13a}$$

$$\det \mathbf{g} = -\frac{\mu^2}{(L^{(\mathfrak{n})}\tau)^2} \det \widetilde{g\!\!\!/}. \tag{8.13b}$$

Moreover, with $\mathrm{dvol}_{\widetilde{g}}, \mathrm{dvol}_{\mathbf{g}}, \mathrm{d}\underline{\varpi}$, and $\mathrm{d}\varpi_{\widetilde{g\!\!\!/}}$ denoting the area and volume forms from Defs. 8.3 and 8.5, we have the following identities relative to the adapted rough coordinates:

$$\mathrm{dvol}_{\widetilde{g}} = |{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}} \mathrm{d}\underline{\varpi} = |{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}} \mathrm{d}\varpi_{\widetilde{g\!\!\!/}}\, \mathrm{d}u', \tag{8.14a}$$

$$\mathrm{dvol}_{\mathbf{g}} = \frac{\mu}{L^{(\mathfrak{n})}\tau} \mathrm{d}\boldsymbol{\varpi} = \frac{\mu}{L^{(\mathfrak{n})}\tau} \mathrm{d}\varpi_{\widetilde{g\!\!\!/}}\, \mathrm{d}u'\, \mathrm{d}\tau. \tag{8.14b}$$

Proof. Let ${}^{(\mathfrak{n})}\hat{R}$ be as in Def. 6.4. Recall (see Prop. 6.7) that ${}^{(\mathfrak{n})}\hat{R}$ is tangent to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ and $\mathbf{g}$-orthogonal to the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. From (5.8c)–(5.8d), (6.5), (6.6), and Lemma 3.9, it follows that ${}^{(\mathfrak{n})}\hat{R}u = \frac{1}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}$. Also considering the identities (6.12)–(6.13) and (6.23b) (where we view the components of RHS (6.23b) as entries of a 3×3 matrix in adapted rough coordinates (u, x^2, x^3) on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$), we carry out straightforward calculations in the adapted rough coordinates to deduce that $\det \widetilde{g}^{-1} = |{}^{(\mathfrak{n})}\breve{R}|^{-2} \det \widetilde{g\!\!\!/}^{-1}$, where $\det \widetilde{g\!\!\!/}^{-1}$ is the determinant of the 2×2 matrix $\left(\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right)\right)_{A,B=2,3}$. The desired result (8.13a) now readily follows. Similarly, recall (see Prop. 6.7) that ${}^{(\mathfrak{n})}\hat{N}$ is the $\mathbf{g}$-unit normal to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$. Since ${}^{(\mathfrak{n})}\hat{R}$ is tangent to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ and ${}^{(\mathfrak{n})}\hat{R}u = \frac{1}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}$, we deduce from the identity (6.20d) that ${}^{(\mathfrak{n})}\hat{N}{}^{(\mathfrak{n})}\tau = \frac{(L^{(\mathfrak{n})}\tau)|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}{\mu}$ and ${}^{(\mathfrak{n})}\hat{N}u = \frac{1}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}$. Also considering the identities (6.12)–(6.13) and (6.23a) (where we view the components of RHS (6.23a) as entries of a 4×4 matrix in adapted rough coordinates), we carry out straightforward calculations in the adapted rough coordinates to deduce that $\det \mathbf{g}^{-1} = -\frac{(L^{(\mathfrak{n})}\tau)^2}{\mu^2} \det \widetilde{g\!\!\!/}^{-1}$, from which (8.13b) readily follows.

The identities (8.14a)–(8.14b) then follow from (8.13a)–(8.13b) and Defs. 8.3 and 8.5.

□

8.2.3. *Geometric L^2 and L^∞ norms.* In this section, we define various L^2 norms with respect to the area and volume forms introduced in Def. 8.3. We also define an L^∞-type norm on the rough tori.

Definition 8.7 (Geometric L^2 norms)**.** Recall that we measure the norm of $\ell_{t,u}$-tangent tensorfields with $g\!\!\!/$, i.e., if ξ is a type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, then $|\xi|_{g\!\!\!/}^2 \stackrel{\mathrm{def}}{=} (g\!\!\!/^{-1})^{\alpha\beta}(g\!\!\!/^{-1})^{\gamma\delta}\xi_{\alpha\gamma}\xi_{\beta\delta}$. Then for scalar functions or $\ell_{t,u}$-tangent tensorfields ξ, we define:

$$\|\xi\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \stackrel{\mathrm{def}}{=} \left(\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} |\xi|_{g\!\!\!/}^2\, \mathrm{d}\varpi_{\widetilde{g\!\!\!/}}\right)^{1/2}, \qquad \|\xi\|_{L^2\left({}^{(\mathfrak{n})}\mathcal{P}_u^J\right)} \stackrel{\mathrm{def}}{=} \left(\int_{{}^{(\mathfrak{n})}\mathcal{P}_u^J} |\xi|_{g\!\!\!/}^2\, \mathrm{d}\overline{\varpi}\right)^{1/2}, \tag{8.15a}$$

$$\|\xi\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^I\right)} \stackrel{\mathrm{def}}{=} \left(\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^I} |\xi|_{g\!\!\!/}^2\, \mathrm{d}\underline{\varpi}\right)^{1/2}, \qquad \|\xi\|_{L^2\left({}^{(\mathfrak{n})}\mathcal{M}_{I,J}\right)} \stackrel{\mathrm{def}}{=} \left(\int_{{}^{(\mathfrak{n})}\mathcal{M}_{I,J}} |\xi|_{g\!\!\!/}^2\, \mathrm{d}\boldsymbol{\varpi}\right)^{1/2}. \tag{8.15b}$$

Definition 8.8 (Geometric L^∞ norms)**.** For scalar functions or $\ell_{t,u}$-tangent tensorfields ξ, we define the following L^∞ norm on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$:

$$\|\xi\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \stackrel{\text{def}}{=} \text{ess sup}_{(x^2,x^3)\in\mathbb{T}^2} |\xi|_{\not g}(\tau,u,x^2,x^3), \tag{8.16}$$

where on RHS (8.16), we are viewing ξ as a function of the adapted rough coordinates (τ,u,x^2,x^3).

Remark 8.9 (Carefully note the role of $\not g$)**.** We stress that $\not g$ is the Riemannian metric on the acoustic tori $\ell_{t,u}$, even though the integrals defining $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}$, $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\mathcal{P}_u^I\right)}$, $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^I\right)}$, and $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\mathcal{M}_{I,J}\right)}$ are over regions and with respect to forms tied to the adapted rough coordinates.

Similarly, on RHS (8.16), $|\xi|_{\not g}$ is the pointwise norm of ξ with respect to the Riemannian metric $\not g$ on the acoustic tori.

8.3. **Strings of commutation vectorfields and vectorfield semi-norms.** Recall that $\mathscr{Z} = \{L, \check{X}, Y_{(2)}, Y_{(3)}\}$, $\mathscr{P} = \{L, Y_{(2)}, Y_{(3)}\}$, and $\mathscr{Y} = \{Y_{(2)}, Y_{(3)}\}$ are the commutation vectorfield sets from (3.16). To simplify the presentation of formulas and estimates, we now introduce notation capturing repeated differentiation with respect to these vectorfields.

Definition 8.10 (Strings of commutation vectorfields and vectorfield semi-norms)**.** Let f be a scalar function and let ξ be an $\ell_{t,u}$-tangent tensorfield.

- $\mathcal{Z}^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{Z}$ applied to f, where the string contains *at most* M factors of $\check{X}$. We set $\mathcal{Z}^{0,0} f = f$. We often write $\mathcal{Z} f$ instead of $\mathcal{Z}^{1;1} f$.
- $\mathcal{P}^N f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{P}$ applied to f. We set $\mathcal{P}^0 f = f$. We often write $\mathcal{P} f$ instead of $\mathcal{P}^1 f$.
- $\mathcal{Y}^N f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{Y}$ applied to f. We set $\mathcal{Y}^0 f = f$.
- $\mathcal{Z}_*^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{Z}$ applied to f, where the string contains *at least* one factor of $\mathcal{P}$ and *at most* M factors of $\check{X}$.
- $\mathcal{P}_*^N$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{P}$ applied to f, where the string contains *at least two factors* of L or *at least one factor of* $Y_{(A)}$.
- $\mathcal{Z}_{**}^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{Z}$ applied to f, where the string contains at least two factors of L or at least one factor of $Y_{(A)}$ and *at most* M factors of $\check{X}$.
- $\mathfrak{P}^{(N)}$ denotes the set of all differential operators of the form $\mathcal{P}^N$.
- $\mathfrak{Y}^{(N)}$ denotes the set of all differential operators of the form $\mathcal{Y}^N$.
- We define order N strings of $\ell_{t,u}$-projected Lie derivatives such as $\not{\mathcal{L}}_{\mathcal{P}}^N$ and $\not{\mathcal{L}}_{\mathcal{Z}}^{N;M}$ in an analogous fashion. Such operators act on $\ell_{t,u}$-tangent tensorfields ξ, e.g., $\not{\mathcal{L}}_{\mathcal{P}}^N \xi$.
- $\not{\mathcal{L}}_{\mathfrak{P}}^{(N)}$ denotes the set of all differential operators of the form $\not{\mathcal{L}}_{\mathcal{P}}^N$.
- $\not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}$ denotes the set of all differential operators of the form $\not{\mathcal{L}}_{\mathcal{Y}}^N$.
- $\mathcal{Z}^{\leq N;M} f$ denotes the array of all terms of the form $\mathcal{Z}^{N';M} f$, where $0 \leq N' \leq N$.
- If $N_1 < N_2$, then $\mathcal{Z}^{[N_1,N_2];M} f$ denotes the array of all terms of the form $\mathcal{Z}^{N';M} f$, where $N_1 \leq N' \leq N_2$.
- We define arrays such as $\mathcal{Y}^{\leq 2} f$, $\not{\mathcal{L}}_{\mathcal{Y}}^{[N_1,N_2]} \xi$, etc. in an analogous fashion.

We also define corresponding pointwise semi-norms:

- $|\mathcal{Z}^{N;M} f|$ denotes the magnitude of $\mathcal{Z}^{N;M} f$ as defined above (there is no summation).
- $|\mathcal{Z}^{\leq N;M} f|$ denotes the sum over all terms of the form $|\mathcal{Z}^{N';M} f|$ with $N' \leq N$.
- $|\mathcal{Z}^{[N_1,N_2];M} f|$ denotes the sum over all terms of the form $|\mathcal{Z}^{N';M} f|$ with $N_1 \leq N' \leq N_2$.
- Terms such as $|\mathcal{P}_*^{[N_1,N_2]} f|$, $|\not{\mathcal{L}}_{\mathcal{Z}}^{\leq N;M} \xi|_{\not g}$, $|\mathcal{Y}^{\leq N} f|$, etc., are defined analogously, e.g., $|\not{\mathcal{L}}_{\mathcal{Z}}^{\leq N;M} \xi|_{\not g}$ is the sum over all terms of the form $|\not{\mathcal{L}}_{\mathcal{Z}}^{N';M} \xi|_{\not g}$ with $N' \leq N$.
- We will freely combine the above definitions with Def. 2.9, e.g.,

$$|\mathcal{P}^N(\Omega,S)| \stackrel{\text{def}}{=} \max\left\{|\mathcal{P}^N \Omega|, |\mathcal{P}^N S|\right\}. \tag{8.17}$$

9. Schematic structure capturing structure and schematic identities

In this section, we introduce schematic notation that will help us succinctly exhibit the important qualitative features of various equations. We then provide a collection of identities expressed in schematic form; they will be helpful when we derive estimates.

9.1. **Some schematic notation.**

Notation 9.1 (Schematic functional dependence)**.** We often use the notation $\mathrm{f}(\xi_{(1)}, \cdots, \xi_{(m)})$ to schematically depict an expression (often tensorial and involving contractions) that depends smoothly on the $\ell_{t,u}$-tangent tensorfields $\xi_{(1)}, \cdots, \xi_{(m)}$. Note that in general, $\mathrm{f}(0) \neq 0$.

Notation 9.2 (Schematic use of the symbol $\mathcal{P}$)**.** Throughout the rest of the paper, $\mathcal{P}$ schematically denotes a differential operator that is tangent to the characteristics $\mathcal{P}_u$. For example, $\mathcal{P} f$ might denote $\not{d} f, L f$, or $Y_{(2)} f$. We use such notation when the details of $\mathcal{P}$ are unimportant.

We use the notation $\vec{x}$ to denote the array of spatial Cartesian coordinates, i.e.,

$$\vec{x} \stackrel{\text{def}}{=} (x^1, x^2, x^3). \tag{9.1}$$

We use the same conventions from Def. 2.12 for differential operators acting on $\vec{x}$, e.g.,

$$\not{d}\vec{x} \stackrel{\text{def}}{=} (\not{d}x^1, \not{d}x^2, \not{d}x^3). \tag{9.2}$$

9.2. **Schematic structure of various tensorfields.**

Proposition 9.1 (Schematic structure of various tensorfields)**.** *Recall that γ and $\underline{\gamma}$ are the arrays from Def. 3.14. The following schematic relations hold for scalar functions ($\alpha, \beta = 0, 1, 2, 3$, $\iota = 0, 1, 2, 3, 4$):*

$$\mathbf{g}_{\alpha\beta}, (\mathbf{g}^{-1})^{\alpha\beta}, \not{g}_{\alpha\beta}, (\not{g}^{-1})^{\alpha\beta}, G^{\iota}_{\alpha\beta}, \not{\Pi}^{\alpha}_{\beta}, L^{\alpha}, X^{\alpha}, \underline{L}^{\alpha}, Y^{\alpha}_{(2)}, Y^{\alpha}_{(3)}, c = \mathrm{f}(\gamma), \tag{9.3a}$$

$$\left(\frac{\partial}{\partial t}\right)^{\alpha}, \left(\frac{\partial}{\partial x^2}\right)^{\alpha}, \left(\frac{\partial}{\partial x^3}\right)^{\alpha} = \mathrm{f}(\gamma), \tag{9.3b}$$

$$\left(\frac{\partial}{\partial u}\right)^{\alpha} = \mathrm{f}(\underline{\gamma}), \tag{9.3c}$$

$$G^{\iota}_{LL}, G^{\iota}_{LX}, G^{\iota}_{XX} = \mathrm{f}(\gamma), \tag{9.3d}$$

$$X^{\alpha}_{(\text{Small})}, Y^{\alpha}_{(2;\text{Small})}, Y^{\alpha}_{(3;\text{Small})}, c-1 = \mathrm{f}(\gamma)\gamma, \tag{9.3e}$$

$$\breve{X}^{\alpha} = \mathrm{f}(\underline{\gamma}). \tag{9.3f}$$

Moreover, if φ is a scalar function, then we have the following schematic relation for its $\not{\nabla}$-Hessian (which is a symmetric type $\binom{0}{2}$-tangent tensorfield):

$$\not{\nabla}^2\varphi = \mathrm{f}(\gamma, \not{d}\vec{x})\mathcal{Y}^2\varphi + \mathrm{f}(\gamma, \not{d}\vec{x}) \cdot \mathcal{Y}\gamma \cdot \mathcal{Y}\varphi. \tag{9.4}$$

Finally, we have the following schematic relations for $\ell_{t,u}$-tangent tensorfields, where $\not{d}\vec{x}$ is defined in (9.2):

$$\not{g}, \vec{\not{G}}_L, \vec{\not{G}}_X, \vec{\not{G}} = \mathrm{f}(\gamma, \not{d}\vec{x}), \tag{9.5a}$$

$$Y_{(2)}, Y_{(3)}, = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x}), \tag{9.5b}$$

$$\chi = \mathrm{f}(\gamma, \not{d}\vec{x})\mathcal{P}\gamma, \tag{9.5c}$$

$$\mathrm{tr}_{\not{g}}\chi = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x})\mathcal{P}\gamma, \tag{9.5d}$$

$$\zeta^{(\text{Tan-}\vec{\Psi})}, \not{k}^{(\text{Tan-}\vec{\Psi})} = \mathrm{f}(\gamma, \not{d}\vec{x})\mathcal{P}\vec{\Psi}, \tag{9.5e}$$

$$\zeta^{(\text{Trans-}\vec{\Psi})}, \not{k}^{(\text{Trans-}\vec{\Psi})} = \mathrm{f}(\gamma, \not{d}\vec{x})\breve{X}\vec{\Psi}. \tag{9.5f}$$

Proof. After one accounts for the third dimension, the same proofs as in [73, Lemma 2.19] hold for (9.3a)–(9.5f), except (9.3a) for $\underline{L}^{\alpha}$ was not stated there, (9.3b)–(9.3c) were not stated there, (9.3e) for $c-1$ was not stated there, and (9.4) was not stated there. The identity $\underline{L}^{\alpha} = \mathrm{f}(\gamma)$ stated in (9.3a) follows from definition (7.1) and the identity (9.3a) for L^{α} and X^{α}. The identity $c - 1 = \mathrm{f}(\gamma)\gamma$ stated in (9.3a) follows easily from (2.5). The identities (9.3b)–(9.3c) follow from (5.8a)–(5.8d) and the remaining identities in the proposition (note that (9.4) is not needed for these proofs). To prove (9.4), we first note that relative to the geometric coordinates (x^2, x^3) on $\ell_{t,u}$, $\nabla\!\!\!/^2 \varphi = \nabla\!\!\!/^2_{AB} \varphi d\!\!/ x^A \otimes d\!\!/ x^B$. Next, we use the Leibniz rule for $\nabla\!\!\!/$ to deduce that $\nabla\!\!\!/^2_{AB}\varphi = \frac{\partial}{\partial x^A}(\frac{\partial}{\partial x^B}\varphi) - (\nabla\!\!\!/_{\frac{\partial}{\partial x^A}} \frac{\partial}{\partial x^B}) \cdot \nabla\!\!\!/ \varphi$. Moreover, computing the Christoffel symbols of $g\!\!\!/$ with respect to the (x^2, x^3) coordinates, we deduce that $(\nabla\!\!\!/_{\frac{\partial}{\partial x^A}} \frac{\partial}{\partial x^B}) \cdot \nabla\!\!\!/ \varphi = \frac{1}{2}(g\!\!\!/^{-1})^{CD}\left(\frac{\partial}{\partial x^A} g\!\!\!/_{CB} + \frac{\partial}{\partial x^B} g\!\!\!/_{CB} - \frac{\partial}{\partial x^C} g\!\!\!/_{AB}\right)\frac{\partial}{\partial x^D}\varphi$. Using (3.31a)–(3.31b) to substitute for the coordinate components of $g\!\!\!/$ and $g\!\!\!/^{-1}$, using (5.8c)–(5.8d) to express all vectorfields $\frac{\partial}{\partial x^A}$ in terms of $Y_{(2)}$ and $Y_{(3)}$, and using the remaining identities in the proposition, we conclude (9.4). □

9.3. **Transversal derivatives in terms of $\mathcal{P}_u$-tangential derivatives and structural properties of g-null forms.** In this section, we use the transport equations from Theorem 2.15 to derive expressions for the $\breve{X}$ derivatives of Ω, S, $\mathcal{C}$, and $\mathcal{D}$ in terms of $\mathcal{P}_u$-tangential derivatives. We also exhibit some crucial structural properties of the inhomogeneous terms in the equations of Theorem 2.15, including the $\mathbf{g}$-null forms.

Lemma 9.2 (Expressions for $\breve{X}\Omega$ and $\breve{X}S$ in terms of $\mathcal{P}_u$-tangential derivatives)**.** *The following schematic identities hold for the $\breve{X}$ derivatives of the Cartesian components of Ω and S:*

$$\breve{X}\Omega^i = -\mu L\Omega + \mathrm{f}(\underline{\gamma}, S, \mathcal{Z}\vec{\Psi}) \cdot (\Omega, S). \tag{9.6a}$$

$$\breve{X}S^i = -\mu LS + \mathrm{f}(\underline{\gamma}, S, \mathcal{Z}\vec{\Psi}) \cdot (\Omega, S). \tag{9.6b}$$

Proof. The identity (9.6a) follows from multiplying the transport equation (2.23a) and by μ, using the identity $\breve{X} = -\mu L + \mu\mathbf{B}$ (see (3.24)), using Lemma 5.6 to write the Cartesian partial derivatives ∂_α in terms of the commutation vectorfields, and using Prop. 9.1. (9.6b) follows from a similar argument based on (2.23c) □

Lemma 9.3 (Crucial structural properties of $\mathbf{g}$-null forms)**.** *The product of* μ *and the terms defined in* (2.26a)–(2.27e) *enjoy the following schematic structure:*[52]

$$\mu\mathfrak{M}^i_{(\mathcal{C})}, \mu\mathfrak{M}_{(\mathcal{D})} = \mathrm{f}\left(\underline{\gamma}, S, \mathcal{Z}\vec{\Psi}\right) \cdot (\mathcal{P}^{\leq 1}\Omega, \mathcal{P}^{\leq 1}S), \tag{9.7a}$$

$$\mu\mathfrak{Q}^i_{(v)}, \mu\mathfrak{Q}_{(\pm)}, \mu\mathfrak{Q}_{(\rho)} = \mathrm{f}(\underline{\gamma}, \mathcal{Z}\vec{\Psi}) \cdot \mathcal{P}\vec{\Psi}, \tag{9.7b}$$

$$\mu\mathfrak{Q}^i_{(\mathcal{C})}, \mu\mathfrak{Q}_{(\mathcal{D})} = \mathrm{f}(\underline{\gamma}, S, \mathcal{Z}\vec{\Psi}) \cdot S. \tag{9.7c}$$

Proof. All the results follow from [52, Lemma 8.2], except the term stemming from last term on RHS (2.26b), which is of the schematic form $\mathrm{f}(\vec{\Psi}, S) \cdot \mu\partial_i\Omega$, was not handled there. To handle this last term, we use Lemma 5.6 to write the Cartesian spatial partial derivatives ∂_i in terms of the commutation vectorfields, use the identity (9.6a) to substitute for the $\breve{X}$ derivatives of Ω, and use Prop. 9.1. □

Lemma 9.4 (Crucial structural properties of the linear inhomogeneous terms)**.** *The product of* μ *and the terms* $\mathcal{C}, \mathcal{D}$*-involving terms on RHSs* (2.22a)–(2.22d)*, as well as the product of* μ *and the terms defined in* (2.28a)–(2.28h) *enjoy the following schematic structure:*

$$\begin{aligned}&\mu c^2 \exp(2\rho)\mathcal{C}^i, \mu\left\{F_{;s}c^2\exp(2\rho) - c\exp(\rho)\frac{p_{;s}}{\bar{\varrho}}\right\}\mathcal{D},\\ &\mu\exp(\rho)\frac{p_{;s}}{\bar{\varrho}}\mathcal{D}, \mu c^2\exp(2\rho)\mathcal{D}\\ &= \mu\mathrm{f}(\vec{\Psi}) \cdot (\mathcal{C}, \mathcal{D}),\end{aligned} \tag{9.8a}$$

$$\begin{aligned}&\mu\mathfrak{L}^i_{(v)}, \mu\mathfrak{L}_{(\pm)}, \mu\mathfrak{L}_{(\rho)}, \mu\mathfrak{L}_{(s)}, \mu\mathfrak{L}^i_{(\Omega)}, \mu\mathfrak{L}^i_{(S)}, \mu\mathfrak{L}_{(\mathrm{div}\Omega)}, \mu\mathfrak{L}^i_{(\mathcal{C})}\\ &= \mathrm{f}(\underline{\gamma}, \Omega, S, \mathcal{Z}\vec{\Psi}) \cdot (\Omega, S).\end{aligned} \tag{9.8b}$$

Proof. We use Lemma 5.6 to write the Cartesian partial derivatives ∂_α in terms of the commutation vectorfields, and we use Prop. 9.1. □

In the next lemma, we provide an analog of Lemma 9.2 for the modified fluid variables.

[52] All of these are $\mathbf{g}$-null forms, except the last term on RHS (2.26b), which turns out to be a harmless error term.

Lemma 9.5 (Expressions for the transversal derivatives of the modified fluid variables in terms of $\mathcal{P}_u$-tangential derivatives)**.** *Recall that the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$ are defined in Def. 2.7. The following schematic identities hold for the $\breve{X}$ derivatives of the Cartesian components of $\mathcal{C}$ and $\mathcal{D}$:*

$$(\breve{X}\mathcal{C}^i, \breve{X}\mathcal{D}) = -\mu(L\mathcal{C}, L\mathcal{D}) + \mathrm{f}(\underline{\gamma}, S, \mathcal{Z}\vec{\Psi}) \cdot \mathcal{P}\vec{\Psi} + \mathrm{f}(\underline{\gamma}, S, \mathcal{Z}\vec{\Psi}) \cdot (\mathcal{P}^{\leq 1}\Omega, \mathcal{P}^{\leq 1}S). \tag{9.9}$$

Proof. The identity (9.9) follows from an argument similar to the one we used to prove Lemma 9.2, based on equations (2.24b) and (2.25a), where we use Lemma 9.3 to handle the $\mathbf{g}$-null forms appearing on the RHSs of (2.24b) and (2.25a). ☐

Lemma 9.6 (Schematic identity for $\mu\mathcal{C}^i$ and $\mu\mathcal{D}$)**.** *The μ-weighted Cartesian components of the modified fluid variables defined in* (2.10a)–(2.10b) *can be expressed as follows:*

$$(\mu\mathcal{C}^i, \mu\mathcal{D}) = \mathrm{f}(\underline{\gamma}, S, \mathcal{Z}\vec{\Psi}) \cdot \mathcal{P}^{\leq 1}(\Omega, S). \tag{9.10}$$

Proof. The identity (9.10) follows from definitions (2.10a)–(2.10b), Prop. 9.1, Lemma 5.6 (which allows us to schematically express $\mu\partial_\alpha = \mathrm{f}(\gamma)\breve{X} + \mu\mathrm{f}(\gamma)\mathcal{P}$), and Lemma 9.2 (which allows us to substitute for the $\breve{X}$ derivatives of Ω and S). ☐

9.4. **Additional schematic identities involving differentiation.** For future use, in this section, we provide some additional schematic identities involving differentiation.

Lemma 9.7 (Schematic identity for $\not{\Delta}\varphi$)**.** *If φ is a scalar function, then its angular Laplacian on $\ell_{t,u}$ can be expressed as follows, where the first term on RHS* (9.11) *is written precisely and the last two are written schematically:*

$$\not{\Delta}\varphi = \sum_{A=2,3} Y_{(A)}(Y_{(A)}\varphi) + \gamma \cdot \mathrm{f}(\gamma)\mathcal{Y}^2\varphi + \mathrm{f}(\gamma) \cdot \mathcal{Y}\gamma \cdot \mathcal{Y}\varphi. \tag{9.11}$$

Proof. Using the identity $\not{g}^{-1} = (\mathbf{g}^{-1})^{\alpha\beta}\not{\Pi}_\alpha^{\ \gamma}\not{\Pi}_\beta^{\ \delta}\partial_\gamma \otimes \partial_\delta$, (1.2), (2.15b), Cor. 5.7, and Prop. 9.1, we find that:

$$\not{g}^{-1} = c^2 \sum_{A=2,3} Y_{(A)} \otimes Y_{(A)} + \gamma \cdot \mathrm{f}(\gamma) \sum_{A,B=2,3} Y_{(A)} \otimes Y_{(B)} = \sum_{A=2,3} Y_{(A)} \otimes Y_{(A)} + \gamma \cdot \mathrm{f}(\gamma) \sum_{A,B=2,3} Y_{(A)} \otimes Y_{(B)}, \tag{9.12}$$

where the first term on RHS (9.12) is written precisely and the second one schematically. Hence, using the Leibniz rule, we deduce that $\not{\Delta}\varphi = \sum_{A=2,3} Y_{(A)}(Y_{(A)}\varphi) + \gamma \cdot \mathrm{f}(\gamma)\sum_{A,B=2,3} Y_{(A)}(Y_{(B)}\varphi)$ plus error terms of the schematic form $\mathrm{f}(\gamma) \cdot \not{g}(\not{\nabla}_{Y_{(A)}} Y_{(B)}, Y_{(C)}) \cdot Y_{(D)}\varphi$. Noting that $\not{g}(\not{\nabla}_{Y_{(A)}} Y_{(B)}, Y_{(C)}) = \mathbf{g}(\mathbf{D}_{Y_{(A)}} Y_{(B)}, Y_{(C)})$, we compute relative to the Cartesian coordinates and use Prop. 9.1 to deduce that:

$$\mathbf{g}(\mathbf{D}_{Y_{(A)}} Y_{(B)}, Y_{(C)}) = \mathbf{g}_{\alpha\beta}(Y_{(A)} Y_{(B)}^\alpha) Y_{(C)}^\beta + Y_{(A)}^\alpha Y_{(B)}^\beta Y_{(C)}^\gamma \mathbf{\Gamma}_{\alpha\gamma\beta} = \mathrm{f}(\gamma) \cdot \mathcal{Y}\gamma, \tag{9.13}$$

where $\mathbf{\Gamma}_{\alpha\gamma\beta} \overset{\text{def}}{=} \frac{1}{2}\left(\partial_\alpha \mathbf{g}_{\gamma\beta} + \partial_\beta \mathbf{g}_{\alpha\gamma} - \partial_\gamma \mathbf{g}_{\alpha\beta}\right)$ are the (lowered) Cartesian Christoffel symbols of $\mathbf{g}$ and the final expression on RHS (9.13) is written schematically. Combining these identities, we conclude (9.11). ☐

Lemma 9.8 (Identity satisfied by $\breve{X}L^i$)**.** *There exist smooth functions, all schematically denoted by "f," such that the following identity holds, where the arrays $\vec{\Psi}$ and $\vec{\Psi}_{(\text{Partial})}$ are defined in Def. 2.8:*

$$\breve{X}L^i = \mathrm{f}(\gamma) \cdot \breve{X}\vec{\Psi} \cdot (-\delta_1^i + X_{(\text{Small})}^i) + \mathrm{f}(\gamma) \cdot \breve{X}\vec{\Psi}_{(\text{Partial})} + \mu\mathrm{f}(\gamma)\mathcal{P}\vec{\Psi} + \mathrm{f}(\gamma)\mathcal{Y}\mu. \tag{9.14}$$

Proof. The same proof of [73, Lemma 2.14] holds with minor modifications that take into account the expressions for $\mathbf{g}_{\alpha\beta}$ and $(\mathbf{g}^{-1})^{\alpha\beta}$ given by (2.15a) and (2.15b), the identity (3.24), and the definition (3.13) of $X_{(\text{Small})}^i$. ☐

9.5. **Deformation tensors.** In our analysis, we encounter the deformation tensors of various vectorfields.

Definition 9.9 (Deformation tensors)**.** Let Z be a spacetime vectorfield. We define the *deformation tensor* ${}^{(Z)}\boldsymbol{\pi}$ of Z (with respect to $\mathbf{g}$) to be the following symmetric type $\binom{0}{2}$ tensorfield:

$${}^{(Z)}\boldsymbol{\pi}_{\alpha\beta} \overset{\text{def}}{=} \mathcal{L}_Z \mathbf{g}_{\alpha\beta} = \mathbf{D}_\alpha Z_\beta + \mathbf{D}_\beta Z_\alpha, \tag{9.15}$$

where the final equality in (9.15) follows from the torsion-free property of $\mathbf{D}$.

9.6. **Identities involving the rough-toroidal components of deformation tensors.** In the next lemma, we derive simple identities relating the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-components of a deformation tensor ${}^{(Z)}\boldsymbol{\pi}$ and the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-components of $\mathcal{L}_Z\widetilde{g}$.

Lemma 9.10 (Relating the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ components of $\mathcal{L}_Z\widetilde{g}$ and ${}^{(Z)}\boldsymbol{\pi}$)**.** *Let Z be a spacetime vectorfield, and let $\widetilde{g}$ be the first fundamental form of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, as in Def. 6.2. Then the following identities hold for $A,B=2,3$:*

$$
{}^{(Z)}\boldsymbol{\pi}\left(\frac{\widetilde{\partial}}{\widetilde{\partial}x^A},\frac{\widetilde{\partial}}{\widetilde{\partial}x^B}\right)\stackrel{def}{=}[\mathcal{L}_Z\mathbf{g}]\left(\frac{\widetilde{\partial}}{\widetilde{\partial}x^A},\frac{\widetilde{\partial}}{\widetilde{\partial}x^B}\right)=[\mathcal{L}_Z\widetilde{g}]\left(\frac{\widetilde{\partial}}{\widetilde{\partial}x^A},\frac{\widetilde{\partial}}{\widetilde{\partial}x^B}\right). \tag{9.16}
$$

Proof. First, using (6.23a) and the Leibniz rule for Lie differentiation, we deduce:

$$
\begin{aligned}
{}^{(Z)}\boldsymbol{\pi}\stackrel{\text{def}}{=}\mathcal{L}_Z\mathbf{g}=&-(\mathcal{L}_Z{}^{(\mathfrak{n})}\hat{N}_\flat)\otimes{}^{(\mathfrak{n})}\hat{N}_\flat-{}^{(\mathfrak{n})}\hat{N}_\flat\otimes\mathcal{L}_Z{}^{(\mathfrak{n})}\hat{N}_\flat\\
&+(\mathcal{L}_Z{}^{(\mathfrak{n})}\hat{R}_\flat)\otimes{}^{(\mathfrak{n})}\hat{R}_\flat+{}^{(\mathfrak{n})}\hat{R}_\flat\otimes\mathcal{L}_Z{}^{(\mathfrak{n})}\hat{R}_\flat+\mathcal{L}_Z\widetilde{g},
\end{aligned} \tag{9.17}
$$

where ${}^{(\mathfrak{n})}\hat{N}_\flat$ denotes the one-form $\mathbf{g}$-dual to ${}^{(\mathfrak{n})}\hat{N}$ and ${}^{(\mathfrak{n})}\hat{R}_\flat$ denotes the one-form $\mathbf{g}$-dual to ${}^{(\mathfrak{n})}\hat{R}$. Contracting (9.17) against $\frac{\widetilde{\partial}}{\widetilde{\partial}x^A}\otimes\frac{\widetilde{\partial}}{\widetilde{\partial}x^B}$ and using that ${}^{(\mathfrak{n})}\hat{N}$ and ${}^{(\mathfrak{n})}\hat{R}$ are $\mathbf{g}$-orthogonal to $\left\{\frac{\widetilde{\partial}}{\widetilde{\partial}x^2},\frac{\widetilde{\partial}}{\widetilde{\partial}x^3}\right\}$, we conclude (9.16). □

9.7. **A schematic rewriting of the wave equations satisfied by $\vec{\Psi}$.** The following lemma shows that $\breve{X}\Psi$ obeys a transport equation with source terms that are small but lose one derivative. We will use it in Sect. 17, when we derive improvements of the auxiliary bootstrap assumptions.

Lemma 9.11 (A schematic rewriting of the wave equations satisfied by $\vec{\Psi}$)**.** *The covariant wave equations* (2.22) *verified by $\Psi\in\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$ can be expressed in the following schematic form:*

$$
L\breve{X}\Psi=\mathrm{f}(\underline{\gamma})\mathcal{P}^2\vec{\Psi}+\mathrm{f}(\underline{\gamma},\mathcal{Z}\vec{\Psi})\mathcal{P}\gamma+\mathrm{f}(\underline{\gamma},S,\mathcal{Z}\vec{\Psi})\cdot\mathcal{P}^{\leq 1}(\Omega,S). \tag{9.18}
$$

Proof. We first decompose $\mu\times$LHS (2.22) using (3.51a), (3.44), Prop. 9.1, and (9.11). We then decompose $\mu\times$RHS (2.22) using Prop. 9.1, (9.7b), and (9.10). □

10. Parameters, their size assumptions, and conventions for constants

In this section, we list and describe the "size-parameters" that appear throughout the paper. These parameters will play a crucial role in Sect. 11, when we describe our assumptions on the data. Then, in Sect. 10.3, we state our conventions for how constants appearing in our analysis, such as C and $C_\blacklozenge$, are allowed to depend on the parameters.

10.1. **Parameters.**

10.1.1. *Parameters of the background simple isentropic plane-symmetric solutions.* First, we recall that in Appendix A, we construct a large family of "admissible" simple isentropic plane-symmetric solutions, where "admissible" means that it has properties such that it falls under the scope of our main results; see Def. A.7 for the precise definition. Each such admissible "background" solution has singularity-forming behavior that is described by the following collection (background solution-dependent) **positive** parameters, which we describe in detail in Appendix A:

$$
U_0,\,U_1,\,U_{\clubsuit},\,U_2,\,\tau_0,\,\mathfrak{n}_0,\,\mathfrak{m}_0^{\text{PS}},\,\mathfrak{m}_1^{\text{PS}},\,\mathring{\delta}_*^{\text{PS}},\,\mathring{\delta}^{\text{PS}},\,\mathring{\alpha}^{\text{PS}},M_2^{\text{PS}}. \tag{10.1}
$$

10.1.2. *Parameters of the perturbed solutions.* The positive parameters listed in (10.1) capture the behavior of various aspects of the background solution near its singular boundary. In Appendix B, we use Cauchy stability arguments to show that there are open sets of initial data on Σ_0, which are close to the data of one of the simple isentropic plane-symmetric solutions, such that the state of the perturbed solution near its singular boundary (but still within the region of classical existence) is described by the following list parameters:

$$
U_0,\,U_1,\,U_{\clubsuit},\,U_2,\,\tau_0,\,\mathfrak{n}_0,\,\mathfrak{m}_0,\,\mathfrak{m}_1,\,\mathring{\delta},\,\mathring{\delta}_*,\,\mathring{\alpha},\,M_2, \tag{10.2}
$$

as well as our main new *smallness parameter*: $\mathring{\epsilon}$. In Appendix B, we show that the parameters U_0, U_1, $U_{\clubsuit}$, U_2, τ_0, $\mathfrak{n}_0$, $\mathfrak{m}_0$ for the perturbed solutions can be chosen to be exactly the same as the ones for the background solutions. On the other hand, we will show (see (B.1)) that the remaining perturbed parameters can be chosen to be close to the background ones in the following sense:

$$\frac{1}{2} \leq \frac{\mathfrak{m}_1}{\mathfrak{m}_1^{\mathrm{PS}}}, \frac{\mathring{\delta}_*}{\mathring{\delta}_*^{\mathrm{PS}}}, \frac{\mathring{\delta}}{\mathring{\delta}^{\mathrm{PS}}}, \frac{\mathring{\alpha}}{\mathring{\alpha}^{\mathrm{PS}}}, \frac{M_2}{M_2^{\mathrm{PS}}} \leq 2, \tag{10.3}$$

$$\mathring{\varepsilon} \geq 0 \text{ can be chosen as small as we want.} \tag{10.4}$$

10.1.3. *Informal description of the parameters.* Many of the parameters listed in Sect. 10.1.2 will not appear until later in the paper, but to help guide the reader, we now provide a summary of their role and that of a few other parameters too. We refer to Fig. 11 for an illustration that shows how some of the parameters are tied to the location of various subsets.

- The background density $\overline{\varrho} > 0$ is fixed throughout the article; see (2.2).
- N_{top} is an integer representing the maximum number of times we commute the equations when we derive energy estimates. In proving our main results, we assume that $N_{\mathrm{top}} \geq 24$.
- The parameters τ_0 and $\mathfrak{m}_0$ are related by $\tau_0 = -\mathfrak{m}_0$ (see Def. 4.8). We view τ_0 to be the "initial rough time," i.e., the value of ${}^{(\mathfrak{n})}\tau$ corresponding to the initial state of the solution near the singularity.
- The parameter $\mathfrak{m}_0$ is the minimum value of μ along the initial rough hypersurface portion ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{☢},U_{☢}]}$. For convenience, we assume that $\mathfrak{m}_0$ is small. While the smallness of $\mathfrak{m}_0$ is not essential, it allows us to focus on studying the solution only near the singularity and allows us to give short proofs of various estimates.
- The parameters $0 < U_1$, $0 < U_2$, $0 < U_{☢} < \min\{U_1, U_2\}$ delineate ranges of values for the eikonal function u in the problem under study. We will study the solution on various intervals of u-values, including: $[-U_1, -U_{☢}]$, $[-U_{☢}, U_{☢}]$, $[U_{☢}, U_2]$, and $[-U_1, U_2]$. **The interesting analysis will happen when $u \in [-U_{☢}, U_{☢}]$, since this region contains the singular boundary portion under study.**
- The parameter $\mathring{\alpha}$ measures the L^∞-size of the amplitude of $\mathcal{R}_{(+)}$ along the initial rough hypersurface ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$.
- The parameter $\mathfrak{m}_1$ quantifies the positivity of μ away from the interesting region, more precisely when $u \notin [-U_{☢}, U_{☢}]$; see (11.20).
- The parameter $\mathring{\delta}_*$ measures the size of the crucial factor that drives the blowup; see definition (11.6).
- The parameter $\mathring{\delta}$ measures the L^∞-size of the transversal derivatives of $\mathcal{R}_{(+)}$ along the initial rough hypersurface ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$. It also controls the L^∞ size of the transversal derivatives of various geometric quantities constructed out of the eikonal function. We make no smallness assumptions on $\mathring{\delta}$.
- The parameter $\mathring{\varepsilon}$ measures the extent to which the solution's data "break the simple isentropic plane-symmetry."
- ε is a small "bootstrap parameter" first appearing in Sect. 12.3.
- The parameter M_2 quantifies the transversal convexity of μ (namely, the positive size of various second-order $\mathcal{P}_u$-transversal derivatives of μ, including $\breve{X}\breve{X}\mu$, ${}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\breve{W}\mu$, etc.) in the interesting region ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{☢},U_{☢}]}$; see, for example, (11.18).
- The parameter $\mathfrak{n}_0$ is such that the transversal convexity mentioned above holds for $\mathfrak{n} \in [0, \mathfrak{n}_0]$.
- The parameter U_0 is defined by

$$U_0 \overset{\mathrm{def}}{=} U_1 + \frac{18}{\mathring{\delta}_*^{\mathrm{PS}}} \tag{10.5}$$

  (see (A.94)) and plays a role only in Appendices A and B and the proof of Lemma 27.3, where we show that there exist open sets of initial data satisfying all of our assumptions; see Sect. 10.2.

10.2. **Parameter size assumptions.** In this section, we state the size assumptions on the parameters that are sufficient for our main results to hold, i.e., for Theorems 31.1 and 34.1 to hold.

For the remainder of the article, when we say that "A is small relative to B," we mean that $A \geq 0$, that $B > 0$, and that there exists a continuous increasing function[53] $f : (0, \infty) \to (0, \infty)$ such that $A < f(B)$. The functions f are allowed to depend on the equation of state.

Assumption 10.1 (Size assumptions on the parameters)**.**

[53]Although we do not specify their form, the functions f could always be chosen to be polynomials with positive coefficients or exponentials of such polynomials.

- To close our estimates, we assume that the regularity-parameter N_{top} is an integer satisfying:

$$N_{\text{top}} \geq 24. \tag{10.6}$$

- We assume that the following parameters are positive, but they do not have to be small or large: $\overline{\varrho}$, U_1, $U_{\clubsuit}$, U_2, $\mathring{\delta}$, $\mathring{\delta}_*$, and $\mathfrak{m}_1$.
- We assume that $0 < M_2 < 1$. We make no other assumptions on M_2.
- We assume that $\mathring{\alpha} > 0$, and that $\mathring{\alpha}$ is small relative to 1 and small relative to the background density $\overline{\varrho}$.
- We assume that $\mathfrak{m}_0$ and $|\tau_0|$ are small relative to 1, $\mathring{\delta}^{-1}$, $\mathring{\delta}_*$, and M_2. This is possible in view of Remark B.1. In particular, we assume:

$$|\tau_0| < 1. \tag{10.7}$$

- We assume that:

$$\mathfrak{n}_0 \leq \frac{M_2 U_{\clubsuit}}{32}. \tag{10.8}$$

- We assume that $\mathring{\epsilon}$ is small relative to $\overline{\varrho}$, 1, $\mathring{\alpha}$, U_1, $U_{\clubsuit}$, U_2, $\mathring{\delta}^{-1}$, $\mathring{\delta}_*$, $\mathfrak{m}_0$, and M_2.
- We assume that $0 < U_{\clubsuit} < \min\{U_1, U_2\}$, $\mathfrak{m}_0 < \frac{\mathfrak{m}_1}{2}$, and $\tau_0 = -\mathfrak{m}_0$.
- Our main results will hold under the assumption that $\varepsilon = C\mathring{\epsilon}$ for some large constant C, where ε is the bootstrap parameter first appearing in Sects. 12.3–12.3.2. This is consistent with the following parameter-size relations, which we assume in order to simplify our bootstrap argument:

$$\mathring{\epsilon} \leq \varepsilon \leq \mathring{\alpha}^2, \tag{10.9a}$$

$$\varepsilon^{3/2} \leq \mathring{\epsilon}. \tag{10.9b}$$

10.3. **Conventions for constants.** In this section, we state our conventions for how the constants C, C_*, $\mathfrak{c}$, and $C_{\blacklozenge}$ appearing in our analysis are allowed to depend on the parameters introduced above.

- The constants C, C_*, and $\mathfrak{c}$ are free to vary from line to line, and we use them in a similar fashion; we mainly use "C" in our estimates, introducing C_* and $\mathfrak{c}$ only in a few arguments in which multiple constants play a role. These constants are allowed to depend on the nonlinearities (i.e., on the equation of state), and they can continuously depend on the quantities $\overline{\varrho}$, U_1, $U_{\clubsuit}$, U_2, $\mathring{\delta}^{-1}$, $\mathring{\delta}_*$, $\mathfrak{m}_1^{-1}$, and M_2^{-1} from Sect. 10.1.3. In particular, C and $\mathfrak{c}$ are allowed, in principle, to *increase* with respect to U_1, $U_{\clubsuit}$, U_2, $\mathring{\delta}^{-1}$, $\mathring{\delta}_*$, $\mathfrak{m}_1^{-1}$, and M_2^{-1}. However, C, C_*, and $\mathfrak{c}$ can be chosen to be **independent of the parameters** $\mathring{\alpha}$, $\mathring{\epsilon}$, ε, τ_0, $\mathfrak{m}_0$, and $\mathfrak{n}_0$ under the smallness assumptions of Sect. 10.2. **In particular, C, C_*, and $\mathfrak{c}$ can be chosen to be independent of the value of $\mathfrak{n}$, as long as $\mathfrak{n} \in [0, \mathfrak{n}_0]$.**
- $A \lesssim B$ means that there exists a constant $C > 0$ (where C has the properties described above) such that $A \leq CB$.
- $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.
- $A = \mathcal{O}(B)$ means that $|A| \lesssim |B|$.
- Constants $C_{\blacklozenge}$ are also allowed to vary from line to line.

  However, unlike C and $\mathfrak{c}$, the $C_{\blacklozenge}$ are **universal** in the sense that under the smallness assumptions of Sect. 10.2, they can be chosen to be **independent** of $\overline{\varrho}$, U_0, U_1, $U_{\clubsuit}$, U_2, τ_0, $\mathfrak{n}_0$, $\mathfrak{m}_0$, $\mathfrak{m}_1$, $\mathring{\delta}$, $\mathring{\delta}_*$, $\mathring{\alpha}$, M_2, and $\mathring{\epsilon}$, and also independent of the equation of state.
- $A = \mathcal{O}_{\blacklozenge}(B)$ means that there exists a constant $C_{\blacklozenge} > 0$ (where $C_{\blacklozenge}$ has the properties described above) such that $|A| \leq C_{\blacklozenge}|B|$.
- As examples, we note that $\mathring{\delta}\mathring{\epsilon} \leq 1 \overset{\text{def}}{=} C_{\blacklozenge}$ and $\mathring{\delta}^2\mathring{\epsilon} \leq 1 \overset{\text{def}}{=} C_{\blacklozenge}$ (because $\mathring{\epsilon}$ is assumed to be small relative to $\mathring{\delta}^{-1}$ and relative to increasing functions of $\mathring{\delta}^{-1}$, such as $\mathring{\delta}^{-2}$), that $10\mathring{\alpha}^2 \leq C_{\blacklozenge}\mathring{\alpha}$, while we have only $\mathring{\delta}\mathring{\alpha} \leq C$ (i.e., our smallness assumptions on $\mathring{\alpha}$ are not strong enough to ensure that $\mathring{\delta}\mathring{\alpha}$ is small because $\mathring{\delta}$ might be large).
- As another example, in our estimates (e.g., the proof of (18.5)), by assuming that $|\tau_0|$ is small and using that C is independent of $|\tau_0|$ (in particular, C does not implicitly contain any factors of $\frac{1}{|\tau_0|}$), we can ensure that $C|\tau_0| \leq \frac{1}{2M_2}$ and $C|\tau_0| \leq \frac{M_2}{2}$.

11. Assumptions on the data

In this section, we state our assumptions on the data in terms of the parameters listed in Sect. 10.1. Moreover, in Appendix B, we show that our assumptions are satisfied by an open set of data that are close to the data of simple isentropic plane-wave solutions. The analysis in Appendix B is based on the construction – carried out in Appendix A – of simple isentropic plane-symmetric solutions that satisfy the assumptions, as well as (mostly) standard Cauchy stability arguments, which we outline.

In the rest of the paper, N_{top} denotes a fixed integer representing the maximum number of times we need to commute the equations of Theorem 2.15 for all estimates to close. The proof of our main results relies on the assumption (10.6).

11.1. **Background solutions $\mathcal{R}_{(+)}^{\mathrm{PS}}$ and bona fide initial data on Σ_0.** Fix any of the "admissible" background (shock-forming) simple isentropic plane-symmetric solutions that we construct in Appendix A, where we define "admissible" in Def. A.7. For such solutions, only a single Riemann invariant is non-vanishing; we denote it by $\mathcal{R}_{(+)}^{\mathrm{PS}}$. In the rest of Sect. 11.1, we view $\mathcal{R}_{(+)}^{\mathrm{PS}}$ as a solution in three spatial dimensions that is independent of the torus coordinates (x^2, x^3).

We now discuss the initial data of a perturbation of one of the background solutions. We consider the "bona fide initial data" of the perturbed solution to be:

$$\vec{\Psi}\big|_{\Sigma_0} = (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)\big|_{\Sigma_0} \stackrel{\text{def}}{=} \left(\mathring{\mathcal{R}}_{(+)}, \mathring{\mathcal{R}}_{(-)}, \mathring{v}^2, \mathring{v}^3, \mathring{s}\right), \tag{11.1}$$

where $\mathring{\mathcal{R}}_{(+)}, \mathring{\mathcal{R}}_{(-)}, \mathring{v}^2, \mathring{v}^3, \mathring{s} : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$ are given scalar-valued functions. Let $\left(\mathring{\Omega}^i, \mathring{S}^i, \mathring{\mathcal{C}}^i, \mathring{\mathcal{D}}\right)_{i=1,2,3}$ respectively denote the initial data on Σ_0 of $\left(\Omega^i, S^i, \mathcal{C}^i, \mathcal{D}\right)_{i=1,2,3}$. Note that these data are determined by $\left(\mathring{\mathcal{R}}_{(+)}, \mathring{\mathcal{R}}_{(-)}, \mathring{v}^2, \mathring{v}^3, \mathring{s}\right)$, the compressible Euler equations (2.6a)–(2.6c), definition (2.7), and Def. 2.7.

Remark 11.1 (The data of the eikonal function quantities on Σ_0 are determined)**.** Recall that the initial condition of the eikonal function is $u|_{\Sigma_0} = -x^1$ (see (3.1)). It is straightforward to check that this initial condition and $\vec{\Psi}\big|_{\Sigma_0}$ together determine the data of all of the auxiliary quantities constructed out of u, such as $\upmu|_{\Sigma_0}$, $L^i|_{\Sigma_0}$, etc.

We next note that relative to the Cartesian coordinates (t, x^1, x^2, x^3), we have (see definition (3.3c)):

$$\ell_{0,u} = \left\{(0, x^1, x^2, x^3) \mid x^1 = -u, (x^2, x^3) \in \mathbb{T}^2\right\}. \tag{11.2}$$

Moreover, we recall (see (3.4a)) that for $u_1 \leq u_2$, relative to the Cartesian coordinates, we have:

$$\Sigma_0^{[u_1,u_2]} = \left\{(0, x^1, x^2, x^3) \mid -x^1 \in [u_1, u_2], (x^2, x^3) \in \mathbb{T}^2\right\}. \tag{11.3}$$

In the next definition, we provide a family of norms of the data perturbations on Σ_0. Under suitable assumptions, smallness of the norms suffices for our main results to hold.

Definition 11.2 (Norms of the data perturbation on Σ_0)**.** Let N_{top} be a fixed integer satisfying (10.6), and let $\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}} \stackrel{\text{def}}{=} \mathcal{R}_{(+)}^{\mathrm{PS}}|_{\Sigma_0}$ denote the initial data of $\mathcal{R}_{(+)}^{\mathrm{PS}}$ on Σ_0. Given real numbers $u_1 < u_2$, we define the $\mathring{\Delta}_{\Sigma_0^{[u_1,u_2]}}^{N_{\mathrm{top}}+1}$ to be the following Sobolev norm of the perturbation of the data from the data of the background solution (that is, we subtract off $\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}$ and then take the norm):

$$\begin{aligned}
\mathring{\Delta}_{\Sigma_0^{[u_1,u_2]}}^{N_{\mathrm{top}}+1} \stackrel{\text{def}}{=} & \left\|\left(\mathring{\mathcal{R}}_{(+)} - \mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}, \mathring{\mathcal{R}}_{(-)}, \mathring{v}^2, \mathring{v}^3, \mathring{s}\right)\right\|_{H_{\mathrm{Cartesian}}^{N_{\mathrm{top}}+1}(\Sigma_0^{[u_1,u_2]})} \\
& + \left\|\left(\mathring{\Omega}^1, \mathring{\Omega}^2, \mathring{\Omega}^3, \mathring{S}^1, \mathring{S}^2, \mathring{S}^3\right)\right\|_{H_{\mathrm{Cartesian}}^{N_{\mathrm{top}}}(\Sigma_0^{[u_1,u_2]})} + \left\|\left(\mathring{\mathcal{C}}^1, \mathring{\mathcal{C}}^2, \mathring{\mathcal{C}}^3, \mathring{\mathcal{D}}\right)\right\|_{H_{\mathrm{Cartesian}}^{N_{\mathrm{top}}}(\Sigma_0^{[u_1,u_2]})} \\
& + \max_{u \in [u_1,u_2]} \sum_{|\vec{I}| \leq N_{\mathrm{top}}} \left\|\left(\partial_{\vec{I}}\mathring{\Omega}^1, \partial_{\vec{I}}\mathring{\Omega}^2, \partial_{\vec{I}}\mathring{\Omega}^3, \partial_{\vec{I}}\mathring{S}^1, \partial_{\vec{I}}\mathring{S}^2, \partial_{\vec{I}}\mathring{S}^3\right)\right\|_{L_{\mathrm{Cartesian}}^2(\ell_{0,u})}.
\end{aligned} \tag{11.4}$$

In (11.4),

$$\|f\|_{H_{\mathrm{Cartesian}}^N(\Sigma_0^{[u_1,u_2]})} \stackrel{\text{def}}{=} \left\{\sum_{|\vec{I}| \leq N} \int_{\Sigma_0^{[u_1,u_2]}} \left[\partial_{\vec{I}} f(t=0, x^1, x^2, x^3)\right]^2 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3\right\}^{1/2}, \tag{11.5a}$$

$$\|f\|_{L_{\mathrm{Cartesian}}^2(\ell_{0,u})} \stackrel{\text{def}}{=} \left\{\int_{\mathbb{T}^2} \left[\partial_{\vec{I}} f(t=0, x^1=-u, x^2, x^3)\right]^2 \mathrm{d}x^2 \mathrm{d}x^3\right\}^{1/2}, \tag{11.5b}$$

where $\vec{I}$ denotes a multi-index of order $|\vec{I}|$ corresponding to repeated partial differentiation with respect to the Cartesian *spatial* coordinates, i.e., repeated differentiation with respect to $\partial_1, \partial_2, \partial_3$. In particular, $\|f\|_{H^N_{\text{Cartesian}}(\Sigma_0^{[u_1,u_2]})}$ is the standard order N Sobolev norm of f along $\Sigma_0^{[u_1,u_2]}$, while the $\|\cdot\|_{L^2_{\text{Cartesian}}(\ell_{0,u})}$ norm sum on the last line of RHS (11.4) controls tangential *and* transversal spatial derivatives of $(\mathring{\Omega}, \mathring{S})$ along $\ell_{0,u}$.

11.2. **The assumptions on the initial data.** In order for our main results to hold, it suffices for $\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ to be sufficiently small, where $\mathring{\delta}_*^{\text{PS}} > 0$ and $U_0 \stackrel{\text{def}}{=} U_1 + \frac{18}{\mathring{\delta}_*^{\text{PS}}} > 0$ are parameters associated to the background solution (see Appendix A) and N_{top} is the fixed integer satisfying (10.6).

For the solutions under study, the analysis is difficult/interesting only near the singularity. Moreover, the structures we use to detect the singular boundary become evident only late in the classical evolution, i.e., close to the singular boundary. For this reason, in Sect. 11.2, we find it convenient to describe the state of the fluid solution and acoustic geometry (e.g., μ, L^i, and χ) on the "late-time" rough hypersurface portion ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$, as well as the null hypersurface portion $\mathcal{P}_{-U_1}^{4\mathring{\delta}_*^{-1}}$, where $\mathring{\delta}_* > 0$ is defined in (11.6). Our description is in terms of *assumed bounds* for various norms of the solution on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$, and $\mathcal{P}_{-U_1}^{4\mathring{\delta}_*^{-1}}$ in terms of the parameters of Sect. 10.2. We refer to Fig. 9 for an illustration of these data-hypersurfaces.

In Appendix B, we use Cauchy stability-type arguments to sketch a proof that if $\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[U_0,U_2]}}$ is sufficiently small, then the assumptions we state in Sect. 11.2 are satisfied, where our main smallness parameter $\mathring{\epsilon}$ (which vanishes for the background solutions) satisfies $\mathring{\epsilon} \lesssim \mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[U_0,U_2]}}$ whenever $\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[U_0,U_2]}}$ is sufficiently small, where the implicit constants depend on the background solution. Since our main results apply whenever $\mathring{\epsilon}$ is sufficiently small, this in particular shows that there are open sets of data for which our main theorem holds.

Remark 11.3 (We don't need the background solutions to control the dynamics)**.** We "use" the plane-symmetric background solutions only to show that there exist open sets of data that satisfy our assumptions on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$, and $\mathcal{P}_{-U_1}^{4\mathring{\delta}_*^{-1}}$. When studying the evolution to the future of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, we never actually have to "subtract off" any background solution or even refer to one at all.

11.2.1. *Quantitative assumptions on the data of the fluid and eikonal function quantities along* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$, *and* $\mathcal{P}_{-U_1}^{4\mathring{\delta}_*^{-1}}$. In this section, we state quantitative assumptions on the data of the fluid variables and the eikonal function quantities along ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$, and $\mathcal{P}_{-U_1}^{4\mathring{\delta}_*^{-1}}$. We again emphasize that our assumptions hold for perturbations of the simple isentropic plane-symmetric solutions from Appendix B.

We start by defining the data-parameter $\mathring{\delta}_*$.

Definition 11.4 (Key data-parameter tied to the Cartesian time of first blowup)**.** We define $\mathring{\delta}_*$ by:

$$\mathring{\delta}_* \stackrel{\text{def}}{=} \sup_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}} \frac{1}{2}\left[(c^{-1}c_{;\rho}+1)\check{X}\mathcal{R}_{(+)}\right]_+. \tag{11.6}$$

We assume that:

$$\mathring{\delta}_* > 0. \tag{11.7}$$

Remark 11.5 (Connection between $\mathring{\delta}_*$ and the Cartesian time of first blowup)**.** For simple isentropic plane-symmetric solutions, the Cartesian time of first shock formation is precisely $\frac{1}{\mathring{\delta}_*}$; see Appendix A. Our main results show that for the perturbed solutions under study, the Cartesian time of first blowup, which we denote here by $t_{\text{First shock}}$, satisfies $t_{\text{First shock}} = \{1 + \mathcal{O}(\mathring{\epsilon})\}\frac{1}{\mathring{\delta}_*}$.

We refer to Sect. 8.3 for notation regarding strings of commutation vectorfields and to Defs. 8.7 and 8.8 for the definitions of our L^2 and L^∞ norms.

L^∞ **assumptions on the wave-variables and their pure transversal derivatives**. For $u \in [-U_1, U_2]$ and $M = 1, 2, 3, 4$, we assume (recall that $\vec{\Psi}$ and $\vec{\Psi}_{(\text{Partial})}$ are defined in Def. 2.8):

$$\left\| \mathcal{R}_{(+)} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\alpha}, \tag{11.8a}$$

$$\left\| \breve{X}^M \mathcal{R}_{(+)} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\delta}, \tag{11.8b}$$

$$\left\| \vec{\Psi}_{(\text{Partial})} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)}, \left\| \breve{X}^M \vec{\Psi}_{(\text{Partial})} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\epsilon}. \tag{11.8c}$$

L^∞ **assumptions involving tangential derivatives of the wave-variables**. For $u \in [-U_1, U_2]$, we assume:

$$\begin{aligned} &\left\| \mathcal{P}^{[1,N_{\text{top}}-10]} \vec{\Psi} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)}, \left\| \mathcal{Z}_*^{[1,N_{\text{top}}-11;1]} \vec{\Psi} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)}, \left\| \mathcal{Z}_*^{[1,6];2} \vec{\Psi} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)}, \\ &\left\| \mathcal{Z}_*^{[1,5];3} \vec{\Psi} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)}, \left\| L \breve{X}\breve{X}\breve{X}\breve{X} \vec{\Psi} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\epsilon}. \end{aligned} \tag{11.9}$$

L^∞ **assumptions involving tangential derivatives of the transport-variables**. For $u \in [-U_1, U_2]$, we assume:

$$\left\| \mathcal{P}^{\leq N_{\text{top}}-11} (\Omega, S) \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)}, \left\| \mathcal{P}^{\leq N_{\text{top}}-12} (\mathcal{C}, \mathcal{D}) \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\epsilon}. \tag{11.10}$$

L^2 **assumptions along** ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$. We assume:

$$\left\| \mathcal{P}^{[1,N_{\text{top}}+1]} \vec{\Psi} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}\right)} \leq \mathring{\epsilon}, \tag{11.11a}$$

$$\left\| \breve{X} \mathcal{P}^{[1,N_{\text{top}}]} \vec{\Psi} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}\right)} \leq \mathring{\epsilon}, \tag{11.11b}$$

$$\left\| \mathcal{P}^{\leq N_{\text{top}}} (\Omega, S) \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}\right)} \leq \mathring{\epsilon}, \tag{11.11c}$$

$$\left\| \mathcal{P}^{\leq N_{\text{top}}} (\mathcal{C}, \mathcal{D}) \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}\right)} \leq \mathring{\epsilon}. \tag{11.11d}$$

L^2 **assumptions along** $\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}$. We assume:

$$\left\| \mathcal{P}^{\leq N_{\text{top}}+1} \vec{\Psi} \right\|_{L^2\left(\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}\right)} \leq \mathring{\epsilon}, \tag{11.12a}$$

$$\left\| \mathcal{P}^{\leq N_{\text{top}}} (\Omega, S) \right\|_{L^2\left(\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}\right)} \leq \mathring{\epsilon}, \tag{11.12b}$$

$$\left\| \mathcal{P}^{\leq N_{\text{top}}} (\mathcal{C}, \mathcal{D}) \right\|_{L^2\left(\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}\right)} \leq \mathring{\epsilon}. \tag{11.12c}$$

L^2 **assumptions along** ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$. For $u \in [-U_1, U_2]$, we assume:

$$\left\| \mathcal{P}^{[1,N_{\text{top}}]} \vec{\Psi} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\epsilon}, \tag{11.13a}$$

$$\left\| \mathcal{P}^{\leq N_{\text{top}}} (\Omega, S) \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\epsilon}, \tag{11.13b}$$

$$\left\| \mathcal{P}^{\leq N_{\text{top}}-1} (\mathcal{C}, \mathcal{D}) \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)} \leq \mathring{\epsilon}. \tag{11.13c}$$

L^{∞} assumptions tied to transversal derivatives of the eikonal function quantities. For $u \in [-U_1, U_2]$ and $M = 0, 1, 2, 3$, we assume:[54]

$$\left\| L\breve{X}^M \upmu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \frac{1}{2} \left\| \breve{X}^M \left\{ c^{-1}(c^{-1}c_{;\rho}+1)\breve{X}\mathcal{R}_{(+)} \right\} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \mathring{\varepsilon}, \tag{11.14a}$$

$$\left\| \breve{X}^M \upmu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \left\| \breve{X}^M \left\{ c^{-1} \right\} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \frac{1}{2\mathring{\delta}_*} \left\| \breve{X}^M \left\{ c^{-1}(c^{-1}c_{;\rho}+1)\breve{X}\mathcal{R}_{(+)} \right\} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \mathring{\varepsilon}. \tag{11.14b}$$

Moreover, for $u \in [-U_1, U_2]$ and $M = 1, 2, 3, 4$, we assume:

$$\left\| L\breve{X}^M L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \mathring{\varepsilon}, \tag{11.14c}$$

$$\left\| \breve{X}^M L^1_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \mathring{\delta}, \tag{11.14d}$$

$$\left\| \breve{X}^M L^A_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \mathring{\varepsilon}. \tag{11.14e}$$

L^{∞} assumptions involving tangential derivatives of the eikonal function quantities. For $u \in [-U_1, U_2]$, we assume:

$$\left\| \mathcal{P}_*^{[1,N_{\text{top}}-12]} \upmu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})}, \left\| \mathcal{Z}_{**}^{[1,5];1} \upmu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})}, \left\| \mathcal{Z}_{**}^{[1,4];2} \upmu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \mathring{\varepsilon}, \tag{11.15a}$$

$$\left\| L^1_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \mathring{\alpha}, \tag{11.15b}$$

$$\begin{aligned} &\left\| L^A_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})}, \left\| \mathcal{P}^{[1,N_{\text{top}}-11]} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})}, \left\| \mathcal{Z}_*^{[1,N_{\text{top}}-12];1} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})}, \\ &\qquad \left\| \mathcal{Z}_*^{[1,5];2} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})}, \left\| \mathcal{Z}_*^{[1,4];3} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \mathring{\varepsilon}. \end{aligned} \tag{11.15c}$$

L^2 assumptions for the eikonal function quantities. We assume:

$$\left\| \mathcal{P}_*^{[1,N_{\text{top}}]} \upmu \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}\right)}, \left\| \mathcal{P}^{[1,N_{\text{top}}]} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}\right)}, \left\| \mathcal{Z}_*^{[1,N_{\text{top}}];1} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}\right)} \leq \mathring{\varepsilon}, \tag{11.16a}$$

$$\left\| \not{\mathcal{L}}_{\mathcal{Z}}^{\leq N_{\text{top}}-1;1} \upchi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}\right)}, \left\| \not{\mathcal{L}}_{\mathcal{P}}^{\leq N_{\text{top}}} \upchi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}\right)} \leq \mathring{\varepsilon}. \tag{11.16b}$$

11.2.2. *Data-assumptions for the size of t and x^1 on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$.* With $\mathring{\delta}_*$ as in (11.6), we assume:

$$\frac{1}{2\mathring{\delta}_*} \leq \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}} t \leq \sup_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}} t \leq \frac{2}{\mathring{\delta}_*}, \tag{11.17a}$$

$$-U_2 + \frac{1}{2\mathring{\delta}_*} \leq \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}} x^1 \leq \sup_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}} x^1 \leq U_1 + \frac{2}{\mathring{\delta}_*}. \tag{11.17b}$$

11.2.3. *Localized assumptions on the data of $\upmu$ and its derivatives on the initial rough hypersurface.* We now make localized quantitative and qualitative assumptions on the behavior of $\upmu$ along ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, where we recall that $0 < U_{\clubsuit} < \min\{U_1, U_2\}$ (see Fig. 11).

[54] Some of these "assumptions" can in fact be derived as a consequence of other assumptions, up to constant factors that we absorb into the parameters $\mathring{\delta}$, $\mathring{\alpha}$, and $\mathring{\varepsilon}$. For convenience, instead of deriving those "assumptions," we just assume them. The same is true for other assumptions on the eikonal function quantities stated below. For example, even though the data of the $\breve{X}$-derivatives of L^i could be controlled via the identity (9.14), we just assume (11.14d). We also refer to (3.44) and (3.46) for intuition behind the form of RHS (11.14a), to (A.10) for intuition behind the first term on RHS (11.14b), and to Remark 11.5 for intuition behind the factor $\mathring{\delta}_*^{-1}$ in the second term on RHS (11.14b).

Transversal convexity of $\upmu$. We assume that there exists a constant M_2 satisfying $0 < M_2 < 1$ such that:[55]

$$
\begin{aligned}
M_2 \leq \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]}} & \Big\{ {}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\breve{W}\upmu,\ {}^{(\mathfrak{n})}\breve{W}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu - \frac{(\breve{X}\upmu)L\breve{X}\upmu}{L\upmu}, \\
& \breve{X}\breve{X}\upmu + \frac{\mathfrak{n}L\breve{X}\upmu}{L\upmu},\ \frac{\partial}{\partial u}\breve{X}\upmu,\ \frac{\partial}{\partial u}\breve{X}\upmu - \frac{(\frac{\partial}{\partial u}\upmu)\frac{\partial}{\partial t}\breve{X}\upmu}{\frac{\partial}{\partial t}\upmu},\ \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\upmu \Big\} \\
\leq \max_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]}} & \Big\{ {}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\breve{W}\upmu,\ {}^{(\mathfrak{n})}\breve{W}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu - \frac{(\breve{X}\upmu)L\breve{X}\upmu}{L\upmu}, \\
& \breve{X}\breve{X}\upmu + \frac{\mathfrak{n}L\breve{X}\upmu}{L\upmu},\ \frac{\partial}{\partial u}\breve{X}\upmu,\ \frac{\partial}{\partial u}\breve{X}\upmu - \frac{(\frac{\partial}{\partial u}\upmu)\frac{\partial}{\partial t}\breve{X}\upmu}{\frac{\partial}{\partial t}\upmu},\ \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\upmu \Big\} \\
\leq \frac{1}{M_2}. &
\end{aligned}
\tag{11.18}
$$

$\breve{\mathbb{X}}_{-\mathfrak{n}}$ is located near $\{u=0\}$. With $\breve{\mathbb{X}}_{-\mathfrak{n}}$ as in (4.3b), we assume that:

$$\breve{\mathbb{X}}_{-\mathfrak{n}} \subset {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-\frac{1}{4}U_{\star},\frac{1}{4}U_{\star}]}, \tag{11.19a}$$

$$\min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]} \setminus {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-\frac{1}{2}U_{\star},\frac{1}{2}U_{\star}]}} |\breve{X}\upmu + \mathfrak{n}| \geq \frac{M_2 U_{\star}}{4}. \tag{11.19b}$$

Quantitative positivity of $\upmu$ away from the interesting region. We assume that there is a constant $\mathfrak{m}_1 > 0$ such that:

$$\min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]} \setminus {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]}} \upmu \geq \mathfrak{m}_1. \tag{11.20}$$

We also assume (recall that $\mathfrak{m}_0 = -\tau_0$):

$$\frac{\mathfrak{m}_1}{2} > \mathfrak{m}_0. \tag{11.21}$$

Quantitative negativity of $L\upmu$. We assume that the following inequalities hold, where $\mathring{\delta}_*$ is defined in (11.6):

$$-\frac{17}{16}\mathring{\delta}_* \leq \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]}} L\upmu \leq \max_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]}} L\upmu \leq -\frac{15}{16}\mathring{\delta}_*. \tag{11.22}$$

Quantitative bounds on the Jacobian ${}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}$. We assume that the Jacobian matrix ${}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}(q)$ defined in (5.4b) is invertible for every $q \in \{\tau_0\} \times [-U_{\star}, U_{\star}] \times \mathbb{T}^2$ and that:

$$\sup_{q_1,q_2 \in \{\tau_0\}\times[-U_{\star},U_{\star}]\times\mathbb{T}^2} \left| {}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}^{-1}(q_1)\,{}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}(q_2) - \mathrm{ID} \right|_{\mathrm{Euc}} \leq \frac{1}{3}, \tag{11.23}$$

where $|\cdot|_{\mathrm{Euc}}$ is the standard Frobenius norm on matrices (equal to the square root of the sum of the squares of the matrix entries) and ID denotes the 4×4 identity matrix.

Quantitative bounds on the Jacobian ${}^{(\breve{\mathscr{M}})}\mathbf{J}$. We assume that the Jacobian matrix ${}^{(\breve{\mathscr{M}})}\mathbf{J}(p)$ defined in (5.3b) is invertible for every $p \in {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]}$ and that the following holds, where $\breve{\mathbf{T}}_{-\tau_0,0} \subset {}^{(0)}\widetilde{\Sigma}_{\tau_0}^{[-U_{\star},U_{\star}]}$ is the $\upmu$-adapted torus defined in (4.3c):

$$\sup_{\substack{p_1 \in \breve{\mathbf{T}}_{-\tau_0,0} \\ p_2 \in {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-\frac{1}{4}U_{\star},\frac{1}{4}U_{\star}]}}} \left| {}^{(\breve{\mathscr{M}})}\mathbf{J}(p_1)\,{}^{(\breve{\mathscr{M}})}\mathbf{J}^{-1}(p_2) - \mathrm{ID} \right|_{\mathrm{Euc}} \leq \frac{1}{3}, \tag{11.24}$$

[55] Most of the estimates assumed in (11.18) are redundant in the sense that they could derived, up to constant factors, as consequences of other assumptions; it is only for convenience that we have chosen to make assumptions on all these quantities.

where $|\cdot|_{\text{Euc}}$ is the standard Frobenius norm on matrices (equal to the square root of the sum of the squares of the matrix entries).

12. The bootstrap assumptions, except those concerning the wave energies

In proving our main results, we rely on a continuity argument based on deriving improvements of a set of bootstrap assumptions for the solution on a region of the form ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. In this section, we set up the bootstrap argument and state all the bootstrap assumptions, except for the ones concerning L^2-type energies, which we provide in Sect. 24.3.

12.1. The start of the bootstrap argument: the bootstrap time interval $[\tau_0,\tau_{\text{Boot}})$ and $\mathfrak{m}_{\text{Boot}}$.

From now until Sect. 31, we assume that there is a classical solution on an "open-at-the-top" region of the form ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, and we will state our bootstrap assumptions on the same region. Here and throughout,

$$\tau_{\text{Boot}} \in (\frac{3}{4}\tau_0, 0] \tag{12.1}$$

is the "bootstrap rough-time," where we recall that the small parameter $\tau_0 < 0$ is the most negative value achieved by the rough time function ${}^{(\mathfrak{n})}\tau$. In Appendix B, we provide Cauchy stability arguments guaranteeing the existence of a region of classical existence of the form ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ for some τ_{Boot} satisfying (12.1). Hence, at the start[56] of our bootstrap argument, ${}^{(\mathfrak{n})}\tau$ has range $[\tau_0,\tau_{\text{Boot}})$, which contains $[\tau_0,\frac{3}{4}\tau_0]$. We also set:

$$\mathfrak{m}_{\text{Boot}} \overset{\text{def}}{=} -\tau_{\text{Boot}} \geq 0. \tag{12.2}$$

12.2. Bootstrap assumptions tied to the fundamental scaffolding of the analysis.

The bootstrap assumptions in this section ensure that various fundamental aspects of our approach (such as the change of variables maps from Sect. 5.1) are well-defined and enjoy basic properties that we use throughout the rest of the paper.

12.2.1. *Bootstrap assumptions for the inverse foliation density.*

1. We assume that the following estimate holds on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:

$$\mu > 0. \tag{\textbf{BA} $\mu > 0$}$$

2. We assume that $L\mu$ and $\frac{\partial}{\partial t}\mu$ are quantitatively negative in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}$, where $\mathring{\delta}_*$ is defined in (11.6):

$$-\frac{5}{4}\delta_* \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}} L\mu \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}} L\mu \leq -\frac{3}{4}\delta_*, \tag{\textbf{BA} $L\mu$ neg}$$

$$-\frac{5}{4}\delta_* \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}} \frac{\partial}{\partial t}\mu \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}} \frac{\partial}{\partial t}\mu \leq -\frac{3}{4}\delta_*. \tag{\textbf{BA} $\frac{\partial}{\partial t}\mu$ neg}$$

[56] In Lemma 15.5, we will show that ${}^{(\mathfrak{n})}\tau$ can be suitably extended to have range $[\tau_0,\tau_{\text{Boot}}]$, and in Theorem 31.1, we will show that it can be suitably extended to have range $[\tau_0, 0]$.

3. We assume that near $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$, $\upmu$ is quantitatively convex in directions transversal to $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$. That is, we assume that the following estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}$, where $0 < M_2 < 1$ is as in (11.18):

$$\begin{aligned} \frac{M_2}{4} &\leq \inf_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}} \Big\{ {}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\breve{W}\upmu,\ {}^{(\mathfrak{n})}\breve{W}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu - \frac{(\breve{X}\upmu)L\breve{X}\upmu}{L\upmu}, \\ &\qquad \breve{X}\breve{X}\upmu + \frac{\mathfrak{n}L\breve{X}\upmu}{L\upmu},\ \frac{\partial}{\partial u}\breve{X}\upmu,\ \frac{\partial}{\partial u}\breve{X}\upmu - \frac{(\frac{\partial}{\partial u}\upmu)\frac{\partial}{\partial t}\breve{X}\upmu}{\frac{\partial}{\partial t}\upmu},\ \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\upmu \Big\} \\ &\leq \sup_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_{☢},U_{☢}]}} \Big\{ {}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\breve{W}\upmu,\ {}^{(\mathfrak{n})}\breve{W}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu,\ \breve{X}\breve{X}\upmu - \frac{(\breve{X}\upmu)L\breve{X}\upmu}{L\upmu}, \qquad (\textbf{BA}\ \upmu\ \text{cnvx}) \\ &\qquad \breve{X}\breve{X}\upmu + \frac{\mathfrak{n}L\breve{X}\upmu}{L\upmu},\ \frac{\partial}{\partial u}\breve{X}\upmu,\ \frac{\partial}{\partial u}\breve{X}\upmu - \frac{(\frac{\partial}{\partial u}\upmu)\frac{\partial}{\partial t}\breve{X}\upmu}{\frac{\partial}{\partial t}\upmu},\ \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\upmu \Big\} \\ &\leq \frac{4}{M_2}. \end{aligned}$$

12.2.2. *Bootstrap assumptions for the rough time function.*

1. We assume that in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, we have:

$$\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau > 0. \qquad (\ \textbf{BA}\ \tfrac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau > 0)$$

2. We assume that the following estimates hold, where $\mathring{\delta}_*$ is defined in (11.6):

$$\frac{3}{4}\mathring{\delta}_* \leq L{}^{(\mathfrak{n})}\tau \leq \frac{5}{4}\mathring{\delta}_*, \qquad \text{on } {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}. \qquad (\ \textbf{BA}\ L^{(\mathfrak{n})}\tau)$$

12.2.3. *Bootstrap assumptions for change of variables maps.* We make the following assumptions for various change of variables maps.

1. The change of variables map $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ defined in (5.1) satisfies $\|\Upsilon\|_{C^{3,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]})} < \infty$ and is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ onto its image.
2. The change of variables map ${}^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\mathfrak{n})}\tau,u,x^2,x^3)$ defined in (5.2) satisfies $\left\|{}^{(\mathfrak{n})}\mathscr{T}\right\|_{C^{2,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]})} < \infty$ and is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ onto its image, which is $[\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2] \times \mathbb{T}^2$.
3. The change of variables map ${}^{(\mathfrak{n})}\Phi$ defined in (5.4a) satisfies $\left\|{}^{(\mathfrak{n})}\Phi\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]\times\mathbb{T}^2)} < \infty$ and is a diffeomorphism from $[\tau_0,\tau_{\text{Boot}}) \times [-U_{☢},U_{☢}] \times \mathbb{T}^2$ onto its image. Furthermore, the Jacobian matrix ${}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}(q)$ defined in (5.4b) is invertible for every $q \in [\tau_0,\tau_{\text{Boot}}) \times [-U_{☢},U_{☢}] \times \mathbb{T}^2$ and satisfies:

$$\max_{q_1,q_2\in[\tau_0,\tau_{\text{Boot}})\times[-U_{☢},U_{☢}]\times\mathbb{T}^2} \left| {}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}^{-1}(q_1)\,{}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}(q_2) - \text{ID} \right|_{\text{Euc}} \leq \frac{2}{3}. \tag{12.3}$$

12.2.4. *Bootstrap assumptions for the structure and location of* $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *and* $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$. Recall that the $\upmu$-adapted tori $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ are defined in (4.3c) and that the hypersurface portions $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ are defined in (4.7b). Also recall that $0 \leq \mathfrak{m}_{\text{Boot}} = -\tau_{\text{Boot}} < \mathfrak{m}_0 = -\tau_0$.

1. We assume that for each $\mathfrak{m} \in (\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]$, there exist scalar functions $\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}, \mathfrak{U}_{\mathfrak{m},-\mathfrak{n}} \in W^{2,\infty}(\mathbb{T}^2)$, depending on $\mathfrak{m}$ and $\mathfrak{n}$, such that relative to the geometric coordinates (t,u,x^2,x^3), we have:

$$\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} = \Big\{ \Big(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3), \mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3), x^2, x^3 \Big) \mid (x^2,x^3) \in \mathbb{T}^2 \Big\}. \qquad (\textbf{BA}\ \upmu - \textbf{TORI STRUCTURE})$$

   In particular, in geometric coordinates, $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ is a $W^{2,\infty}$ graph over $\mathbb{T}^2$.
2. We assume that for each fixed $\tau \in [\tau_0,\tau_{\text{Boot}}) = [-\mathfrak{m}_0,-\mathfrak{m}_{\text{Boot}})$,

$$\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} \subset {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]}. \qquad (\textbf{BA}\ \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} - \textbf{LOCATION})$$

3. We assume that:

$$\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]}. \qquad (\textbf{BA } \breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})} - \textbf{LOCATION})$$

4. We assume that the map ${}^{(\mathfrak{n})}E : (\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0] \times \mathbb{T}^2 \to {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]}$ defined by:

$$ {}^{(\mathfrak{n})}E(\mathfrak{m}, x^2, x^3) \stackrel{\text{def}}{=} \left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3), \mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3), x^2, x^3\right) \tag{12.4}$$

is a diffeomorphism from $(\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0] \times \mathbb{T}^2$ onto $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ such that for every $\mathfrak{m}' \in (\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0)$, we have:

$$ {}^{(\mathfrak{n})}E \in C^{1,1}([\mathfrak{m}', \mathfrak{m}_0] \times \mathbb{T}^2), \tag{12.5}$$

and such that:

$$-\infty < \inf_{(\mathfrak{m},x^2,x^3)\in(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2} \frac{\partial}{\partial \mathfrak{m}} \mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3) \le \sup_{(\mathfrak{m},x^2,x^3)\in(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2} \frac{\partial}{\partial \mathfrak{m}} \mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3) < 0. \tag{12.6}$$

In particular, $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ is a $C^{1,1}$ embedded sub-manifold-with-boundary of ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]}$ whose boundary is $\breve{\mathbf{T}}_{\mathfrak{m}_0,-\mathfrak{n}}$, and:

$$\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})} = \bigcup_{\mathfrak{m}\in(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]} \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}. \qquad (\textbf{BA } \breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})} - \textbf{FOLIATED})$$

12.2.5. *Bootstrap assumptions for the size of t and x^1 on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$.* With $\mathring{\delta}_*$ as in (11.6), we assume that the following inequalities hold for $\tau \in [\tau_0, \tau_{\text{Boot}})$:

$$\frac{1}{4\mathring{\delta}_*} \le \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}} t \le \max_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}} t \le \frac{4}{\mathring{\delta}_*}, \qquad (\textbf{BA } t - \textbf{SIZE})$$

$$-U_2 + \frac{1}{4\mathring{\delta}_*} \le \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}} x^1 \le \max_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}} x^1 \le U_1 + \frac{4}{\mathring{\delta}_*}. \qquad (\textbf{BA } x^1 - \textbf{SIZE})$$

12.2.6. *Soft bootstrap assumptions concerning regularity.* We assume that for every $\tau \in (\tau_0, \tau_{\text{Boot}})$, we have:

$$\vec{\Psi}, \Omega^i, S^i, \mathcal{C}^i, \mathcal{D} \in C^{3,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,U_2]}), \qquad (\textbf{BA Fluid Regularity})$$

$$\Upsilon \in C^{3,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,U_2]}), \qquad (\textbf{BA } \Upsilon \textbf{ Regularity})$$

$$L^i, \mu \in C^{2,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,U_2]}). \qquad (\textbf{BA Geometry Regularity})$$

Remark 12.1. In (**BA Fluid Regularity**)–(**BA Geometry Regularity**), we are not making any quantitative assumptions on the size of the norms. That is, we are assuming only that the norms are finite, e.g., $\|\mu\|_{C^{2,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,U_2]})} < \infty$. In Lemma 15.6, we will show that all of these norms are $\le C$.

12.3. **The main quantitative bootstrap assumptions.** We now state our main quantitative bootstrap assumptions. In the rest of the paper, $\varepsilon \ge 0$ denotes a small "bootstrap" parameter whose smallness we described in Sect. 10.2. Later on, we will close our bootstrap argument by setting $\varepsilon = C\mathring{\epsilon}$ for some large constant C, where $\mathring{\epsilon}$ is the data-size parameter from Sect. 11.2.1; see, in particular, Prop. 30.1.

12.3.1. *Fundamental quantitative bootstrap assumptions.* Our fundamental quantitative bootstrap assumptions for $\vec{\Psi}$, Ω, S, $\mathcal{C}$, and $\mathcal{D}$ are that the following inequalities hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:

$$\left\|\mathcal{P}^{[1,N_{\text{top}}-10]}\vec{\Psi}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\|\mathcal{P}^{\le N_{\text{top}}-11}(\Omega,S)\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\|\mathcal{P}^{\le N_{\text{top}}-12}(\mathcal{C},\mathcal{D})\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \le \varepsilon. \qquad (\textbf{BA } L^\infty \textbf{ FUND})$$

12.3.2. *Auxiliary bootstrap assumptions.* To derive pointwise and L^∞ estimates, we find it convenient to make the following auxiliary bootstrap assumptions.

Auxiliary bootstrap assumptions for small quantities. We assume that the following inequalities hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$ (recall that $\vec{\Psi}$ and $\vec{\Psi}_{(\text{Partial})}$ are defined in Def. 2.8 and that ${}^{(\mathfrak{n})}\widetilde{L}$ is defined in (6.3)):

$$\left\| \mathcal{R}_{(+)} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\alpha}^{1/2} + \varepsilon^{1/2}, \qquad \textbf{(AUX } \mathcal{R}_{(+)} \textbf{ SMALL)}$$

$$\left\| \vec{\Psi}_{(\text{Partial})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2}, \qquad \textbf{(AUX } \vec{\Psi}_{(\text{Partial})} \textbf{ SMALL)}$$

$$\begin{aligned}
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{Z}^{\leq N_{\text{top}}-11;1} \vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,N_{\text{top}}-11];1} \vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{P}^{\leq 4} \breve{X}\breve{X} \vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,6];2} \vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{P}^{\leq 2} \breve{X}\breve{X}\breve{X} \vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,5];3} \vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \breve{X}\breve{X}\breve{X}\breve{X} \vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2},
\end{aligned} \qquad \textbf{(AUX } \vec{\Psi} \textbf{ SMALL)}$$

$$\begin{aligned}
&\left\| \mathcal{Z}^{\leq N_{\text{top}}-11;1} (\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| \mathcal{Z}^{\leq 6;2} (\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| \mathcal{Z}^{\leq 5;3} (\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| \breve{X}\breve{X}\breve{X}\breve{X} (\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2},
\end{aligned} \qquad \textbf{(AUX } (\Omega, S) \textbf{ SMALL)}$$

$$\begin{aligned}
&\left\| \mathcal{Z}^{\leq N_{\text{top}}-12;1} (\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| \mathcal{Z}^{\leq 6;2} (\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| \mathcal{Z}^{\leq 5;3} (\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| \breve{X}\breve{X}\breve{X}\breve{X} (\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2},
\end{aligned} \qquad \textbf{(AUX } (\mathcal{C}, \mathcal{D}) \textbf{ SMALL)}$$

$$\begin{aligned}
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{P}^{[1,N_{\text{top}}-12]} \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{P}_*^{[1,N_{\text{top}}-12]} \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{Z}_*^{[1,5];1} \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_{**}^{[1,5];1} \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{Z}_*^{[1,4];2} \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_{**}^{[1,4];2} \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2},
\end{aligned} \qquad \textbf{(AUX } \mu \textbf{ SMALL)}$$

$$\left\| L^1_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\alpha}^{1/2}, \qquad \textbf{(AUX } L^1_{(\text{Small})} \textbf{ SMALL)}$$

$$\left\| L^A_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2}, \qquad \textbf{(AUX } L^A_{(\text{Small})} \textbf{ SMALL)}$$

$$\begin{aligned}
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{P}^{\leq N_{\text{top}}-11} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{P}^{[1,N_{\text{top}}-11]} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{Z}^{[1,N_{\text{top}}-12];1} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,N_{\text{top}}-12];1} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{Z}^{[1,5];2} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,5];2} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\
&\left\| {}^{(\mathfrak{n})}\widetilde{L} \mathcal{Z}^{[1,4];3} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,4];3} L^i_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2}.
\end{aligned} \qquad \textbf{(AUX } \mathcal{P}L^i_{(\text{Small})} \textbf{ SMALL)}$$

Auxiliary bootstrap assumptions tied to pure transversal derivatives. We assume that the following inequalities hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:

$$\left\| \breve{X}^M \mathcal{R}_{(+)} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\delta} + \varepsilon^{1/2}, \qquad 1 \leq M \leq 4, \tag{AUX $\breve{X}^M \mathcal{R}_{(+)}$ LARGE}$$

$$\left\| \breve{X}^M \vec{\Psi}_{(\text{Partial})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2}, \qquad 1 \leq M \leq 4, \tag{AUX $\breve{X}^M \vec{\Psi}_{(\text{Partial})}$ SMALL}$$

$$\left\| \breve{X}^M L^1_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\delta} + \varepsilon^{1/2}, \qquad 1 \leq M \leq 3, \tag{AUX $\breve{X}^{[1,3]} L^1_{(\text{Small})}$ LARGE}$$

$$\left\| \breve{X}^M L^A_{(\text{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \varepsilon^{1/2}, \qquad 1 \leq M \leq 3, \tag{AUX $\breve{X}^{[1,3]} L^A_{(\text{Small})}$ SMALL}$$

$$\left\| \breve{X}^M \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \left\| \breve{X}^M \left\{ c^{-1} \right\} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \frac{3}{2\mathring{\delta}_*} \left\| \breve{X}^M \left\{ c^{-1}(c^{-1} c_{;\rho} + 1) \breve{X} \mathcal{R}_{(+)} \right\} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \varepsilon^{1/2}, \quad 0 \leq M \leq 3, \tag{AUX $\breve{X}^{\leq 3} \mu$ LARGE}$$

$$\left\| {}^{(\mathfrak{n})}\widetilde{L} \breve{X}^M \mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \frac{1}{\delta_*} \left\| \breve{X}^M \left\{ c^{-1}(c^{-1} c_{;\rho} + 1) \breve{X} \mathcal{R}_{(+)} \right\} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \varepsilon^{1/2}, \qquad 0 \leq M \leq 3. \tag{AUX ${}^{(\mathfrak{n})}\widetilde{L} \breve{X}^{\leq 3} \mu$ LARGE}$$

12.4. **Key running assumptions.** From now until Sect. 31, we will often silently use the parameter-size and initial data assumptions of Sects. 10.2 and 11.2 and the bootstrap assumptions of Sects. 12.2 and 12.3. In particular, when we state lemmas, propositions, and corollaries, we will not explicitly restate these assumptions.

12.5. **A summary of the forthcoming derivation of improvements of the bootstrap assumptions.** In the subsequent sections, we will derive strict improvements of all the bootstrap assumptions that we made throughout Sect. 12. For the reader's convenience, here we state all the forthcoming results that yield the desired strict improvements. Here we clarify that by "strict improvements," we mean one or more of the following three things:

1. (**Quantitative improvement**) By this, we mean that some quantity Q was assumed to satisfy $A_1 \leq Q \leq A_2$ in the bootstrap assumptions (where A_1, A_2 are real numbers), and we derive the improved bound $B_1 \leq Q \leq B_2$, where $A_1 < B_1 \leq B_2 < A_2$.
2. (**From soft to quantitative**) By this, we mean that in the bootstrap assumptions, we assumed that some function f belongs to some function space and has a finite norm in that space, and our improvement is a quantitative estimate for the norm of f.
3. (**Extension to the closure**) By this, we mean that our bootstrap assumptions involved an assumption on the "open-at-the-top" domain ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, and we derive an improved result showing that the assumption holds on the closed domain ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$.

Here are the precise spots in the article where we derive improvements of the bootstrap assumptions.

- Regarding the bootstrap assumptions of Sect. 12.2.1: we derive improvements of (**BA** $\mu > 0$) in (18.1), of (**BA** $L\mu$ neg) in (18.8a), of (**BA** $\frac{\partial}{\partial t}\mu$ neg) in (18.8b), and of (**BA** μ cnvx) in (18.5).
- We derive improvements of the bootstrap assumptions of Sect. 12.2.2 in (18.9a)–(18.9b).
- We derive improvements of the bootstrap assumptions of Sect. 12.2.3 in Lemmas 15.5 and 15.7 and in Prop. 18.4.
- Regarding the bootstrap assumptions of Sect. 12.2.4: we derive improvements of Items 1 and 4 of in Cor. 15.8, of (**BA** $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **– LOCATION**) in (18.3b), and of (**BA** $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ **– LOCATION**) in (18.3a).
- We derive improvements of the bootstrap assumptions of Sect. 12.2.5 in Lemma 18.5.
- We derive improvements of the bootstrap assumptions of Sect. 12.2.6 in Lemma 15.6.
- We derive improvements of the fundamental quantitative bootstrap assumptions of Sect. 12.3.1 in Prop. 30.1.
- We derive improvements of the auxiliary bootstrap assumptions of Sect. 12.3.2 in Prop. 17.1.
- In Sect. 24.3, we state bootstrap assumptions for the L^2-type energies for the wave-variables $\vec{\Psi}$. We derive improvements of them in Prop. 24.1.

13. Preliminary pointwise, commutator, and differential operator comparison estimates

We operate under the assumptions of Sect. 12.4. In this section, we use the bootstrap assumptions to derive several preliminary estimates, including pointwise estimates, commutator identities and estimates, and differential operator estimates comparing $\not{\mathcal{L}}$ and $\not\nabla$.

13.1. The norm of the $\ell_{t,u}$-tangent commutation vectorfields and simple comparison estimates.

Lemma 13.1 (The norm of the $\ell_{t,u}$-tangent commutation vectorfields and simple comparison estimates)**.** *Assume the parameter- and data-size assumptions of Sects. 10.2 and 11.2, and assume that the bootstrap assumptions of Sects. 12.2 and 12.3 hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. *Then the $\ell_{t,u}$-tangent commutation vectorfields $\mathscr{Y} = \{Y_{(2)}, Y_{(3)}\}$ (see Def. 3.8) satisfy the following pointwise estimates on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:

$$|Y_{(A)}|_{\not g} = 1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2}). \tag{13.1}$$

Moreover, for any type $\binom{m}{n}$ $\ell_{t,u}$-tangent tensorfield $\xi^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}$, the following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, *where the $\xi^{A_1\cdots A_n}_{B_1\cdots B_n}$ denote the components of ξ with respect to the geometric coordinates (x^2,x^3) on $\ell_{t,u}$ (see Notation 3.1):*

$$\begin{aligned}
|\xi|^2_{\not g} &= \left\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})\right\} \sum_{\substack{{}^{(1)}U,\cdots,{}^{(m)}U\in\mathscr{Y}\\ {}^{(1)}V,\cdots,{}^{(n)}V\in\mathscr{Y}}} \left| {}^{(1)}U_{\alpha_1}\cdots{}^{(m)}U_{\alpha_m}\,{}^{(1)}V^{\beta_1}\cdots{}^{(n)}V^{\beta_n}\,\xi^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\right|^2 \\
&= \left\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})\right\} \sum_{\substack{A_1,\cdots,A_m=2,3\\ B_1,\cdots,B_n=2,3}} \left|\xi^{A_1\cdots A_n}_{B_1\cdots B_n}\right|^2.
\end{aligned} \tag{13.2}$$

Proof. We first prove (13.2). We only prove the result for $\ell_{t,u}$-tangent one-forms ξ, because arbitrary type $\binom{m}{n}$ $\ell_{t,u}$-tangent tensorfields can be handled via similar arguments. Let ξ be an $\ell_{t,u}$-tangent one form. Then by (3.31b), we have that $|\xi|^2_{\not g} = c^2 \sum_{A=2,3}(\xi_A)^2 - (X^A\xi_A)^2$, where we recall that $\xi_A \overset{\text{def}}{=} \xi\cdot\frac{\partial}{\partial x^A}$. Next, using the bootstrap assumptions, Prop. 9.1, and (10.9a), we deduce that $c-1, X^2, X^3 = \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})$ and thus $|\xi|^2_{\not g} = \left\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})\right\}\sum_{A=2,3}(\xi_A)^2$. Using in addition (5.8c)–(5.8d) and the fact that $X^1 + 1 = X^1_{(\text{Small})} = \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})$ (see (3.13)), we have that $\xi_A = \xi\cdot Y_{(A)} + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})\sum_{A=2,3}|\xi\cdot Y_{(A)}|$. Combining these results, we conclude (13.2) for $\ell_{t,u}$-tangent one-forms ξ.

To prove (13.1), we first use (13.2) to deduce that $|Y_{(A)}|^2_{\not g} = \left\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})\right\}\sum_{B=2,3}(Y^B_{(A)})^2$. Moreover, the identities (5.7c)–(5.7d) and the estimates noted in the previous paragraph yield $\sum_{B=2,3}(Y^B_{(A)})^2 = 1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})$. Combining these estimates, we conclude (13.1).

□

13.2. Basic facts and assumptions that we use silently throughout the paper.

In the rest of the paper, we silently use the following basic facts and assumptions.

1. All of the estimates we derive hold on the bootstrap region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. Moreover, when deriving estimates, we often silently rely on the parameter- and data-size assumptions of Sects. 10.2 and 11.2 and the bootstrap assumptions of Sects. 12.2 and 12.3. In particular, we often silently use (10.9a).
2. All quantities that we estimate can be controlled in terms of the variables $\underline{\gamma} = \{\vec{\Psi}, \mu - 1, L^1_{(\text{Small})}, L^2_{(\text{Small})}, L^3_{(\text{Small})}\}$, Ω, and S (though we also rely on the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$ to help control Ω and S).
3. We use the Leibniz rule for the operators $\not{\mathcal{L}}_Z$ and $\not\nabla$ when deriving pointwise estimates for the $\not{\mathcal{L}}_Z$- and $\not\nabla$-derivatives of tensor products of the schematic form $\prod_{i=1}^m \xi_{(i)}$, where the $\xi_{(i)}$ are scalar functions or $\ell_{t,u}$-tangent tensors. Thanks to our assumption (10.6) on N_{top}, our derivative counts are such that all the $\xi_{(i)}$ except at most one are uniformly bounded in L^∞ on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. Thus, our pointwise estimates often explicitly feature (on the right-hand sides) only one factor with many derivatives on it, multiplied by a constant that uniformly bounds the other factors. In some estimates, the right-hand sides also gain a smallness factor such as $\varepsilon^{1/2}$, generated by the remaining $\xi'_{(i)}s$.
4. We use the conventions for constants $C, \mathfrak{c}$, and $C_{\blacklozenge}$ stated in Sect. 10.3.
5. We use the conventions for strings of commutation vectorfields stated in Sect. 8.3.
6. We use the schematic identities stated in Prop. 9.1.

7. We use the comparison estimates of Lemma 13.1.

13.3. **Pointwise estimates for Cartesian components of geometric vectorfields.** In this section, we provide simple pointwise estimates for the Cartesian components of the vectorfields $\{L, \breve{X}, Y_{(2)}, Y_{(3)}\}$ and their derivatives.

Lemma 13.2 (Pointwise estimates for x^i and the Cartesian components of the vectorfields $\{L, \breve{X}, Y_{(2)}, Y_{(3)}\}$)**.** *Let $P \in \{L, Y_{(2)}, Y_{(3)}\}$. For $i = 1, 2, 3$, the following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:*

$$|P^i| \lesssim 1 + |\gamma|, \tag{13.3a}$$
$$|\mathcal{P}^{[1,N]} P^i| \lesssim |\mathcal{P}^{[1,N]}\gamma|, \tag{13.3b}$$
$$|\mathcal{Z}_*^{[1,N];1} P^i| \lesssim |\mathcal{Z}_*^{[1,N];1}\gamma|, \tag{13.3c}$$
$$|\mathcal{Z}^{[1,N];1} P^i| \lesssim |\mathcal{Z}^{[1,N];1}\gamma|, \tag{13.3d}$$
$$|\breve{X}^i| \lesssim 1 + |\underline{\gamma}|, \tag{13.3e}$$
$$|\mathcal{P}^{[1,N]} \breve{X}^i| \lesssim |\mathcal{P}^{[1,N]}\underline{\gamma}|, \tag{13.3f}$$
$$|\mathcal{Z}_*^{[1,N];1} \breve{X}^i| \lesssim |\mathcal{Z}_*^{[1,N];1}\underline{\gamma}|, \tag{13.3g}$$
$$|\mathcal{Z}^{[1,N];1} \breve{X}^i| \lesssim |\mathcal{Z}^{[1,N];1}\underline{\gamma}|, \tag{13.3h}$$
$$|{d\!\!\!/} x^i|_{g\!\!\!/} \lesssim 1 + |\gamma|, \tag{13.3i}$$
$$|{d\!\!\!/} \mathcal{P}^{[1,N]} x^i|_{g\!\!\!/} \lesssim |\mathcal{P}^{[1,N]}\gamma|, \tag{13.3j}$$
$$|{d\!\!\!/} \mathcal{Z}^{[1,N];1} x^i|_{g\!\!\!/} \lesssim |\mathcal{Z}_*^{[1,N];1}\gamma| + |\mathcal{P}_*^{[1,N]}\underline{\gamma}|. \tag{13.3k}$$

Proof. The lemma follows from the same arguments given in [50, Lemma 8.4], with minor modifications accounting for the third dimension. ☐

13.4. **Pointwise estimates for various $\ell_{t,u}$-tangent tensorfields.** In this section, we record several pointwise estimates for various $\ell_{t,u}$-tangent tensorfields.

Lemma 13.3 (Crude pointwise estimates for the Lie derivatives of $g\!\!\!/$ and χ)**.** *The following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:*

$$|{\mathcal{L}\!\!\!/}_{\mathcal{P}}^{N+1} g\!\!\!/|_{g\!\!\!/}, |{\mathcal{L}\!\!\!/}_{\mathcal{P}}^{N+1} g\!\!\!/^{-1}|_{g\!\!\!/}, |{\mathcal{L}\!\!\!/}_{\mathcal{P}}^{N} \chi|_{g\!\!\!/}, |\mathcal{P}^N \operatorname{tr}_{g\!\!\!/}\chi| \lesssim |\mathcal{P}^{[1,N+1]}\gamma|, \tag{13.4a}$$
$$|{\mathcal{L}\!\!\!/}_{\mathcal{Z}_*}^{N+1;1} g\!\!\!/|_{g\!\!\!/}, |{\mathcal{L}\!\!\!/}_{\mathcal{Z}_*}^{N+1;1} g\!\!\!/^{-1}|_{g\!\!\!/}, |{\mathcal{L}\!\!\!/}_{\mathcal{Z}}^{N;1} \chi|_{g\!\!\!/}, |\mathcal{Z}^{N;1} \operatorname{tr}_{g\!\!\!/}\chi| \lesssim |\mathcal{Z}_*^{[1,N+1];1}\gamma| + |\mathcal{P}_*^{[1,N+1]}\underline{\gamma}|, \tag{13.4b}$$
$$|{\mathcal{L}\!\!\!/}_{\mathcal{Z}}^{N+1;1} g\!\!\!/|_{g\!\!\!/}, |{\mathcal{L}\!\!\!/}_{\mathcal{Z}}^{N+1;1} g\!\!\!/^{-1}|_{g\!\!\!/} \lesssim |\mathcal{Z}^{[1,N+1];1}\gamma| + |\mathcal{P}_*^{[1,N+1]}\underline{\gamma}|. \tag{13.4c}$$

Proof. The same proof of [50, Lemma 8.5] holds with minor modifications to account for the third spatial dimension. ☐

13.5. **Commutator identities and estimates.**

Lemma 13.4 (Simple commutator identities)**.** *For any $\mathcal{P}_u$-tangent vectorfields $P, P_1, P_2 \in \{L, Y_{(2)}, Y_{(3)}\}$, the commutators $[P_1, P_2]$ and $[\breve{X}, P]$ are $\ell_{t,u}$-tangent. Moreover, there exist smooth functions, all schematically denoted by "f," such that the following identity holds:*

$$[P_1, P_2] = \mathrm{f}(\mathcal{P}^{\leq 1}\gamma)Y_{(2)} + \mathrm{f}(\mathcal{P}^{\leq 1}\gamma)Y_{(3)}. \tag{13.5a}$$

For each $P \in \{L, Y_{(2)}, Y_{(3)}\}$, there exist smooth functions, all schematically denoted by "f," such that the following identity holds:

$$[P, \breve{X}] = \mathrm{f}(\mathcal{P}^{\leq 1}\underline{\gamma}, \breve{X}\vec{\Psi})Y_{(2)} + \mathrm{f}(\mathcal{P}^{\leq 1}\underline{\gamma}, \breve{X}\vec{\Psi})Y_{(3)}. \tag{13.5b}$$

Proof. Let $P \in \{L, Y_{(2)}, Y_{(3)}\}$. From Lemma 3.9 and the fact that the $Y_{(A)}$ are $\ell_{t,u}$-tangent, we find deduce that $[P, \breve{X}]t = [P, \breve{X}]u = 0$, i.e, that $[P, \breve{X}]$ is $\ell_{t,u}$-tangent. Similarly, if $P_1, P_2 \in \{L, Y_{(2)}, Y_{(3)}\}$, then $[P_1, P_2]t = [P_1, P_2]u = 0$, and hence $[P_1, P_2]$ is $\ell_{t,u}$-tangent.

We now prove (13.5b). From the above observations and the fact that $\{\frac{\partial}{\partial x^A}\}_{A=2,3}$ spans the tangent space of $\ell_{t,u}$, we see that $[P, \breve{X}] = [P, \breve{X}]^A \frac{\partial}{\partial x^A}$. From this identity, (5.8c)–(5.8d), and Prop. 9.1, we deduce that $[P, \breve{X}]$ can be written as a linear combination of $Y_{(2)}, Y_{(3)}$ with coefficients that are smooth functions of $\mathcal{P}^{\leq 1}\underline{\gamma}$, of the Cartesian components $\breve{X}L^i$, and of $\breve{X}\vec{\Psi}$. We then use (9.14) to substitute for $\breve{X}L^i$, which in total yields (13.5b).

The identity (13.5a) can be proved through similar but simpler arguments that do not involve factors of μ or $\breve{X}$ differentiations, and we omit the details. □

The following proposition provides various vectorfield commutator estimates that we use in our analysis.

Proposition 13.5 (Pointwise commutator estimates)**.** *Let $1 \leq N \leq N_{\text{top}}$ be an integer, and let φ be a scalar function on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. For any $P \in \mathscr{P} \overset{def}{=} \{L, Y_{(2)}, Y_{(3)}\}$, iterated commutators can be pointwise bounded as follows on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:*

$$\left|[P,\mathcal{P}^N]\varphi\right| \lesssim \varepsilon^{1/2}\left|\mathcal{P}^{[1,N]}\varphi\right| + \underbrace{\sum_{\substack{N_1+N_2\leq N+1\\ N_1,N_2\leq N}}\left|\mathcal{P}^{[2,N_1]}\gamma\right|\left|\mathcal{P}^{[1,N_2]}\varphi\right|}_{\textit{Absent if } N=1}, \tag{13.6a}$$

$$\begin{aligned}\left|[\breve{X},\mathcal{P}^N]\varphi\right|, \left|[P,\mathcal{Z}^{N;1}]\varphi\right| &\lesssim \left|\mathcal{P}^{[1,N]}\varphi\right| + \underbrace{\sum_{\substack{N_1+N_2\leq N+1\\ N_1,N_2\leq N}}\left|\mathcal{P}_*^{[2,N_1]}\underline{\gamma}\right|\left|\mathcal{P}^{[1,N_2]}\varphi\right|}_{\textit{Absent if } N=1}\\ &\quad + \underbrace{\sum_{\substack{N_1+N_2\leq N\\ N_1\leq N-1}}\left|\mathcal{P}^{[1,N_1]}\breve{X}\vec{\Psi}\right|\left|\mathcal{P}^{[1,N_2]}\varphi\right|}_{\textit{Absent if } N=1}.\end{aligned} \tag{13.6b}$$

In particular, for any $P \in \{L, Y_{(2)}, Y_{(3)}\}$, we have the following pointwise estimates on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:

$$\left|[P,\mathcal{P}^N]\varphi\right| \lesssim \varepsilon^{1/2}\left|\mathcal{P}^{[1,N]}\varphi\right|, \qquad \textit{if } 1 \leq N \leq N_{\text{top}} - 11, \tag{13.7a}$$

$$\left|[\breve{X},\mathcal{P}^N]\varphi\right|, \left|[P,\mathcal{Z}^{N;1}]\varphi\right| \lesssim \left|\mathcal{P}^{[1,N]}\varphi\right|, \qquad \textit{if } 1 \leq N \leq N_{\text{top}} - 12. \tag{13.7b}$$

Moreover, the following pointwise estimates hold:

$$\left|[P,\mathcal{Z}^{N;2}]\varphi\right|, \left|[\breve{X},\mathcal{Z}^{N;1}]\varphi\right| \lesssim \left|\mathcal{Z}_*^{[1,N];1}\varphi\right|, \qquad \textit{if } 1 \leq N \leq 5, \tag{13.8a}$$

$$\left|[P,\mathcal{Z}^{N;3}]\varphi\right|, \left|[\breve{X},\mathcal{Z}^{N;2}]\varphi\right| \lesssim \left|\mathcal{Z}_*^{[1,N];2}\varphi\right|, \qquad \textit{if } 1 \leq N \leq 4. \tag{13.8b}$$

Finally, if ξ is an $\ell_{t,u}$-tangent type $\binom{0}{n}$-tensorfield and $P \in \{L, Y_{(2)}, Y_{(3)}\}$, then the following pointwise estimates hold:

$$\left|[\not{\mathcal{L}}_P,\not{\mathcal{L}}_{\mathcal{P}}^N]\xi\right|_{\not{g}} \lesssim \varepsilon^{1/2}\left|\not{\mathcal{L}}_{\mathcal{P}}^{\leq N}\xi\right|_{\not{g}} + \underbrace{\sum_{\substack{N_1+N_2\leq N+1\\ N_2\leq N}}\left|\mathcal{P}_*^{[2,N_1]}\gamma\right|\left|\not{\mathcal{L}}_{\mathcal{P}}^{[1,N_2]}\xi\right|_{\not{g}}}_{\textit{Absent if } N=1}, \tag{13.9a}$$

$$\begin{aligned}\left|[\not{\mathcal{L}}_P,\not{\mathcal{L}}_{\mathcal{Z}}^{N;1}]\xi\right|_{\not{g}} &\lesssim \left|\not{\mathcal{L}}_{\mathcal{P}}^{\leq N}\xi\right|_{\not{g}} + \underbrace{\sum_{\substack{N_1+N_2\leq N+1\\ N_2\leq N}}\left|\mathcal{P}_*^{[2,N_1]}\underline{\gamma}\right|\left|\not{\mathcal{L}}_{\mathcal{P}}^{[1,N_2]}\xi\right|_{\not{g}}}_{\textit{Absent if } N=1}\\ &\quad + \underbrace{\sum_{\substack{N_1+N_2\leq N\\ N_1\leq N-1}}\left|\mathcal{P}^{[1,N_1]}\breve{X}\vec{\Psi}\right|\left|\not{\mathcal{L}}_{\mathcal{P}}^{[1,N_2]}\xi\right|_{\not{g}}}_{\textit{Absent if } N=1}.\end{aligned} \tag{13.9b}$$

Proof. We first prove (13.6a). Iterating (13.5a) and using Prop. 9.1 and the bootstrap assumptions, we find that $\left|[P,\mathcal{P}^N]\varphi\right| \lesssim \sum_{\substack{N_1+N_2\leq N+1\\ N_1,N_2\leq N}} \left|\mathcal{P}^{[1,N_1]}\gamma\right|\left|\mathcal{P}^{[1,N_2]}\varphi\right|$. From this bound and the fact that the bootstrap assumptions imply that $|\mathcal{P}\gamma| \lesssim \varepsilon^{1/2}$, we arrive at (13.6a).

To prove (13.6b) for $\left|[\breve{X},\mathcal{P}^N]\varphi\right|$, we use a similar argument that also relies on (13.5b) to deduce $\left|[\breve{X},\mathcal{P}^N]\varphi\right| \lesssim \sum_{\substack{N_1+N_2\leq N+1\\ N_1,N_2\leq N}} \left|\mathcal{P}^{[1,N_1]}\underline{\gamma}\right|\left|\mathcal{P}^{[1,N_2]}\varphi\right| + \sum_{\substack{N_1+N_2\leq N\\ N_1\leq N-1}} \left|\mathcal{P}^{\leq N_1}\breve{X}\vec{\Psi}\right|\left|\mathcal{P}^{[1,N_2]}\varphi\right|$. From this bound and the fact that the bootstrap assumptions imply that $|\mathcal{P}\underline{\gamma}| \lesssim 1$ and $\left|\breve{X}\vec{\Psi}\right| \lesssim 1$, we conclude the desired bounds for $\left|[\breve{X},\mathcal{P}^N]\varphi\right|$. The estimate (13.6b) for $\left|[P,\mathcal{Z}^{N;1}]\varphi\right|$ can be proved through a similar argument, and we omit the details.

Except for (13.9a)–(13.9b), the remaining estimates in the proposition follow from similar arguments that take into account the details of the auxiliary bootstrap assumptions of Sect. 12.3.2, in particular the L^∞ regularity of the solution variables with respect to $\breve{X}$ and $\mathcal{P}_u$-tangential differentiations.

We now prove (13.9a). We first consider the case in which $\xi = \xi_A \not{d} x^A$ is an $\ell_{t,u}$-tangent one-form, where by our usual conventions, $\xi_A \overset{\text{def}}{=} \xi \cdot \frac{\partial}{\partial x^A}$. By Lemma 3.13 and the Leibniz rule, for any $P \in \mathscr{P}$, we have:

$$\not{\mathcal{L}}_P \xi = (P\xi_A)\not{d} x^A + \xi_A \not{d}(P x^A). \tag{13.10}$$

Differentiating (13.10) with $\ell_{t,u}$-projected Lie derivatives, using the Leibniz rule, using Lemma 3.13, using the bootstrap assumptions, using (13.6a) with the scalar functions ξ_A in the role of φ, and using (13.3i)–(13.3j), we see that $\left|[\not{\mathcal{L}}_P,\not{\mathcal{L}}_{\mathcal{P}}^N]\xi\right|_{\not{g}} \lesssim \sum_{A=2,3}\varepsilon^{1/2}\left|\mathcal{P}^{[1,N]}\xi_A\right| + \sum_{A=2,3}\sum_{\substack{N_1+N_2\leq N+1\\ N_2\leq N}} \left|\mathcal{P}_*^{[2,N_1]}\gamma\right|\left|\mathcal{P}^{\leq N_2}\xi_A\right|$. We clarify that we have used the fact that the set of products arising in the Leibniz expansions of $\not{\mathcal{L}}_P \not{\mathcal{L}}_{\mathcal{P}}^N(\xi_A \not{d} x^A)$ and $\not{\mathcal{L}}_{\mathcal{P}}^N \not{\mathcal{L}}_P(\xi_A \not{d} x^A)$ is the same, except that the vectorfield differential operators can appear in a different order in the two sets of products. Next, by considering the coordinate components of the one-forms on each side of (13.10), using the bootstrap assumptions, and using (13.2) and (13.3j), we find that $\sum_{A=2,3}|P\xi_A| \lesssim |(P\xi_A)\not{d} x^A|_{\not{g}} \lesssim |\not{\mathcal{L}}_P\xi|_{\not{g}} + |\xi|_{\not{g}}$. Differentiating (13.10) and using similar arguments, we use induction in M to deduce that for $1 \leq M \leq N_{\text{top}}$, we have $\sum_{A=2,3}|\mathcal{P}^M\xi_A| \lesssim \left|\not{\mathcal{L}}_{\mathcal{P}}^{\leq M}\xi\right|_{\not{g}} + \sum_{\substack{M_1+M_2\leq M\\ M_2\leq M-1}} \left|\mathcal{P}^{[1,M_1]}\gamma\right|\left|\not{\mathcal{L}}_{\mathcal{P}}^{\leq M_2}\xi\right|_{\not{g}}$. Combining these estimates and using the bootstrap assumptions, we conclude (13.9a) for $\ell_{t,u}$-tangent one-forms ξ. For any $n \geq 2$, the estimate (13.9a) for type $\binom{0}{n}$ $\ell_{t,u}$-tangent tensorfields ξ can be proved using similar arguments. We have therefore proved (13.9a). The commutator estimate (13.9b) follows from similar arguments that rely on (13.3k), (13.4c), (13.6b), and the identity (9.14) (we use the latter to eliminate the explicit presence of $\breve{X}L^i$-dependent terms on RHS (13.9b), which otherwise would have arisen from the first terms on RHSs (13.3k) and (13.4c)); we omit the details. □

13.6. **Differential operator estimates comparing $\not{\mathcal{L}}$ and $\not{\nabla}$.** In this section, we provide some pointwise estimates that compare various differential operators.

Lemma 13.6 (Differential operator pointwise comparison estimates)**.** *Let f be a scalar function on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. Then the following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:*

$$|\not{d} f|_{\not{g}}^2 = \left\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})\right\}\sum_{A=2}^{3}|Y_{(A)}f|^2, \tag{13.11a}$$

$$|\not{\Delta} f|^2 \leq 2(1 + C_{\blacklozenge}\mathring{\alpha}^{1/2})\sum_{A=2}^{3}|\not{d} Y_{(A)}f|_{\not{g}}^2 + C\varepsilon^{1/2}|\not{d} f|_{\not{g}}^2. \tag{13.11b}$$

Proof. (13.11a) follows from (13.2) with $\xi \overset{\text{def}}{=} \not{d} f$.

We now prove (13.11b). We first note that (13.1)–(13.2), the Leibniz rule, and the fact that $\not{g}(\not{\nabla}_{Y_{(A)}}Y_{(B)},Y_{(C)}) = \mathbf{g}(\mathbf{D}_{Y_{(A)}}Y_{(B)},Y_{(C)})$ yield the following estimate for any scalar function f:

$$|\not{\nabla}^2 f|_{\not{g}}^2 \leq \left\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}^{1/2})\right\}\sum_{A=2}^{3}|\not{d} Y_{(A)}f|_{\not{g}}^2 + C\sum_{A,B,C=2}^{3}|\mathbf{g}(\mathbf{D}_{Y_{(A)}}Y_{(B)},Y_{(C)})|^2|\not{d} f|_{\not{g}}^2. \tag{13.12}$$

Next, we use (9.13) and the bootstrap assumptions to deduce that $|\mathbf{g}(\mathbf{D}_{Y_{(A)}} Y_{(B)}, Y_{(C)})| \lesssim \varepsilon^{1/2}$. Inserting this bound into RHS (13.12) and using the inequality $|\Delta\!\!\!/ f|^2 \leq 2|\nabla\!\!\!\!/^2 f|^2_{g\!\!\!/}$, we arrive at the desired estimate (13.11b).

□

13.7. **Transport inequalities for the eikonal function quantities.** In this section, we provide transport inequalities satisfied by the eikonal function quantities $\mu, L^i_{(\text{Small})}, \chi$, and $\text{tr}_{g\!\!\!/}\chi$. We also provide pointwise estimates for the differentiated quantity $\mathcal{L}\!\!\!/_L \mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi$. These estimates *involve a loss of one order of differentiability* relative to $\vec{\Psi}$ in the sense that the right-hand sides of the transport equations that we use to derive the inequalities depend on the first-order derivatives of $\vec{\Psi}$. In Sect. 25, we use the transport inequalities to derive below-top-order energy estimates for the eikonal function quantities.

Proposition 13.7 (Transport inequalities for the eikonal function quantities)**.** *The following pointwise estimates hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$*:*

$$|L\mu| \lesssim |\mathcal{Z}\vec{\Psi}|, \tag{13.13a}$$

$$|L\mathcal{P}^N_*\mu|, |\mathcal{P}^N_* L\mu| \lesssim |\mathcal{Z}^{[1,N+1];1}_*\vec{\Psi}| + |\mathcal{P}^{[1,N]}\gamma| + \varepsilon^{1/2}|\mathcal{P}^{[1,N]}_*\underline{\gamma}|, \qquad \text{if } 1 \leq N \leq N_{\text{top}}, \tag{13.13b}$$

$$|L\mathcal{P}^N L^i_{(\text{Small})}|, |\mathcal{P}^N L L^i_{(\text{Small})}| \lesssim |\mathcal{P}^{[1,N+1]}\vec{\Psi}| + \varepsilon^{1/2}|\mathcal{P}^{[1,N]}\gamma|, \qquad \text{if } 0 \leq N \leq N_{\text{top}}, \tag{13.13c}$$

$$|L\mathcal{P}^{N-1}\text{tr}_{g\!\!\!/}\chi|, |\mathcal{P}^{N-1}L\text{tr}_{g\!\!\!/}\chi| \lesssim |\mathcal{P}^{[1,N+1]}\vec{\Psi}| + \varepsilon^{1/2}|\mathcal{P}^{[1,N]}\gamma|, \qquad \text{if } 1 \leq N \leq N_{\text{top}}, \tag{13.13d}$$

$$|\mathcal{L}\!\!\!/_L\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi|_{g\!\!\!/}, |\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathcal{L}\!\!\!/_L\chi|_{g\!\!\!/} \lesssim |\mathcal{P}^{[1,N+1]}\vec{\Psi}| + \varepsilon^{1/2}|\mathcal{P}^{[1,N]}\gamma|, \qquad \text{if } 1 \leq N \leq N_{\text{top}}, \tag{13.13e}$$

$$|L\mathcal{Z}^{N;1}L^i_{(\text{Small})}|, |\mathcal{Z}^{N;1}LL^i_{(\text{Small})}| \lesssim |\mathcal{Z}^{[1,N+1];1}_*\vec{\Psi}| + |\mathcal{Z}^{[1,N];1}_*\gamma| + \varepsilon^{1/2}|\mathcal{P}^{[1,N]}_*\underline{\gamma}|, \qquad \text{if } 1 \leq N \leq N_{\text{top}}, \tag{13.13f}$$

$$|L\mathcal{Z}^{N-1;1}\text{tr}_{g\!\!\!/}\chi|, |\mathcal{Z}^{N-1;1}L\text{tr}_{g\!\!\!/}\chi| \lesssim |\mathcal{Z}^{[1,N+1];1}_*\vec{\Psi}| + |\mathcal{Z}^{[1,N];1}_*\gamma| + \varepsilon^{1/2}|\mathcal{P}^{[1,N]}_*\underline{\gamma}|, \qquad \text{if } 2 \leq N \leq N_{\text{top}}, \tag{13.13g}$$

$$|\mathcal{L}\!\!\!/_L\mathcal{L}\!\!\!/_{\mathcal{Z}}^{N-1;1}\chi|_{g\!\!\!/}, |\mathcal{L}\!\!\!/_{\mathcal{Z}}^{N-1;1}\mathcal{L}\!\!\!/_L\chi|_{g\!\!\!/} \lesssim |\mathcal{Z}^{[1,N+1];1}_*\vec{\Psi}| + |\mathcal{Z}^{[1,N];1}_*\gamma| + \varepsilon^{1/2}|\mathcal{P}^{[1,N]}_*\underline{\gamma}|, \qquad \text{if } 2 \leq N \leq N_{\text{top}}. \tag{13.13h}$$

Proof. Thanks to the transport equations of Lemma 3.21, the identity (3.49b), and the commutator estimates of Prop. 13.5, the estimates (13.13a)–(13.13d), (13.13f), and (13.13g) follow from the same arguments given in [50, Proposition 8.13], and we omit the details. In the rest of the proof, we will silently use the comparison estimates stated in (13.2).

We now prove (13.13e) for $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathcal{L}\!\!\!/_L\chi$. Since $\ell_{t,u}$-projected Lie differentiation with respect to the elements of $\mathscr{Z}$ commutes with $d\!\!\!/$ (see (3.29)), equation (3.49a), the chain rule, (2.18a), and Prop. 9.1 imply the following relation, where the last three products on RHS (13.14) are written schematically:

$$\begin{aligned} \mathcal{L}\!\!\!/_L\chi = (\vec{G}_{ab} \diamond L\vec{\Psi})d\!\!\!/ L^a \otimes d\!\!\!/ x^b + \mathbf{g}_{ab}d\!\!\!/ L^a \otimes d\!\!\!/ L^b + \mathbf{g}_{ab}d\!\!\!/ LL^a \otimes d\!\!\!/ x^b \\ + \mathrm{f}(\gamma, d\!\!\!/\vec{x}) \cdot PP\vec{\Psi} + \mathrm{f}(\gamma, d\!\!\!/\vec{x}) \cdot P\vec{\Psi} \cdot P\gamma + \mathrm{f}(\gamma, d\!\!\!/\vec{x}) \cdot P\vec{\Psi} \cdot d\!\!\!/ P\vec{x}. \end{aligned} \tag{13.14}$$

Substituting RHS (3.45) for LL^a on RHS (13.14), taking $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}$ derivatives of the resulting expression, and using (3.29), Prop. 9.1, (13.3i)–(13.3j), and the bootstrap assumptions, we conclude (13.13e) for $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathcal{L}\!\!\!/_L\chi$. The estimate (13.13e) for $\mathcal{L}\!\!\!/_L\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi$ then follows from the commutator estimate (13.9a), the crude estimate (13.4a) for the $\ell_{t,u}$-projected Lie derivatives of χ, and the bootstrap assumptions. With the help of the estimates (13.3k) and (13.4b) and the commutator estimate (13.9b), the estimates stated in (13.13h) follow from similar arguments, and we omit the details. □

13.8. **Pointwise commutator estimates for χ tied to a Codazzi-type identity.** We will use the results of this section in Sect. 29.3, when we derive top-order L^2 estimates for χ with the help of elliptic estimates. A key point of Lemmas 13.8–13.9 is that RHSs (13.15)–(13.16) depend on *one fewer derivative* of γ compared to the crude results of Lemma 13.3.

Lemma 13.8 (Codazzi-type identity for χ)**.** *There exist smooth functions, all schematically denoted by "*f*," such that the following identity holds:*

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\chi - d\!\!\!/\,\text{tr}_{g\!\!\!/}\chi = \mathrm{f}(\mathcal{P}^{\leq 1}\gamma, d\!\!\!/\vec{x})\mathcal{P}\gamma + \mathrm{f}(\gamma, d\!\!\!/\vec{x})\mathcal{P}^2\vec{\Psi}. \tag{13.15}$$

Proof. We view both sides of (3.49a) as a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield such that the components of the identity with respect to the geometric coordinates (x^2, x^3) take the following form $\chi_{AB} = \mathbf{g}_{ab}\frac{\partial}{\partial x^A}L^a \otimes \frac{\partial}{\partial x^B}x^b + \cdots$, where $\cdots$ denotes terms that do not depend on the derivatives of L. For clarity, we note that the first, third, and fourth product on RHS (3.49a) are not symmetric, even though the sum of all terms on RHS (3.49a) must be symmetric. We apply $\nabla\!\!\!\!/^B$

(which, in view of Def. 3.11 and the symmetry of χ_{AB}, is the same as the operator $\mathrm{d}\!\!/\mathrm{iv}$) to the identity, and we apply $\nabla\!\!\!/_A$ to (3.49b) and note that the RHSs of the resulting equations are $\ell_{t,u}$-tangent one-forms such that the terms involving the second-order derivatives of L have the A-components $\mathbf{g}_{ab}(g\!\!\!/^{-1})^{BC}[\nabla\!\!\!/^2_{BA}L^a]\mathrm{d}\!\!\!/_C x^b$, i.e., the second-order-in-L terms agree (and therefore cancel from LHS (13.15)). Also using Lemmas 9.1 and 9.7, we conclude (13.15). □

Lemma 13.9 (Pointwise commutator estimates for χ tied to a Codazzi-type identity)**.** *Let* $1 \leq N \leq N_{\mathrm{top}}$. *Then the following pointwise estimates hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$, *where on LHS* (13.16), $\mathcal{P}^{N-1}$ *denotes the same order* $N-1$ *string of commutation vectorfields in each of the two terms:*

$$\left|\mathrm{d}\!\!/\mathrm{iv}\,\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi - \mathrm{d}\!\!\!/\mathcal{P}^{N-1}\mathrm{tr}_{g\!\!\!/}\chi\right|_{g\!\!\!/} \lesssim \left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \left|\mathcal{P}^{[1,N]}\gamma\right|. \tag{13.16}$$

Proof. Throughout the proof, we silently use the comparison estimates in (13.2). We apply $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}$ to (13.15). By Lemma 3.13, the operator $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}$ commutes under the operators $\mathrm{d}\!\!\!/$ on each side of (13.15). This yields the main term $\mathrm{d}\!\!\!/\mathcal{P}^{N-1}\mathrm{tr}_{g\!\!\!/}\chi$ on LHS (13.16), while the bootstrap assumptions and Lemma 9.7 yield that $\left|\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathrm{RHS}\ (13.15)\right|_{g\!\!\!/} \lesssim \mathrm{RHS}$ (13.16) as desired. Hence, to complete the proof, we must show that $\left|\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathrm{d}\!\!/\mathrm{iv}\,\chi - \mathrm{d}\!\!/\mathrm{iv}\,\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi\right|_{g\!\!\!/} \lesssim \mathrm{RHS}$ (13.16). To this end, we apply $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathrm{d}\!\!/\mathrm{iv}$ and $\mathrm{d}\!\!/\mathrm{iv}\,\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}$ to (3.49a), thereby obtaining expressions for $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathrm{d}\!\!/\mathrm{iv}\,\chi$ and $\mathrm{d}\!\!/\mathrm{iv}\,\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi$ respectively. Unlike in the proof of (13.15), here we are viewing $\mathrm{d}\!\!/\mathrm{iv}\,\chi$ to be $\nabla\!\!\!/^A\chi_{AB}$, i.e., the covariant derivative acts on the first slot of χ. Similarly, in the expression $\mathrm{d}\!\!/\mathrm{iv}\,\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi$, it is understood that the covariant derivative acts on the first slot of $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi$. As above, Lemmas 3.13, 5.5 and 9.7 and the bootstrap assumptions imply that all terms except the principal ones, i.e., the ones involving the order $N+1$ derivatives of L^i, are bounded in the norm $|\cdot|_{g\!\!\!/}$ by $\lesssim$ RHS (13.16). The principal terms in $\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathrm{d}\!\!/\mathrm{iv}\,\chi$ and $\mathrm{d}\!\!/\mathrm{iv}\,\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi$ are respectively $\mathbf{g}_{ab}(\mathcal{P}^{N-1}\Delta\!\!\!\!/\,L^a)\mathrm{d}\!\!\!/x^b$ and $\mathbf{g}_{ab}(\Delta\!\!\!\!/\,\mathcal{P}^{N-1}L^a)\mathrm{d}\!\!\!/x^b$. Hence, using Lemma 9.7, the commutator estimate (13.6a), and the bootstrap assumptions, we see that the principal terms in the difference $\mathbf{g}_{ab}(\mathcal{P}^{N-1}\Delta\!\!\!\!/\,L^a)\mathrm{d}\!\!\!/x^b - \mathbf{g}_{ab}(\Delta\!\!\!\!/\,\mathcal{P}^{N-1}L^a)\mathrm{d}\!\!\!/x^b$ cancel and that $\left|\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\mathrm{d}\!\!/\mathrm{iv}\,\chi - \mathrm{d}\!\!/\mathrm{iv}\,\mathcal{L}\!\!\!/_{\mathcal{P}}^{N-1}\chi\right|_{g\!\!\!/} \lesssim \mathrm{RHS}$ (13.16) as desired. □

13.9. **Pointwise estimates for the inhomogeneous terms in the commuted equations.** In this section, we derive pointwise estimates for the derivatives of the inhomogeneous terms in the equations of Theorem 2.15. These pointwise estimates are straightforward to derive and serve as preliminary ingredients in our derivation of energy estimates. We remark that we will derive pointwise estimates for the commutator error terms later in the article. The commutator estimates for the wave equations are particularly difficult, and we derive them in Sect. 22, after we derive sharp estimates for μ (see Sect. 18.1).

13.9.1. *Pointwise estimates for the derivatives of the null forms.*

Lemma 13.10 (Pointwise estimates for the derivatives of the null forms)**.** *Let* $N \leq N_{\mathrm{top}}$. *The* $\mathcal{P}^N$*-derivatives of the product of* μ *and the terms defined in* (2.26a)–(2.27e) *satisfy the following pointwise estimates on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$*:*

$$\begin{aligned}\left|\mathcal{P}^N(\mu\mathfrak{M}^i_{(\mathcal{C})})\right|, \left|\mathcal{P}^N(\mu\mathfrak{M}_{(\mathcal{D})})\right| &\lesssim \left|\mathcal{P}^{\leq N+1}(\Omega,S)\right| \\ &\quad + \varepsilon\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|,\end{aligned} \tag{13.17a}$$

$$\left|\mathcal{P}^N(\mu\mathfrak{Q}^i_{(v)})\right|, \left|\mathcal{P}^N(\mu\mathfrak{Q}_{(\pm)})\right|, \left|\mathcal{P}^N(\mu\mathfrak{Q}_{(\rho)})\right| \lesssim \varepsilon\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|, \tag{13.17b}$$

$$\begin{aligned}\left|\mathcal{P}^N(\mu\mathfrak{Q}^i_{(\mathcal{C})})\right|, \left|\mathcal{P}^N(\mu\mathfrak{Q}_{(\mathcal{D})})\right| &\lesssim \left|\mathcal{P}^{\leq N}S\right| \\ &\quad + \varepsilon\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|.\end{aligned} \tag{13.17c}$$

Proof. We differentiate the identities of Lemma 9.3 with $\mathcal{P}^N$, use the bootstrap assumptions, and use the commutator estimate (13.6b) so that on the RHSs of the estimates, all terms featuring any $\breve{X}$-differentiation of $\vec{\Psi}$ are such that the $\breve{X}$ operator acts last. □

13.9.2. *Pointwise estimates for the derivatives of the linear inhomogeneous terms.*

Lemma 13.11 (Pointwise estimates for the derivatives of the linear inhomogeneous terms)**.** *Let* $N \leq N_{\mathrm{top}}$. *Consider the product of* μ *and the terms* $\mathcal{C},\mathcal{D}$*-involving terms on RHSs* (2.22a)–(2.22d), *as well as the product of* μ *and the terms defined in*

(2.28a)–(2.28h). *Then the $\mathcal{P}^N$-derivatives of these terms satisfy the following pointwise estimates on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:

$$\begin{aligned}
&\left|\mathcal{P}^N\left(\mu c^2\exp(2\rho)\mathcal{C}^i,\, \mu\left\{F_{;s}c^2\exp(2\rho) - c\exp(\rho)\frac{p_{;s}}{\overline{\varrho}}\right\}\mathcal{D}\right)\right|,\\
&\left|\mathcal{P}^N\left(\mu\exp(\rho)\frac{p_{;s}}{\overline{\varrho}}\mathcal{D},\, \mu c^2\exp(2\rho)\mathcal{D}\right)\right|\\
&\lesssim \mu\left|\mathcal{P}^N(\mathcal{C},\mathcal{D})\right| + \left|\mathcal{P}^{\leq N-1}(\mathcal{C},\mathcal{D})\right| + \varepsilon\left|\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|,
\end{aligned} \tag{13.18a}$$

$$\begin{aligned}
&\left|\mathcal{P}^N\left(\mu\mathfrak{L}_{(v)}^i,\, \mu\mathfrak{L}_{(\pm)},\, \mu\mathfrak{L}_{(\rho)},\, \mu\mathfrak{L}_{(s)},\, \mu\mathfrak{L}_{(\Omega)}^i,\, \mu\mathfrak{L}_{(S)}^i,\, \mu\mathfrak{L}_{(\operatorname{div}\Omega)},\, \mu\mathfrak{L}_{(\mathcal{C})}^i\right)\right|\\
&\lesssim \left|\mathcal{P}^{\leq N}(\Omega,S)\right| + \varepsilon\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|.
\end{aligned} \tag{13.18b}$$

Proof. We apply the same reasoning used in the proof of Lemma 13.10 to the identities provided by Lemma 9.4. □

13.9.3. *Pointwise estimates for the derivatives of the inhomogeneous terms in the commuted wave equations.*

Corollary 13.12 (Pointwise estimates for the derivatives of the inhomogeneous terms in the commuted wave equations)**.** *Let* $\vec{\Psi} \overset{def}{=} (\Psi_0,\Psi_1,\Psi_2,\Psi_3,\Psi_4) \overset{def}{=} (\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s)$ *be the solutions to the covariant wave equations* (2.22). *We denote the product of* μ *and the RHS of the covariant wave equation satisfied by* Ψ_ι *by* $\mathfrak{G}_\iota$, *i.e.,* $\mu\square_{\mathbf{g}}\Psi_\iota = \mathfrak{G}_\iota$. *Let* $1 \leq N \leq N_{\text{top}}$. *Then the following pointwise estimates hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:

$$\left|\mathcal{P}^N\mathfrak{G}_\iota\right| \lesssim \mu\left|\mathcal{P}^N(\mathcal{C},\mathcal{D})\right| + \left|\mathcal{P}^{\leq N-1}(\mathcal{C},\mathcal{D})\right| + \left|\mathcal{P}^{\leq N}(\Omega,S)\right| + \varepsilon\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|. \tag{13.19}$$

Proof. The corollary is a direct consequence of the estimates (13.17b) and (13.18a)–(13.18b).

□

14. Embeddings of $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ and the flow map of ${}^{(\mathfrak{n})}\breve{W}$

We continue to work under the assumptions of Sect. 13.2. In this section, we derive quantitative control over how the level-sets $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ (see definition (4.7b)) are embedded in the spacetime region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. We also derive quantitative control of the flow map of ${}^{(\mathfrak{n})}\breve{W}$. We will use these results in Sect. 15 to demonstrate the viability of the "transversal initial value" problem (4.4a)–(4.4b) that we used to construct the rough time function ${}^{(\mathfrak{n})}\tau$; see Remark 15.2 concerning some subtleties tied to the regularity theory in the construction.

14.1. **Embedded sub-manifolds and quantitative control over the embeddings.**

Lemma 14.1 (Embedded sub-manifolds and quantitative control over the embeddings)**.** *Let* $\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)$ *and* $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)$ *be the functions on* $\mathbb{T}^2$ *from Sect. 12.2.4. Then for* $\mathfrak{m} \in (\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]$, *we have:*

$$\|\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}\|_{W^{2,\infty}(\mathbb{T}^2)},\, \|\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}\|_{W^{2,\infty}(\mathbb{T}^2)} \leq C, \tag{14.1a}$$

$$\left\|\left(\frac{\partial}{\partial x^2}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}},\, \frac{\partial}{\partial x^3}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}\right)\right\|_{W^{1,\infty}(\mathbb{T}^2)},\, \left\|\left(\frac{\partial}{\partial x^2}\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}},\, \frac{\partial}{\partial x^3}\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}\right)\right\|_{W^{1,\infty}(\mathbb{T}^2)} \leq C\varepsilon^{1/2}. \tag{14.1b}$$

Moreover,

$$\frac{\partial}{\partial\mathfrak{m}}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}} = \frac{1}{\frac{\partial}{\partial t}\mu - \frac{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu}{\frac{\partial}{\partial u}\breve{X}\mu}}, \tag{14.2}$$

and there is a $C > 1$ *such that following estimate holds:*

$$-C < \inf_{(\mathfrak{m},x^2,x^3)\in(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2} \frac{\partial}{\partial\mathfrak{m}}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3) \leq \sup_{(\mathfrak{m},x^2,x^3)\in(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2} \frac{\partial}{\partial\mathfrak{m}}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3) < -\frac{1}{C}. \tag{14.3}$$

$\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ ***is a graph****. Let*

$${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]} \overset{def}{=} \left\{(t,x^2,x^3) \in \mathbb{R}\times\mathbb{T}^2 \mid \mathfrak{T}_{\mathfrak{m}_0,-\mathfrak{n}}(x^2,x^3) \leq t < \mathfrak{T}_{\mathfrak{m}_{\text{Boot}},-\mathfrak{n}}(x^2,x^3)\right\}. \tag{14.4}$$

Then ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ *is precompact, and there exists an embedding* ${}^{(\mathfrak{n})}H : {}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]} \to {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-\frac{3}{4}U_{\text{☢}},\frac{3}{4}U_{\text{☢}}]}$ *of the form* ${}^{(\mathfrak{n})}H(t,x^2,x^3) = \left(t, {}^{(\mathfrak{n})}h(t,x^2,x^3), x^2, x^3\right)$ *such that* ${}^{(\mathfrak{n})}H \in W^{2,\infty}(\text{int}\left({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right))$ *and such that* ${}^{(\mathfrak{n})}H$ *is a diffeomorphism from* ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ *onto* $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$*, where:*

$$\text{int}\left({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right) \overset{def}{=} \left\{(t,x^2,x^3) \in \mathbb{R}\times\mathbb{T}^2 \mid \mathfrak{T}_{\mathfrak{m}_0,-\mathfrak{n}}(x^2,x^3) < t < \mathfrak{T}_{\mathfrak{m}_{\text{Boot}},-\mathfrak{n}}(x^2,x^3)\right\} \tag{14.5}$$

is the interior of ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$*. In particular, relative to the geometric coordinates* (t,u,x^2,x^3)*, we have:*

$$\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})} = \left\{\left(t, {}^{(\mathfrak{n})}h(t,x^2,x^3), x^2, x^3\right) \in {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]} \mid (t,x^2,x^3) \subset {}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right\}. \tag{14.6}$$

Moreover, ${}^{(\mathfrak{n})}H$ *is* $C^{1,1}$ *on every compact subset of* ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$*, and the following estimates hold:*

$$\left\|{}^{(\mathfrak{n})}H\right\|_{W^{2,\infty}\left(\text{int}\left({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right)\right)} \leq C, \tag{14.7a}$$

$$\left\|\left(\frac{\partial}{\partial x^2}{}^{(\mathfrak{n})}h, \frac{\partial}{\partial x^3}{}^{(\mathfrak{n})}h\right)\right\|_{W^{1,\infty}\left(\text{int}\left({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right)\right)} \leq C\varepsilon^{1/2}. \tag{14.7b}$$

Finally, on ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$*, the following estimate holds:*

$$\frac{\partial}{\partial t}{}^{(\mathfrak{n})}h = -\frac{L\breve{X}\mu \circ {}^{(\mathfrak{n})}H}{\breve{X}\breve{X}\mu \circ {}^{(\mathfrak{n})}H} + \mathcal{O}(\varepsilon^{1/2}). \tag{14.8}$$

Proof. Throughout this proof, we silently use the soft bootstrap assumptions of Sect. 12.2.6, which guarantee our needed qualitative regularity. When proving quantitative estimates, we will use the bootstrap assumptions of Sects. 12.3.1 and 12.3.2.

To derive the existence of the embedding of the form ${}^{(\mathfrak{n})}H(t,x^2,x^3) = \left(t, {}^{(\mathfrak{n})}h(t,x^2,x^3), x^2, x^3\right)$, we first note that by (12.5), (12.6), and the inverse function theorem, the map $(\mathfrak{m},x^2,x^3) \to \left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right)$ on $(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2$ is a diffeomorphism onto ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ whose inverse is of the form $(t,x^2,x^3) \to \left({}^{(\mathfrak{n})}I(t,x^2,x^3),x^2,x^3\right)$, where for any $\mathfrak{m}' \in (\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]$, the map $(t,x^2,x^3) \to {}^{(\mathfrak{n})}I(t,x^2,x^3)$ is $C^{1,1}$ on $\left\{(t,x^2,x^3) \in \mathbb{R}\times\mathbb{T}^2 \mid \mathfrak{T}_{\mathfrak{m}_0,-\mathfrak{n}}(x^2,x^3) \leq t \leq \mathfrak{T}_{\mathfrak{m}',-\mathfrak{n}}(x^2,x^3)\right\}$. Thus, the desired embedding ${}^{(\mathfrak{n})}H$ (into ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-\frac{3}{4}U_{\text{☢}},\frac{3}{4}U_{\text{☢}}]}$ by (**BA** $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ **– LOCATION**)) is the composition of the diffeomorphism $(t,x^2,x^3) \to \left({}^{(\mathfrak{n})}I(t,x^2,x^3),x^2,x^3\right)$ with the embedding ${}^{(\mathfrak{n})}E$ defined in (12.4).

We now prove (14.7a)–(14.7b) and (14.8). The estimate $\|{}^{(\mathfrak{n})}h\|_{L^\infty\left(\text{int}\left({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right)\right)} \leq C$ follows trivially from the fact that ${}^{(\mathfrak{n})}h\left({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right) \subset [-\frac{3}{4}U_{\text{☢}},\frac{3}{4}U_{\text{☢}}]$ (since ${}^{(\mathfrak{n})}H\left({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right) \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-\frac{3}{4}U_{\text{☢}},\frac{3}{4}U_{\text{☢}}]}$). To control the derivatives of ${}^{(\mathfrak{n})}h$, we differentiate the equation $[\breve{X}\mu]\left(t, {}^{(\mathfrak{n})}h(t,x^2,x^3),x^2,x^3\right) = -\mathfrak{n}$ (which holds since $\breve{X}\mu|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}} = -\mathfrak{n}$) with $\frac{\partial}{\partial t}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$, use the chain rule to algebraically solve for the derivatives of ${}^{(\mathfrak{n})}h$ relevant for the estimates (14.7a)–(14.7b) and (14.8), and then use the bootstrap assumptions. More precisely, we use the following consequences of Lemmas 9.1 and 5.5 and the bootstrap assumptions: $L = \frac{\partial}{\partial t} + \mathcal{O}(\varepsilon^{1/2})\frac{\partial}{\partial x^2} + \mathcal{O}(\varepsilon^{1/2})\frac{\partial}{\partial x^3}$, $\breve{X} = \frac{\partial}{\partial u} + \mathcal{O}(\varepsilon^{1/2})\frac{\partial}{\partial x^2} + \mathcal{O}(\varepsilon^{1/2})\frac{\partial}{\partial x^3}$, $\|\breve{X}\mu\|_{W^{2,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \leq C$, $\left\|\left(\frac{\partial}{\partial x^2}\breve{X}\mu, \frac{\partial}{\partial x^3}\breve{X}\mu\right)\right\|_{W^{1,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \leq C\varepsilon^{1/2}$, $\frac{\partial}{\partial t}\breve{X}\mu = L\breve{X}\mu + \mathcal{O}(\varepsilon^{1/2})$, $\frac{\partial}{\partial u}\breve{X}\mu = \breve{X}\breve{X}\mu + \mathcal{O}(\varepsilon^{1/2})$, and $\frac{\partial}{\partial u}\breve{X}\mu \approx 1$ along $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ (see (**BA** $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ **– LOCATION**) and (**BA** μ cnvx)). We have therefore proved (14.7a)–(14.7b) and (14.8).

We now prove (14.1a)–(14.1b). The estimate $\|\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}\|_{L^\infty(\mathbb{T}^2)} \leq C$ follows trivially from (**BA** t **– SIZE**), as does the precompactness of ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$. The estimate $\|\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}\|_{L^\infty(\mathbb{T}^2)} \leq C$ follows trivially from (**BA** $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **– LOCATION**), which implies that $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(\mathbb{T}^2) \subset [-\frac{3}{4}U_{\text{☢}},\frac{3}{4}U_{\text{☢}}]$. To control the derivatives of $\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}$ and $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}$ that are relevant for (14.1a)–(14.1b), we implicitly differentiate the identities $(\mu,\breve{X}\mu)\circ\left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right) = (\mathfrak{m},-\mathfrak{n})$ with $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial x^3}$ and argue as in the proof of (14.7a)–(14.7b), using in addition the fact that $-\frac{\partial}{\partial t}\mu \approx 1$ along $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ (see (**BA** $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ **– LOCATION**) and (**BA** $\frac{\partial}{\partial t}\mu$ neg)). We have therefore proved (14.1a)–(14.1b). Similarly, to prove (14.2)–(14.3), we implicitly differentiate the identities $(\mu,\breve{X}\mu)\circ\left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right) = (\mathfrak{m},-\mathfrak{n})$ with $\frac{\partial}{\partial\mathfrak{m}}$ and use the bootstrap assumptions, including (**BA** $\frac{\partial}{\partial t}\mu$ neg) and (**BA** μ cnvx).

To deduce that ${}^{(\mathfrak{n})}H$ is $C^{1,1}$ on every compact subset of ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$, we first note that (12.6) implies that for any compact subset $\mathfrak{K}$ of ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$, there is a $\mathfrak{m}' \in (\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0]$ and a compact set

$$
{}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}',\mathfrak{m}_0]} \stackrel{\text{def}}{=} \left\{(t,x^2,x^3) \in \mathbb{R}\times\mathbb{T}^2 \mid \Upsilon_{\mathfrak{m}_0,-\mathfrak{n}}(x^2,x^3) \le t \le \Upsilon_{\mathfrak{m}',-\mathfrak{n}}(x^2,x^3)\right\}
$$

such that $\mathfrak{K} \subset {}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}',\mathfrak{m}_0]}$. Since (12.5) implies that ${}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}',\mathfrak{m}_0]}$ has a C^1 boundary,[57] it is a standard Sobolev embedding result (see [39, Theorem 5 in Section 5.6]) that $W^{1,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}',\mathfrak{m}_0)}) \hookrightarrow C^{0,1}({}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}',\mathfrak{m}_0]})$, where by (12.6), ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}',\mathfrak{m}_0)} \stackrel{\text{def}}{=} \left\{(t,x^2,x^3) \in \mathbb{R}\times\mathbb{T}^2 \mid \Upsilon_{\mathfrak{m}_0,-\mathfrak{n}}(x^2,x^3) < t < \Upsilon_{\mathfrak{m}',-\mathfrak{n}}(x^2,x^3)\right\}$ is the interior of ${}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}',\mathfrak{m}_0]}$. Thus, in view of (14.7a), we conclude that ${}^{(\mathfrak{n})}H$ is $C^{1,1}$ on ${}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}',\mathfrak{m}_0]}$ as desired. □

14.2. Properties of the flow map of ${}^{(\mathfrak{n})}\breve{W}$ and the viability of the data-hypersurface $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}$.

Lemma 14.2 (Properties of the flow map of ${}^{(\mathfrak{n})}\breve{W}$ and the viability of the data-hypersurface $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}}]}_{-\mathfrak{n}}$).

Properties of the flow map of ${}^{(\mathfrak{n})}\breve{W}$. *Let* ${}^{(\mathfrak{n})}\breve{W}$ *be the vectorfield defined in* (4.2)*, and recall that* ${}^{(\mathfrak{n})}\breve{W}\tau = 0$ *and* ${}^{(\mathfrak{n})}\breve{W}u = 1$*. Let* $(\Delta u, t, u, x^2, x^3) \to {}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3)$ *denote the flow map of* ${}^{(\mathfrak{n})}\breve{W}$*, i.e., at each fixed* $(t,u,x^2,x^3) \in {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$*, the components of* ${}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3)$ *solve the following ODE system initial value problem on the flow interval* $\Delta u \in [-U_1-u, U_2-u]$*:*

$$
\frac{\partial}{\partial \Delta u}{}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3) = {}^{(\mathfrak{n})}\breve{W} \circ {}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3), \qquad {}^{(\mathfrak{n})}\iota_0(t,u,x^2,x^3) = (t,u,x^2,x^3). \tag{14.9}
$$

Then for each fixed $\tau \in [\tau_0,\tau_{\text{Boot}})$ *and each pair* $u_1,u_2 \in [-U_1,U_2]$*,* ${}^{(\mathfrak{n})}\iota_{u_2-u_1}$ *is a diffeomorphism from the rough torus* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u_1}$ *(defined in* (4.6b)*) onto the rough torus* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u_2}$*. In particular, the integral curves of* ${}^{(\mathfrak{n})}\breve{W}$ *thread* ${}^{(\mathfrak{n})}\widetilde{\Sigma}^{[-U_1,U_2]}_{\tau}$*. Moreover, for every fixed* $\tau \in [\tau_0,\tau_{\text{Boot}})$*, each integral curve of* ${}^{(\mathfrak{n})}\breve{W}$ *passes through precisely one point on the* μ*-adapted torus* $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ *defined in* (4.3c)*.*

Moreover, with d_{geo} *denoting the differential with respect to the geometric coordinates, we have the following bounds, where the implicit constants in* (14.11) *are independent of all* Δu *such that* $|\Delta u| \le U_1 + U_2$*:*

$$
\sup_{|\Delta u| \le U_1+U_2} \|d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}\|_{W^{2,\infty}_{\text{geo}}\left({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)} \cap {}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1-\Delta u,U_2-\Delta u)}\right)} \le C, \tag{14.10}
$$

$$
\det\left(d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}\right) \approx 1 \text{ on } {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]} \cap {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1-\Delta u,U_2-\Delta u]}. \tag{14.11}
$$

Estimates tied to the flow of $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}$ ***by*** ${}^{(\mathfrak{n})}\breve{W}$. *Let* ${}^{(\mathfrak{n})}H : {}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]} \to {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]}$ *be the embedding of* $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}$ *from Lemma 14.1, which is of the form* ${}^{(\mathfrak{n})}H(t,x^2,x^3) = \left(t, {}^{(\mathfrak{n})}h(t,x^2,x^3), x^2, x^3\right)$*, and let* ${}^{(\mathfrak{n})}F$ *be the map and set defined by:*

$$
{}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3) \stackrel{def}{=} {}^{(\mathfrak{n})}\iota_{\Delta u} \circ {}^{(\mathfrak{n})}H(t,x^2,x^3), \tag{14.12}
$$

$$
{}^{(\mathfrak{n})}\mathscr{F} \stackrel{def}{=} \left\{(\Delta u,t,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid (t,x^2,x^3) \in {}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]} \text{ and } \Delta u \in \left[-U_1 - {}^{(\mathfrak{n})}h(t,x^2,x^3), U_2 - {}^{(\mathfrak{n})}h(t,x^2,x^3)\right]\right\}. \tag{14.13}
$$

Then ${}^{(\mathfrak{n})}F : {}^{(\mathfrak{n})}\mathscr{F} \to {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ *is a diffeomorphism such that* ${}^{(\mathfrak{n})}F$ *and its inverse function* ${}^{(\mathfrak{n})}F^{-1}$ *satisfy the following bounds:*

$$
\|{}^{(\mathfrak{n})}F\|_{W^{2,\infty}(\text{int}({}^{(\mathfrak{n})}\mathscr{F}))} \le C, \tag{14.14}
$$

$$
\|{}^{(\mathfrak{n})}F^{-1}\|_{W^{2,\infty}_{\text{geo}}\left({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)}\right)} \le C, \tag{14.15}
$$

where:

$$
\begin{aligned}
\text{int}\left({}^{(\mathfrak{n})}\mathscr{F}\right) = \Big\{&(\Delta u,t,x^2,x^3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid (x^2,x^3) \in \mathbb{T}^2, \Upsilon_{\mathfrak{m}_0,-\mathfrak{n}}(x^2,x^3) < t < \Upsilon_{\mathfrak{m}_{\text{Boot}},-\mathfrak{n}}(x^2,x^3), \\
&\text{and } \Delta u \in \left(-U_1 - {}^{(\mathfrak{n})}h(t,x^2,x^3), U_2 - {}^{(\mathfrak{n})}h(t,x^2,x^3)\right)\Big\}
\end{aligned} \tag{14.16}
$$

57 In fact, the boundary is $C^{1,1}$, though we do not need this fact here.

is the interior of ${}^{(\mathfrak{n})}\mathscr{F}$, *and* $\mathfrak{T}_{\mathfrak{m}_0,-\mathfrak{n}}$ *and* $\mathfrak{T}_{\mathfrak{m}_{\text{Boot}},-\mathfrak{n}}$ *are the functions appearing in* (14.6). *Finally,* ${}^{(\mathfrak{n})}F$ *is* $C^{1,1}$ *on every compact subset of* ${}^{(\mathfrak{n})}\mathscr{F}$, *and* ${}^{(\mathfrak{n})}F^{-1}$ *is* $C^{1,1}$ *on every compact subset of* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$.

Proof. Throughout this proof, we silently use the soft bootstrap assumptions of Sect. 12.2.6, which guarantee sufficient qualitative regularity. For quantitative estimates, we will use the bootstrap assumptions of Sects. 12.3.1 and 12.3.2.

From (**BA** μ **– TORI STRUCTURE**), (**BA** $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **– LOCATION**), and the facts that ${}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\tau = 0$ and ${}^{(\mathfrak{n})}\breve{W}u = 1$, it follows that for each fixed $\tau \in [\tau_0, \tau_{\text{Boot}})$, every integral curve of ${}^{(\mathfrak{n})}\breve{W}$ in ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}$ must intersect $\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ at one or more points in ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-\frac{3}{4}U_{\maltese},\frac{3}{4}U_{\maltese}]}$. Recalling that $\mu|_{\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}} = -\tau$ and ${}^{(\mathfrak{n})}\breve{W}\mu|_{\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}} = 0$, and using the transversal convexity bootstrap assumption (**BA** μ cnvx) for ${}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\breve{W}\mu$, we see that the intersection occurs at a unique point.

Next, we differentiate the evolution equation in (14.9) and use the chain rule to deduce that $d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}$ is the solution to the following linear ODE system initial value problem:

$$\frac{\partial}{\partial \Delta u} d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u} = (d_{\text{geo}}{}^{(\mathfrak{n})}\breve{W}) \circ {}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3) \cdot d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3), \tag{14.17}$$

$$d_{\text{geo}}{}^{(\mathfrak{n})}\iota_0 = \text{diag}(1,1,1,1). \tag{14.18}$$

Definition (4.2), Lemma 5.5, Prop. 9.1, and the bootstrap assumptions imply that:

$$\left\| \left({}^{(\mathfrak{n})}\breve{W}t, {}^{(\mathfrak{n})}\breve{W}u, {}^{(\mathfrak{n})}\breve{W}^2, {}^{(\mathfrak{n})}\breve{W}^3 \right) \right\|_{W^{3,\infty}_{\text{geo}}\left({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)}\right)} \leq C \tag{14.19}$$

and thus:

$$\|d_{\text{geo}}{}^{(\mathfrak{n})}\breve{W}\|_{W^{2,\infty}_{\text{geo}}\left({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)}\right)} \leq C, \tag{14.20}$$

where we are viewing $d_{\text{geo}}{}^{(\mathfrak{n})}\breve{W}$ to be the Jacobian matrix of the map $(t,u,x^2,x^3) \to \left({}^{(\mathfrak{n})}\breve{W}t, {}^{(\mathfrak{n})}\breve{W}u, {}^{(\mathfrak{n})}\breve{W}^2, {}^{(\mathfrak{n})}\breve{W}^3\right)$. We now integrate (14.17) with respect to Δu starting from 0 and use the initial condition (14.18), the estimate (14.20), Grönwall's inequality, and the fact that $|\Delta u| \leq U_1 + U_2$ in the region under study, thereby concluding (14.10).

The estimate (14.11) follows from a similar argument based on the linear ODE system:

$$\frac{\partial}{\partial \Delta u} \ln\det\left(\text{d}_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}\right) = \text{tr}(\text{d}_{\text{geo}}{}^{(\mathfrak{n})}\breve{W}) \circ {}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3), \qquad \det\left(\text{d}_{\text{geo}}{}^{(\mathfrak{n})}\iota_0\right) = 1 \tag{14.21}$$

and the estimate $\text{tr}(\text{d}_{\text{geo}}{}^{(\mathfrak{n})}\breve{W}) = \mathcal{O}(\varepsilon^{1/2})$, which is a consequence of definition (4.2), Lemma 5.5, Prop. 9.1, and the bootstrap assumptions.

The estimate (14.14) follows from (14.7a), (14.10), the chain rule, and finally from using the equation (14.9) and the aforementioned estimate $\|\left({}^{(\mathfrak{n})}\breve{W}t, {}^{(\mathfrak{n})}\breve{W}u, {}^{(\mathfrak{n})}\breve{W}^2, {}^{(\mathfrak{n})}\breve{W}^3\right)\|_{W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \leq C$ to control the partial derivatives of ${}^{(\mathfrak{n})}F$ with respect to Δu.

Next, we highlight that the map $\Delta u \to {}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3)$ is just a parameterization of the integral curve of ${}^{(\mathfrak{n})}\breve{W}$ that passes through the point (t,u,x^2,x^3). Moreover, we recall that we have already shown that every integral curve of ${}^{(\mathfrak{n})}\breve{W}$ in ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}$ must intersect $\check{\mathbf{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ at a unique point (more precisely, a point in the torus $\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}$, which is contained in $\check{\mathbf{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$), where $\check{\mathbf{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ is the image of the set ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ under the embedding ${}^{(\mathfrak{n})}H$ (see Lemma 14.1). It follows that ${}^{(\mathfrak{n})}F$ is a bijection from ${}^{(\mathfrak{n})}\mathscr{F}$ onto ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. Thus, to conclude that ${}^{(\mathfrak{n})}F$ is a diffeomorphism on ${}^{(\mathfrak{n})}\mathscr{F}$, it remains only for us to show that its Jacobian matrix $d_{(\Delta u,t,x^2,x^3)}{}^{(\mathfrak{n})}F$ has non-vanishing determinant. To this end, we note that the standard theory of flow maps, the identity ${}^{(\mathfrak{n})}\breve{W} \circ {}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3) = [d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}(t,u,x^2,x^3)] \cdot {}^{(\mathfrak{n})}\breve{W}(t,u,x^2,x^3)$, and the chain rule yield the identity $d_{(\Delta u,t,x^2,x^3)}{}^{(\mathfrak{n})}F(\Delta u,t,x^2,x^3) = [(d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}) \circ {}^{(\mathfrak{n})}H(t,x^2,x^3)] \cdot {}^{(\mathfrak{n})}M(t,x^2,x^3)$, where ${}^{(\mathfrak{n})}M$ is the 4×4 matrix-valued function on ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ whose first column is $\left({}^{(\mathfrak{n})}\breve{W}t, {}^{(\mathfrak{n})}\breve{W}u, {}^{(\mathfrak{n})}\breve{W}^2, {}^{(\mathfrak{n})}\breve{W}^3\right)^\top \circ {}^{(\mathfrak{n})}H$ and whose last three columns form the Jacobian matrix $d_{(t,x^2,x^3)}{}^{(\mathfrak{n})}H$. This implies, in particular, that $\det\left(d_{(\Delta u,t,x^2,x^3)}{}^{(\mathfrak{n})}F(\Delta u,t,x^2,x^3)\right) = \det\left((d_{\text{geo}}{}^{(\mathfrak{n})}\iota_{\Delta u}) \circ {}^{(\mathfrak{n})}H\right) \cdot \det\left({}^{(\mathfrak{n})}M(t,x^2,x^3)\right)$. Thus, we see from (14.11) that to prove:

$$|\det\left(d_{(\Delta u,t,x^2,x^3)}{}^{(\mathfrak{n})}F\right)| \approx 1, \qquad \text{on the domain } {}^{(\mathfrak{n})}\mathscr{F}, \tag{14.22}$$

we need only to show that $|\det {}^{(\mathfrak{n})}M| \approx 1$ on the domain ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$. To this end, we use Lemma 14.1 (in particular (14.7b) and (14.8)), the identities (see (4.2)) ${}^{(\mathfrak{n})}\breve{W}t = \frac{\phi\mathfrak{n}}{L\mu}$, ${}^{(\mathfrak{n})}\breve{W}u = 1$, and ${}^{(\mathfrak{n})}\breve{W}x^A = \mu X^A + \frac{\phi\mathfrak{n}}{L\mu}L^A$, Prop. 9.1, and the bootstrap assumptions to deduce that:

$$ {}^{(\mathfrak{n})}M = \begin{pmatrix} [\phi\frac{\mathfrak{n}}{L\mu}] \circ {}^{(\mathfrak{n})}H & 1 & 0 & 0 \\ 1 & -\frac{L\breve{X}\mu \circ {}^{(\mathfrak{n})}H}{\breve{X}\breve{X}\mu \circ {}^{(\mathfrak{n})}H} + * & * & * \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{pmatrix}, \tag{14.23} $$

where "$*$" denotes $\mathcal{O}(\varepsilon^{1/2})$ quantities. From (14.23) and the bootstrap assumptions (notably (**BA** μ cnvx)), we see that $-C \leq \det {}^{(\mathfrak{n})}M = -\frac{1}{C}$. We have thus shown that ${}^{(\mathfrak{n})}F$ is a diffeomorphism from ${}^{(\mathfrak{n})}\mathscr{F}$ onto ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$.

(14.15) follows from differentiating the identity ${}^{(\mathfrak{n})}F \circ {}^{(\mathfrak{n})}F^{-1}(t,u,x^2,x^3) = (t,u,x^2,x^3)$ and using (14.14), (14.22), and the chain rule.

To show that ${}^{(\mathfrak{n})}F$ is $C^{1,1}$ on every compact subset of ${}^{(\mathfrak{n})}\mathscr{F}$, we first note that arguments similar to the ones we used to prove (14.10), but based on the soft regularity assumptions of Sect. 12.2.6, imply that ${}^{(\mathfrak{n})}\iota_{\Delta u}$ is $C^{2,1}$ on every compact subset of $\left\{(\Delta u, t, u, x^2, x^3) \mid (t,u,x^2,x^3) \in {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]} \cap {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1-\Delta u,U_2-\Delta u]}, |\Delta u| \leq U_1 + U_2\right\}$. Since Lemma 14.1 shows that ${}^{(\mathfrak{n})}H$ is $C^{1,1}$ on every compact subset of ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$, the desired $C^{1,1}$ result for ${}^{(\mathfrak{n})}F$ follows from definition (14.12) and the fact that the composition of two $C^{1,1}$ maps is also $C^{1,1}$. The fact that ${}^{(\mathfrak{n})}F^{-1}$ is $C^{1,1}$ on every compact subset of ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ then follows from differentiating the identity ${}^{(\mathfrak{n})}F \circ {}^{(\mathfrak{n})}F^{-1}(t,u,x^2,x^3) = (t,u,x^2,x^3)$, using the $C^{1,1}$ result for ${}^{(\mathfrak{n})}F$, and using the fact that diffeomorphisms map compact subsets to compact subsets.

□

15. Estimates for the rough time function, continuous extensions, and various diffeomorphisms and homeomorphisms

We continue to work under the assumptions of Sect. 13.2. In this section, we use the results of Sect. 14 to derive estimates for the rough time function ${}^{(\mathfrak{n})}\tau$. We then show that the map ${}^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$ extends to a diffeomorphism on the closure of ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, which is ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$. Next, we show that various solution variables extend to the compact set ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ as functions with substantial Hölder regularity, which we will exploit throughout the paper. Finally, we establish related results for the map ${}^{(\mathfrak{n})}\Phi({}^{(\mathfrak{n})}\tau, u, x^2, x^3) = (\mu, \breve{X}\mu, x^2, x^3)$ from Def. 5.1.

15.1. Transport equation solutions that are smoother than the data-hypersurface.

We will use the following lemma to derive $W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})$ estimates for the rough time function. There is one nonstandard aspect that we carefully handle: the initial data are smoother than the hypersurface on which they are posed.

Lemma 15.1 (Transport equation solutions that are smoother than the data-hypersurface)**.** *Consider the initial value problem for a scalar function φ:*

$$ {}^{(\mathfrak{n})}\breve{W}\varphi = 0, \qquad \varphi|_{\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}} = \mathscr{A}|_{\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}}, \tag{15.1} $$

and assume that $\mathscr{A} \in W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})$ is an "ambient" spacetime function of the geometric coordinates satisfying $({}^{(\mathfrak{n})}\breve{W}\mathscr{A})|_{\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}} = 0$, where the hypersurface $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}$ is $W^{2,\infty}$ (as was shown in Lemma 14.1 via the embedding ${}^{(\mathfrak{n})}H$). Then there exists a unique solution $\varphi \in W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})$ satisfying the following estimates:

$$ \|\varphi\|_{W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \lesssim \|\mathscr{A}\|_{W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})}, \tag{15.2a} $$

$$ \left\|\left(\frac{\partial}{\partial x^2}\varphi, \frac{\partial}{\partial x^3}\varphi\right)\right\|_{W^{2,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \lesssim \left\|\left(\frac{\partial}{\partial x^2}\mathscr{A}, \frac{\partial}{\partial x^3}\mathscr{A}\right)\right\|_{W^{2,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})}. \tag{15.2b} $$

Remark 15.2 (The solution is more regular than the data-hypersurface)**.** The main point of the lemma is that the solution has the same regularity as $\mathscr{A}$, even though the embedded data-hypersurface $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}})}_{-\mathfrak{n}}$ is one degree less regular, that is, the embedding ${}^{(\mathfrak{n})}H$ from Lemma 14.1 satisfies ${}^{(\mathfrak{n})}H \in W^{2,\infty}(\text{int}({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}))$.

Proof. We prove only (15.2a) since (15.2b) can be proved using similar arguments.

Throughout the proof, we silently use the fact that functions $f \in W^{1,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})$, are locally Lipschitz and thus have (by Rademacher's theorem) a.e. differentiable locally Lipschitz traces along the $W^{2,\infty}$ hypersurface $\breve{\mathbb{X}}^{[\tau_0,\tau_{\mathrm{Boot}})}_{-\mathfrak{n}}$ such that $\|f\|_{W^{1,\infty}(\mathrm{int}(\breve{\mathbb{X}}^{[\tau_0,\tau_{\mathrm{Boot}})}_{-\mathfrak{n}}))} \lesssim \|f\|_{W^{1,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})}$.

By Lemma 14.2, (15.1) is equivalent to the following ODE initial value problem:

$$\frac{\partial}{\partial \Delta u}\left(\varphi \circ {}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3)\right) = 0, \tag{15.3}$$

$$\varphi \circ {}^{(\mathfrak{n})}F(0, t, x^2, x^3) = \mathscr{A} \circ {}^{(\mathfrak{n})}H(t, x^2, x^3), \tag{15.4}$$

where ${}^{(\mathfrak{n})}F$ is the diffeomorphism from ${}^{(\mathfrak{n})}\mathscr{F}$ onto ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$ from the lemma. The solution to (15.3)–(15.4) is $\varphi \circ {}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3) = \mathscr{A} \circ {}^{(\mathfrak{n})}H(t, x^2, x^3)$. From this formula, the assumptions of Lemma 15.1, (14.7a), and (14.14), it immediately follows that $\varphi \circ {}^{(\mathfrak{n})}F \in W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{F}))$, where we stress that the norm on $W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{F}))$ is with respect to the coordinates $(\Delta u, t, x^2, x^3)$. Composing $\varphi \circ {}^{(\mathfrak{n})}F$ with ${}^{(\mathfrak{n})}F^{-1}$ (in particular, using that ${}^{(\mathfrak{n})}F$ and ${}^{(\mathfrak{n})}F^{-1}$ are $W^{2,\infty}$ diffeomorphisms by (14.14) and (14.15)), we see that $\|\varphi\|_{W^{2,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})} \lesssim \|\mathscr{A}\|_{W^{2,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})}$.

To complete the proof, it suffices for us to show:

$$\vec{\partial}_{\mathrm{geo}}\varphi \in W^{2,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}), \qquad \|\vec{\partial}_{\mathrm{geo}}\varphi\|_{W^{2,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})} \lesssim \|\mathscr{A}\|_{W^{3,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})}, \tag{15.5}$$

where $\vec{\partial}_{\mathrm{geo}}\varphi \stackrel{\mathrm{def}}{=} \left(\frac{\partial}{\partial t}\varphi, \frac{\partial}{\partial u}\varphi, \frac{\partial}{\partial x^2}\varphi, \frac{\partial}{\partial x^3}\varphi\right)$ is the array of geometric coordinate partial derivatives of φ. Due to the limited regularity ${}^{(\mathfrak{n})}H \in W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}))$, the desired regularity of φ cannot be inferred directly from the formula $\varphi \circ {}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3) = \mathscr{A} \circ {}^{(\mathfrak{n})}H(t, x^2, x^3)$. Instead, we commute (15.1) with the geometric coordinate partial derivative vectorfields to deduce that $\vec{\partial}_{\mathrm{geo}}\varphi$ satisfies the following initial value problem:

$$ {}^{(\mathfrak{n})}\breve{W}\vec{\partial}_{\mathrm{geo}}\varphi = \vec{I} \cdot \vec{\partial}_{\mathrm{geo}}\varphi, \qquad \vec{\partial}_{\mathrm{geo}}\varphi|_{\breve{\mathbb{X}}^{[\tau_0,\tau_{\mathrm{Boot}})}_{-\mathfrak{n}}} = \vec{\partial}_{\mathrm{geo}}\mathscr{A}|_{\breve{\mathbb{X}}^{[\tau_0,\tau_{\mathrm{Boot}})}_{-\mathfrak{n}}}, \tag{15.6}$$

where schematically, we have $\vec{I} = d_{\mathrm{geo}}{}^{(\mathfrak{n})}\breve{W}$ (i.e., $\vec{I}$ schematically comprises the first partial derivatives of the components of ${}^{(\mathfrak{n})}\breve{W}$ in the geometric coordinate system), the validity of the initial condition on RHS (15.6) relies on the compatibility condition assumption $({}^{(\mathfrak{n})}\breve{W}\mathscr{A}) \circ {}^{(\mathfrak{n})}H = 0$ and the fact that ${}^{(\mathfrak{n})}\breve{W}$ is transversal to $\breve{\mathbb{X}}^{[\tau_0,\tau_{\mathrm{Boot}})}_{-\mathfrak{n}}$ (by the bootstrap assumption (**BA** μ cnvx) for ${}^{(\mathfrak{n})}\breve{W}\breve{X}\mu$ and (**BA** $\breve{\mathbb{X}}^{[\tau_0,\tau_{\mathrm{Boot}})}_{-\mathfrak{n}}$ – **LOCATION**)), and by (14.20), we have $\|\vec{I}\|_{W^{2,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})} \lesssim 1$. As in (15.3)–(15.4), we can rewrite (15.6) as the following linear ODE system in the unknowns $(\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3)$:

$$\frac{\partial}{\partial \Delta u}\left[(\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3)\right] = \left[\vec{I} \circ {}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3)\right] \cdot (\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F(\Delta u, t, x^2, x^3), \tag{15.7}$$

$$(\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F(0, t, x^2, x^3) = (\vec{\partial}_{\mathrm{geo}}\mathscr{A}) \circ {}^{(\mathfrak{n})}H(t, x^2, x^3), \tag{15.8}$$

where our assumptions on $\vec{\partial}_{\mathrm{geo}}\mathscr{A}$ and the regularity of ${}^{(\mathfrak{n})}H$ imply that $(\vec{\partial}_{\mathrm{geo}}\mathscr{A}) \circ {}^{(\mathfrak{n})}H \in W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}))$ (here ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}$ is the domain of ${}^{(\mathfrak{n})}H$) with $\|(\vec{\partial}_{\mathrm{geo}}\mathscr{A}) \circ {}^{(\mathfrak{n})}H\|_{W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}))} \lesssim \|\mathscr{A}\|_{W^{3,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})}$. The standard theory of transport equations with $W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{F}))$ coefficients yields that (15.7)–(15.8) has a unique solution (which must be $(\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F$, where φ is the solution to (15.1)) satisfying $(\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F \in W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{F}))$. Moreover, with the help of the above bound for $\|\vec{\partial}_{\mathrm{geo}}\mathscr{A}\|_{W^{2,\infty}(\mathrm{int}({}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}))}$, a standard argument based on commuting (15.7) up to two times with respect to the partial derivatives in the coordinate system $(\Delta u, t, x^2, x^3)$, integrating with respect to Δu, applying Grönwall's inequality, and using that $|\Delta u| \leq U_1 + U_2$ in the region under study yields the bound:

$$\left\|(\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F\right\|_{W^{2,\infty}({}^{(\mathfrak{n})}\mathscr{F})} \lesssim \|\mathscr{A}\|_{W^{3,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})}\left\{1 + \|\vec{I} \circ {}^{(\mathfrak{n})}F\|_{W^{2,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})}\right\}. \tag{15.9}$$

Finally, composing $(\vec{\partial}_{\mathrm{geo}}\varphi) \circ {}^{(\mathfrak{n})}F$ with ${}^{(\mathfrak{n})}F^{-1}$, and using (14.14)–(14.15) as well as the bound $\|\vec{I}\|_{W^{2,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})} \lesssim 1$ and the bound (15.9), we conclude (15.2a).

□

15.2. **Estimates for the rough time function and the change of variables map from adapted rough coordinates to geometric coordinates.** Using Lemma 15.1, we now derive estimates for the rough time function, the change of variables map from geometric coordinates to adapted rough coordinates, and its inverse.

We start with the following simple lemma, which provides an identity for $\mathbf{D}^{(\mathfrak{n})}\tau|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}}$.

Lemma 15.3 (Identity for $\mathbf{D}^{(\mathfrak{n})}\tau$ along $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$)**.** *Recall that $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ is the truncated $\breve{X}\mu$-level-set defined in* (4.7b). *With $\mathbf{D}\varphi$ denoting the spacetime gradient one-form of the scalar function φ, we have the following identity:*

$$\mathbf{D}^{(\mathfrak{n})}\tau|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}} = -\mathbf{D}\mu|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}}. \tag{15.10}$$

Proof. The bootstrap assumptions of Sect. 12.2.4 imply that $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ is a hypersurface portion foliated by $\{\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\}_{\mathfrak{m}\in(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$. From these facts and (4.4b), it follows that ${}^{(\mathfrak{n})}\tau$ and $-\mu$ have the same derivatives in directions tangent to $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$. Moreover, since definition (4.2) implies that ${}^{(\mathfrak{n})}\breve{W}\mu|_{\breve{\mathbb{X}}_{-\mathfrak{n}}} = 0$, we see from (4.4a) that along $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$, ${}^{(\mathfrak{n})}\tau$ and $-\mu$ have the same ${}^{(\mathfrak{n})}\breve{W}$-derivative. Since (**BA** μ cnvx) and (**BA** $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ $-$ **LOCATION**) imply that ${}^{(\mathfrak{n})}\breve{W}$ is transversal to $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$, we conclude the desired identity (15.10). □

Lemma 15.4 (Estimates for ${}^{(\mathfrak{n})}\tau$, ${}^{(\mathfrak{n})}\mathscr{T}$, and ${}^{(\mathfrak{n})}\mathscr{T}^{-1}$)**.** *The following estimates hold.*

Estimates for ${}^{(\mathfrak{n})}\tau$:

$$\left\|{}^{(\mathfrak{n})}\tau\right\|_{W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \lesssim 1, \tag{15.11a}$$

$$\left\|\left(\frac{\partial}{\partial x^2}{}^{(\mathfrak{n})}\tau, \frac{\partial}{\partial x^3}{}^{(\mathfrak{n})}\tau\right)\right\|_{W^{2,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \lesssim \varepsilon^{1/2}. \tag{15.11b}$$

Moreover,

$$\frac{1}{2}\delta_* \leq \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau \leq \frac{3}{2}\delta_*, \qquad on\ {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}, \tag{15.12a}$$

$$\frac{1}{2}\delta_* \leq L^{(\mathfrak{n})}\tau \leq \frac{3}{2}\delta_*, \qquad on\ {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}. \tag{15.12b}$$

Estimates for ${}^{(\mathfrak{n})}\mathscr{T}$ and ${}^{(\mathfrak{n})}\mathscr{T}^{-1}$*: The following estimate holds, where ${}^{(\mathfrak{n})}\mathscr{T}$ is the change of variables map from geometric coordinates to adapted rough coordinates defined in* (5.2):

$$\left\|{}^{(\mathfrak{n})}\mathscr{T}\right\|_{W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \lesssim 1. \tag{15.13}$$

The following estimates hold, where $d_{\text{geo}}{}^{(\mathfrak{n})}\mathscr{T}$ is the Jacobian matrix of ${}^{(\mathfrak{n})}\mathscr{T}$:

$$\det\left(d_{\text{geo}}{}^{(\mathfrak{n})}\mathscr{T}\right) \approx 1, \qquad on\ {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}, \tag{15.14}$$

$$\left\|[d_{\text{geo}}{}^{(\mathfrak{n})}\mathscr{T}]^{-1}\right\|_{W^{2,\infty}\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}\right)} \leq C. \tag{15.15}$$

In addition, the following estimate holds, where ${}^{(\mathfrak{n})}\mathscr{T}^{-1}$ is the change of variables map from adapted rough coordinates to geometric coordinates and $d_{\text{rough}}{}^{(\mathfrak{n})}\mathscr{T}^{-1}$ is its Jacobian matrix:

$$\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}t = \det\left(d_{\text{rough}}{}^{(\mathfrak{n})}\mathscr{T}^{-1}\right) \approx 1, \qquad on\ [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2] \times \mathbb{T}^2. \tag{15.16}$$

In addition, the following estimate holds:

$$\left\|{}^{(\mathfrak{n})}\mathscr{T}^{-1}\right\|_{W^{3,\infty}_{\text{rough}}((\tau_0,\tau_{\text{Boot}})\times(-U_1,U_2)\times\mathbb{T}^2)} \leq C. \tag{15.17}$$

Finally, ${}^{(\mathfrak{n})}\mathscr{T}^{-1}$ extends to a $C^{2,1}([\tau_0,\tau_{\text{Boot}}] \times [-U_1,U_2] \times \mathbb{T}^2)$ map satisfying the estimate:

$$\left\|{}^{(\mathfrak{n})}\mathscr{T}^{-1}\right\|_{C^{2,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C. \tag{15.18}$$

Proof. Recall from Def. 4.5 that ${}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\tau = 0$ and ${}^{(\mathfrak{n})}\tau|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}} = -\mu|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}}$. Hence, (15.11a) follows from applying Lemma 15.1 with $\mathscr{A} = -\mu$ and the bound $\|\mu\|_{W^{3,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \lesssim 1$, which follows from Lemma 5.5, Prop. 9.1, and the bootstrap assumptions.

Similarly, (15.11b) follows from (15.2b) and the bound $\left\|\left(\frac{\partial}{\partial x^2}\mu, \frac{\partial}{\partial x^3}\mu\right)\right\|_{W^{2,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \lesssim \varepsilon^{1/2}$, which follows from Lemma 5.5, Prop. 9.1, and the bootstrap assumptions.

(15.13) follows from (15.11a) and the definition of ${}^{(\mathfrak{n})}\mathscr{T}$.

To prove (15.12a) and (15.12b), we first commute the equation ${}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\tau = 0$ with L to obtain the transport equation ${}^{(\mathfrak{n})}\breve{W}L{}^{(\mathfrak{n})}\tau = [{}^{(\mathfrak{n})}\breve{W}, L]{}^{(\mathfrak{n})}\tau$. Using (4.2), Prop. 13.5, the bootstrap assumptions, and (15.11b), we see that the source term can be pointwise bounded on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ as follows: $\left|[{}^{(\mathfrak{n})}\breve{W}, L]{}^{(\mathfrak{n})}\tau\right| \lesssim \varepsilon^{1/2}$. Moreover, since Lemma 15.3 implies that $L{}^{(\mathfrak{n})}\tau|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}} = -L\mu|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}}$, we can use the bootstrap assumptions (**BA** $L\mu$ neg) and (**BA** $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ **– LOCATION**) to deduce that $-\frac{5}{4}\delta_* \leq -L{}^{(\mathfrak{n})}\tau|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}} \leq -\frac{3}{4}\delta_*$. Hence, recalling that ${}^{(\mathfrak{n})}\breve{W}u = 1$, we can integrate the transport equation for $L{}^{(\mathfrak{n})}\tau$ starting from the ${}^{(\mathfrak{n})}\breve{W}$-transversal data-hypersurface $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}})}$ (see (**BA** μ cnvx)) and use the fact that $|u| \leq U_1 + U_2 \leq C$ on the region under study to conclude that:

$$-\frac{5}{4}\delta_* - C\varepsilon^{1/2} \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}} -L{}^{(\mathfrak{n})}\tau \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}} -L{}^{(\mathfrak{n})}\tau \leq -\frac{3}{4}\delta_* + C\varepsilon^{1/2}, \tag{15.19}$$

which yields (15.12b). Finally, using (15.19), the identity $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau = L{}^{(\mathfrak{n})}\tau - L^A\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau$, and the bound $\left|L^A\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau\right| \lesssim \varepsilon^{1/2}$ implied by the bootstrap assumptions and (15.11b), we conclude (15.12a).

(15.14) now follows from (15.12a) and the simple identity $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau = \det\left(d_{\text{geo}}{}^{(\mathfrak{n})}\mathscr{T}\right)$, which follows easily from the definition of ${}^{(\mathfrak{n})}\mathscr{T}$.

(15.15) now follows from (15.13) and (15.14).

The "=" in (15.16) follows easily from the definition of ${}^{(\mathfrak{n})}\mathscr{T}$. The "≈" in (15.16) follows from the identity

$$\det\left([d_{\text{rough}}{}^{(\mathfrak{n})}\mathscr{T}^{-1}] \circ {}^{(\mathfrak{n})}\mathscr{T}\right) = \left(\det[d_{\text{geo}}{}^{(\mathfrak{n})}\mathscr{T}]\right)^{-1}$$

and (15.14).

(15.17) follows from differentiating (up to two times, with the adapted rough coordinate partial derivative vectorfields) the identity $d_{\text{rough}}{}^{(\mathfrak{n})}\mathscr{T}^{-1} = [d_{\text{geo}}{}^{(\mathfrak{n})}\mathscr{T}]^{-1} \circ [{}^{(\mathfrak{n})}\mathscr{T}^{-1}]$ and using (**BA** t **– SIZE**), (15.15), and the chain rule.

(15.18) follows from (15.17), and the following Sobolev embedding result for scalar functions f (see the proof of [39, Theorem 5 in Section 5.6]), which relies on the convexity of the domain $(\tau_0,\tau_{\text{Boot}})\times(-U_1,U_2)\times\mathbb{T}^2$: $\|f\|_{C^{0,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C\|f\|_{W^{1,\infty}_{\text{rough}}((\tau_0,\tau_{\text{Boot}})\times(-U_1,U_2)\times\mathbb{T}^2)}$.

□

In the next lemma, we continue our analysis of the change of variables map ${}^{(\mathfrak{n})}\mathscr{T}$. More precisely, we exhibit its properties on the closure of ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$. We also derive the quasi-convexity of the closure of ${}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)}$ and, as a consequence, prove a standard Sobolev embedding result.

Lemma 15.5 (Properties of ${}^{(\mathfrak{n})}\mathscr{T}$ on the closure of ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ and quasi-convexity)**.** *The following results hold.*

1. ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ *is precompact in the topology of the geometric coordinates* (t,u,x^2,x^3).
2. *The change of variables map* ${}^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\mathfrak{n})}\tau,u,x^2,x^3)$ *extends to a* $C^{2,1}$ *diffeomorphism on the closure of* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, *which we denote by* $\text{cl}\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}\right)$. *Moreover,* $\text{cl}\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}\right) = {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$, *and* ${}^{(\mathfrak{n})}\mathscr{T}\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}\right) = [\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$.
3. *The following estimates hold for the extended maps:*

$$\frac{1}{2}\delta_* \leq \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau \leq \frac{3}{2}\delta_*, \qquad \text{on } {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}. \tag{15.20}$$

4.

$$\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}t = \det\left(d_{\text{rough}}{}^{(\mathfrak{n})}\mathscr{T}^{-1}\right) \approx 1, \qquad \text{on } [\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2, \tag{15.21}$$

5.

$$\left\| {}^{(\mathfrak{n})}\mathscr{T} \right\|_{C^{2,1}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]})} \leq C, \tag{15.22}$$

$$\left\| {}^{(\mathfrak{n})}\mathscr{T}^{-1} \right\|_{C^{2,1}_{\mathrm{rough}}([\tau_0,\tau_{\mathrm{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C, \tag{15.23}$$

$$\left\| \left(\frac{\partial}{\partial x^2} {}^{(\mathfrak{n})}\tau, \frac{\partial}{\partial x^3} {}^{(\mathfrak{n})}\tau \right) \right\|_{C^{1,1}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]})} \leq C\varepsilon^{1/2}. \tag{15.24}$$

*6. (**Quasi-convexity of** ${}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$). For every pair of points $q_1, q_2 \in (\tau_0,\tau_{\mathrm{Boot}})\times(-U_1,U_2)\times\mathbb{T}^2$, we have:*

$$\mathrm{dist}_{\mathrm{flat}}\left({}^{(\mathfrak{n})}\mathscr{T}^{-1}(q_1), {}^{(\mathfrak{n})}\mathscr{T}^{-1}(q_2) \right) \approx \mathrm{dist}_{\mathrm{flat}}(q_1,q_2), \tag{15.25}$$

where $\mathrm{dist}_{\mathrm{flat}}(q_1,q_2)$ is the standard Euclidean distance between q_1 and q_2 in the flat space $\mathbb{R}_\tau \times \mathbb{R}_u \times \mathbb{T}^2$.

Moreover, ${}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$ is quasi-convex.[58] That is, every pair of points $p_1, p_2 \in {}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$ is connected by a C^1_{geo} curve in ${}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$ whose length with respect to the standard flat Euclidean metric on geometric coordinate space $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$ is $\leq C\mathrm{dist}_{\mathrm{flat}}(p_1,p_2)$.

*7. (**Sobolev embedding**). There is a constant $C > 0$ that is **independent of** τ_{Boot} such that the following Sobolev embedding result holds for scalar functions f on ${}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$:*

$$\|f\|_{C^{0,1}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]})} \leq C\|f\|_{W^{1,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)})}. \tag{15.26}$$

*8. (${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}$ **is a graph**). For $\tau \in [\tau_0,\tau_{\mathrm{Boot}}]$, there exists a function $\mathrm{t}_{\tau,\mathfrak{n}} : [-U_1,U_2]\times\mathbb{T}^2 \to \mathbb{R}$, depending on τ and $\mathfrak{n}$, such that:*

$$\|\mathrm{t}_{\tau,\mathfrak{n}}\|_{C^{2,1}([-U_1,U_2]\times\mathbb{T}^2)} \leq C \tag{15.27}$$

and such that relative to the geometric coordinates, we have:

$${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]} = \left\{ (t,u,x^2,x^3) \mid t = \mathrm{t}_{\tau,\mathfrak{n}}(u,x^2,x^3), (u,x^2,x^3) \in [-U_1,U_2]\times\mathbb{T}^2 \right\}. \tag{15.28}$$

Proof. In (15.18) and just above it, we showed that ${}^{(\mathfrak{n})}\mathscr{T}^{-1}$ extends as a $C^{2,1}_{\mathrm{rough}}$ function to the compact, convex domain $[\tau_0,\tau_{\mathrm{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$ such that (15.16) holds on $[\tau_0,\tau_{\mathrm{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$. In particular, this yields (15.21) and (15.23). From these facts, (15.12a), and the fact that $(t,u,x^2,x^3) = {}^{(\mathfrak{n})}\mathscr{T}^{-1}(\tau,u,x^2,x^3)$, we conclude that the map ${}^{(\mathfrak{n})}\mathscr{T}^{-1}$ with domain $[\tau_0,\tau_{\mathrm{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$ has a global inverse, i.e., that ${}^{(\mathfrak{n})}\mathscr{T}$ extends to the compact domain ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}$ as an invertible map such that ${}^{(\mathfrak{n})}\tau$ satisfies (15.20).

We now prove (15.25). For $i = 1,2$, we set $q_i \stackrel{\mathrm{def}}{=} (\tau_i,u_i,x_i^2,x_i^3)$ and $p_i \stackrel{\mathrm{def}}{=} {}^{(\mathfrak{n})}\mathscr{T}^{-1}(q_i) \stackrel{\mathrm{def}}{=} (t_i,u_i,x_i^2,x_i^3)$. We define $\Delta\tau \stackrel{\mathrm{def}}{=} \tau_2 - \tau_1$, $\Delta u \stackrel{\mathrm{def}}{=} u_2 - u_1$, and $|\Delta q|_{\mathrm{Taxi}} \stackrel{\mathrm{def}}{=} |\Delta\tau| + |\Delta u| + |\Delta x^2|_{\mathbb{T}} + |\Delta x^3|_{\mathbb{T}}$, where for $j = 2,3$, $|\Delta x^j|_{\mathbb{T}}$ is the Euclidean distance between x_2^j and x_1^j in the torus. We similarly define $\Delta t \stackrel{\mathrm{def}}{=} t_2 - t_1$ and $|\Delta p|_{\mathrm{Taxi}} \stackrel{\mathrm{def}}{=} |\Delta t| + |\Delta u| + |\Delta x^2|_{\mathbb{T}} + |\Delta x^3|_{\mathbb{T}}$. Note that $|\Delta q|_{\mathrm{Taxi}} \approx \mathrm{dist}_{\mathrm{flat}}(q_1,q_2)$ and $|\Delta p|_{\mathrm{Taxi}} \approx \mathrm{dist}_{\mathrm{flat}}(p_1,p_2)$. Without loss of generality, we assume $\Delta\tau \geq 0$. Then by (15.21) and (15.23), there is a constant $C > 1$ such that $\frac{1}{C}\Delta\tau - C(|\Delta u| + |\Delta x^2|_{\mathbb{T}} + |\Delta x^3|_{\mathbb{T}}) \leq \Delta t \leq C\Delta\tau + C(|\Delta u| + |\Delta x^2|_{\mathbb{T}} + |\Delta x^3|_{\mathbb{T}})$. Hence, we see that there exists a (possibly different) constant $C > 1$ such that:

$$\left| \frac{1}{C}\Delta\tau - C\left(|\Delta u| + |\Delta x^2|_{\mathbb{T}} + |\Delta x^3|_{\mathbb{T}} \right) \right| + |\Delta u| + |\Delta x^2|_{\mathbb{T}} + |\Delta x^3|_{\mathbb{T}} \leq |\Delta p|_{\mathrm{Taxi}} \leq C\left(|\Delta\tau| + |\Delta u| + |\Delta x^2|_{\mathbb{T}} + |\Delta x^3|_{\mathbb{T}} \right). \tag{15.29}$$

From (15.29), it follows that $|\Delta p|_{\mathrm{Taxi}} \approx |\Delta q|_{\mathrm{Taxi}}$, which implies (15.25).

We now prove the quasi-convexity of ${}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$. Let $p_1, p_2 \in {}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$, and let $q_1, q_2 \in (\tau_0,\tau_{\mathrm{Boot}})\times(-U_1,U_2)\times\mathbb{T}^2$ be the unique points such that $p_i = {}^{(\mathfrak{n})}\mathscr{T}^{-1}(q_i)$, as above. Let ℓ be a straight line in $(\tau_0,\tau_{\mathrm{Boot}})\times(-U_1,U_2)\times\mathbb{T}^2$ whose flat length is equal to $\mathrm{dist}_{\mathrm{flat}}(q_1,q_2)$. From (15.23), it follows that the image curve ${}^{(\mathfrak{n})}\mathscr{T}^{-1}(\ell)$ has a Euclidean length that is $\lesssim \mathrm{dist}_{\mathrm{flat}}(q_1,q_2)$, which by (15.25) is $\approx \mathrm{dist}_{\mathrm{flat}}(p_1,p_2)$ as desired.

(15.26) is a standard Sobolev embedding result (see, for example, [40, Theorem 7]), which relies on the quantitative quasi-convexity of ${}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_1,U_2)}$ proved in the previous paragraph.

(15.22) follows from (15.13) and (15.26).

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[58] Here, we are not just interested in a qualitative version of quasi-convexity, but rather in obtaining control over the constants "C." Similar remarks apply for the Sobolev embedding result (15.26) and for other quasi-convexity and Sobolev embedding results derived throughout the paper.

The existence of the function $\mathfrak{t}_{\tau,\mathfrak{n}}$ such that (15.28) holds, as well as the estimate (15.27), follow from the fact that $(t,u,x^2,x^3) = {}^{(\mathfrak{n})}\mathscr{T}^{-1}(\tau,u,x^2,x^3)$ and the estimate (15.23), i.e., $\mathfrak{t}_{\tau,\mathfrak{n}}(u,x^2,x^3)$ is the first component of ${}^{(\mathfrak{n})}\mathscr{T}^{-1}(\tau,u,x^2,x^3)$. □

15.3. **Hölder-space extensions to the compact set ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$.** With the help of the bootstrap assumptions and Lemma 15.5, we now show that various solution variables extend to the compact set ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ as functions with substantial Hölder regularity relative to the geometric coordinates.

Lemma 15.6 (Hölder-space extensions to the compact set ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$)**.** *The following quantities extend to the compact set ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ as elements of the following Hölder spaces, and their corresponding spacetime Hölder norms on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ are bounded by $\leq C$:*

- $\vec{\Psi}, \Omega^i, S^i, \mathcal{C}^i, \mathcal{D} \in C^{3,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]})$
- $\Upsilon \in C^{3,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]})$
- $L^i, \mu \in C^{2,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]})$
- ${}^{(\mathfrak{n})}\tau \in C^{2,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]})$

Proof. The results for ${}^{(\mathfrak{n})}\tau$ were already proved as (15.22). For the remaining results, we give the proof only for $\vec{\Psi}$ since the other solution variables can be handled using nearly identical arguments. To proceed, we note that Lemma 5.5, Prop. 9.1, and the bootstrap assumptions imply that $\|\vec{\Psi}\|_{W^{4,\infty}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\text{Boot}}),(-U_1,U_2)})} \leq C$. From this bound and (15.26), we conclude that $\|\vec{\Psi}\|_{C^{3,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]})} \leq C$ as desired. □

15.4. **Properties of ${}^{(\mathfrak{n})}\Phi$ and related maps.** In this section, we provide a detailed analysis of the change of variables map ${}^{(\mathfrak{n})}\Phi(\tau,u,x^2,x^3) = (\mu,\breve{X}\mu,x^2,x^3)$ and its Jacobian matrix ${}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}$. We also reveal some key consequences of the properties of ${}^{(\mathfrak{n})}\Phi$, and we study some related maps.

Lemma 15.7 (Properties of ${}^{(\mathfrak{n})}\Phi$ and related maps)**.**

Estimates for and diffeomorphism properties of ${}^{(\mathfrak{n})}\Phi$. *The map ${}^{(\mathfrak{n})}\Phi(\tau,u,x^2,x^3) = (\mu,\breve{X}\mu,x^2,x^3)$ from Def. 5.1 extends to a $C^{1,1}_{\text{rough}}$ map on $[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$ satisfying:*

$$\left\|{}^{(\mathfrak{n})}\Phi\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C. \tag{15.30}$$

Moreover, the Jacobian matrix ${}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}(\tau,u,x^2,x^3) = \frac{\partial(\mu,\breve{X}\mu,x^2,x^3)}{\partial(\tau,u,x^2,x^3)}$ is invertible at every point $q \in [\tau_0,\tau_{\text{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2$ and satisfies:

$$-C \leq \det {}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J} \leq -\frac{1}{C}, \quad \text{on } [\tau_0,\tau_{\text{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2, \tag{15.31a}$$

$$\max_{q_1,q_2\in[\tau_0,\tau_{\text{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2} \left|{}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}^{-1}(q_1)\,{}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}(q_2) - \mathrm{ID}\right|_{\text{Euc}} \leq \frac{1}{2}, \tag{15.31b}$$

where $|\cdot|_{\text{Euc}}$ is the standard Frobenius norm on matrices (equal to the square root of the sum of the squares of the matrix entries) and ID *denotes the 4×4 identity matrix.*

Furthermore, ${}^{(\mathfrak{n})}\Phi$ is a diffeomorphism from the compact, convex set $[\tau_0,\tau_{\text{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2$ onto its (compact) image ${}^{(\mathfrak{n})}\Phi\left([\tau_0,\tau_{\text{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2\right)$, where ${}^{(\mathfrak{n})}\Phi\left([\tau_0,\tau_{\text{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2\right)$ enjoys the following properties, and we recall that $\mathfrak{m}_{\text{Boot}} = -\tau_{\text{Boot}}$ and $\mathfrak{m}_0 = -\tau_0$:

1. *It contains $[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\{-\mathfrak{n}\}\times\mathbb{T}^2$.*
2. *It contains $(\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0)\times\{-\mathfrak{n}\}\times\mathbb{T}^2$ in its interior.*
3. *We have the following quasi-convexity result: every pair of points $r_1, r_2 \in {}^{(\mathfrak{n})}\Phi\left((\tau_0,\tau_{\text{Boot}})\times(-U_{☢},U_{☢})\times\mathbb{T}^2\right)$ is connected by a C^1 curve in ${}^{(\mathfrak{n})}\Phi\left((\tau_0,\tau_{\text{Boot}})\times(-U_{☢},U_{☢})\times\mathbb{T}^2\right)$ whose length with respect to the standard Euclidean metric on $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$ is $\lesssim \text{dist}_{\text{flat}}(r_1,r_2)$, where $\text{dist}_{\text{flat}}(r_1,r_2)$ is the standard Euclidean distance between r_1 and r_2 in the flat space $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$.*

Properties of $^{(\mathfrak{n})}\Phi^{-1}$. *With* $^{(\mathfrak{n})}\Phi^{-1}$ *denoting the inverse map and* $^{(\mathfrak{n})}\mathscr{T}^{-1}$ *denoting the inverse of the change of variables map* $^{(\mathfrak{n})}\mathscr{T}$ *defined in* (5.2), *the following holds for* $\mathfrak{m} \in [\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0]$:

$$^{(\mathfrak{n})}\mathscr{T}^{-1} \circ {}^{(\mathfrak{n})}\Phi^{-1}\left(\{\mathfrak{m}\} \times \{-\mathfrak{n}\} \times \mathbb{T}^2\right) = \check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} \subset {}^{(\mathfrak{n})}\widetilde{\Sigma}_{-\mathfrak{m}}^{[-\frac{3}{4}U_{☢}, \frac{3}{4}U_{☢}]}, \tag{15.32}$$

and:

$$^{(\mathfrak{n})}\mathscr{T}^{-1} \circ {}^{(\mathfrak{n})}\Phi^{-1}\left([\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0] \times \{-\mathfrak{n}\} \times \mathbb{T}^2\right) = \check{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0, \tau_{\text{Boot}}]} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0, \tau_{\text{Boot}}], [-\frac{3}{4}U_{☢}, \frac{3}{4}U_{☢}]}. \tag{15.33}$$

In addition, $^{(\mathfrak{n})}\Phi^{-1}$ *satisfies the following estimate:*

$$\left\| {}^{(\mathfrak{n})}\Phi^{-1} \right\|_{C^{1,1}\left({}^{(\mathfrak{n})}\Phi\left([\tau_0, \tau_{\text{Boot}}] \times [-U_{☢}, U_{☢}] \times \mathbb{T}^2\right)\right)} \leq C. \tag{15.34}$$

Monotonicity and properties of the map $u \to \breve{X}\mu(\tau, u, x^2, x^3)$ ***and consequences***. *For each fixed* $(\tau, x^2, x^3) \in [\tau_0, \tau_{\text{Boot}}] \times \mathbb{T}^2$, *the map* $u \to \breve{X}\mu(\tau, u, x^2, x^3)$ *is strictly increasing on* $[-U_{☢}, U_{☢}]$.

Furthermore, the map $(\tau, u, x^2, x^3) \to \left(\tau, \breve{X}\mu, x^2, x^3\right)$ *is a* $C^{1,1}_{\text{rough}}$ *diffeomorphism from* $[\tau_0, \tau_{\text{Boot}}] \times [-U_{☢}, U_{☢}] \times \mathbb{T}^2$ *onto its image, which contains* $[\tau_0, \tau_{\text{Boot}}] \times \left[-\mathfrak{n} - \frac{M_2 U_{☢}}{16}, -\mathfrak{n} + \frac{M_2 U_{☢}}{16}\right] \times \mathbb{T}^2$. *In particular, by* (10.8), *the image set contains* $[\tau_0, \tau_{\text{Boot}}] \times [-2\mathfrak{n}_0, \mathfrak{n}_0] \times \mathbb{T}^2$. *Moreover,*

$$\min_{{}^{(\mathfrak{n})}\mathcal{P}_{U_{☢}}^{[\tau_0, \tau_{\text{Boot}}]}} \breve{X}\mu \geq \frac{1}{16} M_2 U_{☢} - \mathfrak{n} \geq \frac{1}{32} M_2 U_{☢}, \tag{15.35a}$$

$$\max_{{}^{(\mathfrak{n})}\mathcal{P}_{-U_{☢}}^{[\tau_0, \tau_{\text{Boot}}]}} \breve{X}\mu \leq -\mathfrak{n} - \frac{1}{16} M_2 U_{☢} \leq -\frac{1}{16} M_2 U_{☢}. \tag{15.35b}$$

Properties of the μ***-adapted tori as graphs over*** $\mathbb{T}^2$ ***in adapted rough coordinates***. *There exists a family of functions* $\{\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}\}_{\mathfrak{m} \in [\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0]}$ *on* $\mathbb{T}^2$ *that, for* $\mathfrak{m} \in (\mathfrak{m}_{\text{Boot}}, \mathfrak{m}_0]$, *are equal to the functions from Sect. 12.2.4, such that for each* $\tau \in [\tau_0, \tau_{\text{Boot}}]$, *we have:*

$$\sup_{\tau \in [\tau_0, \tau_{\text{Boot}}]} \|\mathfrak{U}_{-\tau,-\mathfrak{n}}\|_{C^{1,1}(\mathbb{T}^2)} \leq C, \tag{15.36}$$

$$^{(\mathfrak{n})}\mathscr{T}\left(\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}\right) = \left\{\left(\tau, \mathfrak{U}_{-\tau,-\mathfrak{n}}(x^2, x^3), x^2, x^3\right) \mid (x^2, x^3) \in \mathbb{T}^2\right\}, \tag{15.37}$$

where $^{(\mathfrak{n})}\mathscr{T}$ *is the change of variables map defined in* (5.2). *In particular,*

$$\check{\mathbf{T}}_{-\tau_{\text{Boot}},-\mathfrak{n}} \subset {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-\frac{3}{4}U_{☢}, \frac{3}{4}U_{☢}]}. \tag{15.38}$$

Proof.
Proof of (15.30): Lemmas 5.5, 9.1, 15.5, and 15.6 yield (15.30).

Proof of (15.31a)–(15.31b): We first use Lemma 15.5 (in particular, (15.21)), Lemma 15.6, (**BA** $\frac{\partial}{\partial t}\mu$ neg), and (**BA** μ cnvx) to deduce that on the compact, convex set $\mathscr{D} \stackrel{\text{def}}{=} [\tau_0, \tau_{\text{Boot}}] \times [-U_{☢}, U_{☢}] \times \mathbb{T}^2$, the Jacobian matrix $^{(^{(\mathfrak{n})}\Phi)}\mathbf{J} = \frac{\partial(\mu, \breve{X}\mu, x^2, x^3)}{\partial(\tau, u, x^2, x^3)}$ satisfies $\|^{(^{(\mathfrak{n})}\Phi)}\mathbf{J}\|_{C^{0,1}_{\text{rough}}(\mathscr{D})} \leq C$ and:

$$\begin{aligned}
\det {}^{(^{(\mathfrak{n})}\Phi)}\mathbf{J} &= \det\left(d_{\text{rough}}{}^{(\mathfrak{n})}\mathscr{T}^{-1}\right) \det \frac{\partial(\mu, \breve{X}\mu, x^2, x^3)}{\partial(t, u, x^2, x^3)} \\
&= \det\left(d_{\text{rough}}{}^{(\mathfrak{n})}\mathscr{T}^{-1}\right) \left\{\left(\frac{\partial}{\partial t}\mu\right) \frac{\partial}{\partial u}\breve{X}\mu - \left(\frac{\partial}{\partial u}\mu\right) \frac{\partial}{\partial t}\breve{X}\mu\right\} \\
&< -1/C, \qquad (\text{on } \mathscr{D}).
\end{aligned} \tag{15.39}$$

(15.39) and the bound $\|^{(^{(\mathfrak{n})}\Phi)}\mathbf{J}\|_{C^{0,1}_{\text{rough}}(\mathscr{D})} \leq C$ yield (15.31a) and imply that $^{(^{(\mathfrak{n})}\Phi)}\mathbf{J}$ is invertible on $\mathscr{D}$. Also using the definition of Lipschitz continuity, we deduce the pointwise bound $\left|^{(^{(\mathfrak{n})}\Phi)}\mathbf{J}(\tau, u, x^2, x^3) - {}^{(^{(\mathfrak{n})}\Phi)}\mathbf{J}(\tau_0, u, x^2, x^3)\right|_{\text{Euc}} \leq C|\tau - \tau_0| \leq C\mathfrak{m}_0$. From these bounds and the data-assumption (11.23), we conclude (15.31b) whenever $\mathfrak{m}_0$ is sufficiently small (see Remark B.1).

Proof of the diffeomorphism properties of ${}^{(\mathfrak{n})}\Phi$ on $\mathscr{D}$ and the quasi-convexity of ${}^{(\mathfrak{n})}\Phi\,(\mathrm{int}(\mathscr{D}))$: Using (15.30), the inverse function theorem, and (15.31a)–(15.31b), we deduce that ${}^{(\mathfrak{n})}\Phi$ is a $C^{1,1}_{\mathrm{rough}}$ diffeomorphism from $\mathscr{D}$ onto its image such that ${}^{(\mathfrak{n})}\Phi\,(\mathrm{int}(\mathscr{D}))$ is quasi-convex (where $\mathrm{int}(\mathscr{D})$ is the interior of $\mathscr{D}$). We clarify that (15.30), (15.31a)–(15.31b), and the convexity of $\mathscr{D}$ together guarantee the injectivity of ${}^{(\mathfrak{n})}\Phi$ on $\mathscr{D}$ and the quasi-convexity of ${}^{(\mathfrak{n})}\Phi\,(\mathrm{int}(\mathscr{D}))$.

Next, we use (**BA** $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **– LOCATION**), the fact that ${}^{(\mathfrak{n})}\Phi$ is a diffeomorphism on $\mathscr{D}$, and the facts that $\mu|_{\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}} = -\tau$ and ${}^{(\mathfrak{n})}\breve{W}\mu|_{\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}} = 0$ (i.e., that $\breve{X}\mu|_{\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}} = -\mathfrak{n}$) to deduce that ${}^{(\mathfrak{n})}\Phi(\mathscr{D})$ contains $[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]\times\{-\mathfrak{n}\}\times\mathbb{T}^2$ and that $(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0)\times\{-\mathfrak{n}\}\times\mathbb{T}^2$ is contained in the interior of ${}^{(\mathfrak{n})}\Phi(\mathscr{D})$.

Proof of (15.32)–(15.33): These results follow from the bootstrap assumptions (**BA** $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **– LOCATION**)–(**BA** $\check{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}})}$ **– LOCATION**) and (**BA** $\check{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}})}$ **– FOLIATED**), definitions (4.3c) and (4.7b), definitions (5.2) and (5.3a) (see also (5.5)), the fact that ${}^{(\mathfrak{n})}\mathscr{T}$ is a diffeomorphism on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}$ (see Lemma 15.5), and the fact that ${}^{(\mathfrak{n})}\Phi$ is a diffeomorphism from $[\tau_0,\tau_{\mathrm{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2$ onto a set containing $[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]\times\{-\mathfrak{n}\}\times\mathbb{T}^2$.

Proof of (15.34): The estimate (15.34) follows from (15.30), (15.31a), the fact that ${}^{(\mathfrak{n})}\Phi$ is a $C^{1,1}_{\mathrm{rough}}$ diffeomorphism from $\mathscr{D}$ onto its image, and the quasi-convexity ${}^{(\mathfrak{n})}\Phi(\mathscr{D})$.

Proof of the properties of the map $u \to \breve{X}\mu(\tau,u,x^2,x^3)$ and (15.35a)–(15.35b): We first recall an estimate from the bootstrap assumption (**BA** μ cnvx):

$$\frac{M_2}{4} \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_{☢},U_{☢}]}} \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\mu \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_{☢},U_{☢}]}} \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\mu \leq \frac{4}{M_2}. \tag{15.40}$$

From (15.40), it follows that for each fixed $(\tau,x^2,x^3)\in[\tau_0,\tau_{\mathrm{Boot}}]\times\mathbb{T}^2$, the map $u\to\breve{X}\mu(\tau,u,x^2,x^3)$ is strictly increasing on $[-U_{☢},U_{☢}]$. Hence, the map $(\tau,u,x^2,x^3)\to\left(\tau,\breve{X}\mu,x^2,x^3\right)$ is a $C^{1,1}$ diffeomorphism from $[\tau_0,\tau_{\mathrm{Boot}}]\times[-U_{☢},U_{☢}]\times\mathbb{T}^2$ onto its image. Since (**BA** μ **– TORI STRUCTURE**) and (**BA** $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **– LOCATION**) imply that there is a $u_*\in[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]$ such that the image of (τ,u_*,x^2,x^3) under the map is $(\tau,-\mathfrak{n},x^2,x^3)$, we further deduce from the first inequality in (15.40) and the mean value theorem that the image of $[-U_{☢},U_{☢}]\times\mathbb{T}^2$ under the map contains $[\tau_0,\tau_{\mathrm{Boot}}]\times[-\mathfrak{n}-\frac{M_2U_{☢}}{16},-\mathfrak{n}+\frac{M_2U_{☢}}{16}]\times\mathbb{T}^2$, as is desired.

To prove (15.35a), we fix $\tau\in[\tau_0,\tau_{\mathrm{Boot}}]$, and we let $u_*\in[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]$ be as above. Using the estimate (15.40) and the mean value theorem, we deduce that $\breve{X}\mu(\tau,U_{☢},x^2,x^3)\geq\breve{X}\mu(\tau,u_*,x^2,x^3)+\frac{M_2}{4}(U_{☢}-u_*)\geq-\mathfrak{n}+\frac{1}{16}M_2U_{☢}$. From this estimate and (10.8), we conclude (15.35a). Similarly, to prove (15.35b), we use (15.40) and the mean value theorem to deduce that $\breve{X}\mu(\tau,-U_{☢},x^2,x^3)\leq\breve{X}\mu(\tau,u_*,x^2,x^3)+\frac{M_2}{4}(-U_{☢}-u_*)\leq-\mathfrak{n}-\frac{1}{16}M_2U_{☢}$. From this estimate and our assumption $\mathfrak{n}\geq0$, we conclude (15.35b).

Proof of the properties of $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}$ and the estimates (15.36)–(15.38): These results follow from the form (5.4a) of ${}^{(\mathfrak{n})}\Phi$, the fact that ${}^{(\mathfrak{n})}\Phi^{-1}$ is $C^{1,1}$ on the compact, quasi-convex set ${}^{(\mathfrak{n})}\Phi(\mathscr{D})$, and the bootstrap assumption (**BA** $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **– LOCATION**), which implies that for $\mathfrak{m}\in[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]$ and $(x^2,x^3)\in\mathbb{T}^2$, we have ${}^{(\mathfrak{n})}\Phi^{-1}(\mathfrak{m},-\mathfrak{n},x^2,x^3)\in{}^{(\mathfrak{n})}\widetilde{\Sigma}_{-\mathfrak{m}}^{[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]}$. □

Corollary 15.8 (Quantitative control of the embeddings on the closures of their domains)**.**
Control over ${}^{(\mathfrak{n})}E$. *The map* ${}^{(\mathfrak{n})}E(\mathfrak{m},x^2,x^3)=\left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right)\in\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *from* (12.4) *extends to a* $C^{1,1}$ *embedding from* $[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2$ *onto its image, which is* $\check{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}}]}$. *In addition, there is a $C>1$ such that the extended embedding satisfies:*

$$\|{}^{(\mathfrak{n})}E\|_{C^{1,1}([\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2)}\leq C, \tag{15.41}$$

and:

$$-C<\min_{(\mathfrak{m},x^2,x^3)\in[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2}\frac{\partial}{\partial\mathfrak{m}}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)\leq\max_{(\mathfrak{m},x^2,x^3)\in[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2}\frac{\partial}{\partial\mathfrak{m}}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)<-\frac{1}{C}. \tag{15.42}$$

Furthermore, for $\mathfrak{m}\in[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]$, *we have:*

$$\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}=\left\{\left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right)\mid(x^2,x^3)\in\mathbb{T}^2\right\} \tag{15.43}$$

and:

$$\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]} = \bigcup_{\mathfrak{m}\in[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]} \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}. \tag{15.44}$$

Control over $^{(\mathfrak{n})}H$ ***and the quasi-convexity of*** $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$. *The map* $^{(\mathfrak{n})}H(t,x^2,x^3) = \left(t, {}^{(\mathfrak{n})}h(t,x^2,x^3),x^2,x^3\right)$ *from Lemma 14.1 extends to a* $C^{1,1}$ *embedding from* $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]} \overset{def}{=} \{(t,x^2,x^3)\in\mathbb{R}\times\mathbb{T}^2 \mid \Upsilon_{\mathfrak{m}_0,-\mathfrak{n}}(x^2,x^3)\leq t\leq \Upsilon_{\mathfrak{m}_{\text{Boot}},-\mathfrak{n}}(x^2,x^3)\}$ *onto its image, which is* $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}$.

Moreover, $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ *is a quasi-convex subset of* $\mathbb{R}_t\times\mathbb{T}^2$ *in the following sense: every pair of points* $r_1,r_2\in {}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ *is connected by a* C^1 *curve in* $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ *whose length with respect to the standard Euclidean metric on* $\mathbb{R}_t\times\mathbb{T}^2$ *is* $\lesssim \text{dist}_{\text{flat}}(r_1,r_2)$, *where* $\text{dist}_{\text{flat}}(r_1,r_2)$ *is the standard Euclidean distance between* r_1 *and* r_2 *in the flat space* $\mathbb{R}_t\times\mathbb{T}^2$.

Finally, the extended embedding satisfies the following estimate:

$$\|{}^{(\mathfrak{n})}H\|_{C^{1,1}\left({}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right)} \leq C. \tag{15.45}$$

Proof. $^{(\mathfrak{n})}E$ is the composition of the maps $(\mathfrak{m},x^2,x^3)\to {}^{(\mathfrak{n})}\Phi^{-1}(\mathfrak{m},\mathfrak{n},x^2,x^3)$ and $^{(\mathfrak{n})}\mathscr{T}^{-1}$ (see Def. 5.1) and thus all of the desired conclusions except for those concerning $^{(\mathfrak{n})}H$ and $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ follow from combining Lemmas 15.5 and 15.7.

To obtain the results for $^{(\mathfrak{n})}H$ and $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$, we start by considering the map $^{(\mathfrak{n})}\imath(\mathfrak{m},x^2,x^3) \overset{\text{def}}{=} \left(\Upsilon_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right)$ on the domain $[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2$, which has image $^{(\mathfrak{n})}\imath\left([\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2\right) = {}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$. Using the estimates (15.41) and (15.42) and the inverse function theorem, we solve for its global inverse $^{(\mathfrak{n})}\imath^{-1}$ (which has domain equal to $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$), and we deduce that $\|{}^{(\mathfrak{n})}\imath\|_{C^{1,1}([\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2)}\leq C$ and that $\det d_{(\mathfrak{m},x^2,x^3)}{}^{(\mathfrak{n})}\imath \approx -1$ on $[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2$, where $d_{(\mathfrak{m},x^2,x^3)}{}^{(\mathfrak{n})}\imath$ denotes the differential of $^{(\mathfrak{n})}\imath$ with respect to $(\mathfrak{m},x^2,x^3)$. The quasi-convexity of $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ follows by combining the estimate $\|{}^{(\mathfrak{n})}\imath\|_{C^{1,1}([\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2)}\leq C$ and the monotonicity estimate (15.42) with arguments similar to the ones we used to prove (15.25) and the quasi-convexity of $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$; we omit the details. Next, we note that the inverse function theorem, the estimates $\|{}^{(\mathfrak{n})}\imath\|_{C^{1,1}([\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]\times\mathbb{T}^2)}\leq C$ and $\det d_{(\mathfrak{m},x^2,x^3)}{}^{(\mathfrak{n})}\imath\approx -1$, and the quasi-convexity of $^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}$ imply that $\|{}^{(\mathfrak{n})}\imath^{-1}\|_{C^{1,1}\left({}^{(\mathfrak{n})}\mathscr{H}_{[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]}\right)}\leq C$. Combining this estimate with (15.41), and noting that $^{(\mathfrak{n})}H = {}^{(\mathfrak{n})}E\circ {}^{(\mathfrak{n})}\imath^{-1}$, we conclude the desired bound (15.45). ☐

15.5. **Estimates for** $^{(\breve{\mathscr{M}})}\mathbf{J}$**.** Recall that $\breve{\mathscr{M}}(t,u,x^2,x^3) = (\mu,\breve{X}\mu,x^2,x^3)$ is the map defined in (5.3a). In the next lemma, we prove estimates for its Jacobian matrix in the region $\{|u|\leq U_{☢}\}$. Near the end of the paper, in Prop. 32.5, we will use the estimates in our analysis of the invertibility properties of $\breve{\mathscr{M}}$, which ultimately will help us derive the structure of the singular boundary.

Lemma 15.9 (Estimates for $^{(\breve{\mathscr{M}})}\mathbf{J}$)**.** *There exists a* $C>1$ *such that the Jacobian matrix* $^{(\breve{\mathscr{M}})}\mathbf{J}$ *defined in* (5.3b) *is invertible on* $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}$ *and satisfies the following bounds:*

$$-C\leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \det {}^{(\breve{\mathscr{M}})}\mathbf{J} \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \det {}^{(\breve{\mathscr{M}})}\mathbf{J} \leq -\frac{1}{C}, \tag{15.46}$$

$$\sup_{\substack{p_1\in\breve{\mathbf{T}}_{-\tau_0,0}\\ p_2\in {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}}} \left| {}^{(\breve{\mathscr{M}})}\mathbf{J}(p_1)\, {}^{(\breve{\mathscr{M}})}\mathbf{J}^{-1}(p_2) - \text{ID}\right|_{\text{Euc}} \leq \frac{1}{2}, \tag{15.47}$$

where $|\cdot|_{\text{Euc}}$ *is the standard Frobenius norm on matrices (equal to the square root of the sum of the squares of the matrix entries) and* ID *denotes the* 4×4 *identity matrix.*

Proof. Thanks to the initial data assumption (11.24) and the transversal convexity bootstrap assumption (**BA** μ cnvx), the lemma can be proved using the same arguments we used to prove (15.31b) (see especially the estimates in (15.39)), which in particular relied on the smallness of $\mathfrak{m}_0$. ☐

16. Control of the flow map of ${}^{(\mathfrak{n})}\widetilde{L}$

We continue to work under the assumptions of Sect. 13.2. In this section, we derive various estimates tied to the flow map of the vectorfield ${}^{(\mathfrak{n})}\widetilde{L}$. We then use these results to derive preliminary estimates for solutions f to transport equations of the form ${}^{(\mathfrak{n})}\widetilde{L}f = F$.

16.1. Basic properties of the flow map of ${}^{(\mathfrak{n})}\widetilde{L}$.

Lemma 16.1 (Basic properties of the flow map of ${}^{(\mathfrak{n})}\widetilde{L}$)**.** *Let ${}^{(\mathfrak{n})}\widetilde{L}$ be the null vectorfield defined in* (6.3)*, and let ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ be the τ_0-normalized flow map of ${}^{(\mathfrak{n})}\widetilde{L}$ with respect to the adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$, i.e., the solution to the following initial value problem:*

$$\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) = {}^{(\mathfrak{n})}\widetilde{L}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3), \qquad {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_0,u,x^2,x^3) = (\tau_0,u,x^2,x^3). \tag{16.1}$$

Then for $A = 2,3$ there exist functions ${}^{(\mathfrak{n})}\widetilde{\Lambda}^A : [\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2 \to \mathbb{T}$ such that:

$$ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) = \left(\tau,u,{}^{(\mathfrak{n})}\widetilde{\Lambda}^2(\tau,u,x^2,x^3),{}^{(\mathfrak{n})}\widetilde{\Lambda}^3(\tau,u,x^2,x^3)\right). \tag{16.2}$$

Moreover, ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ is a $C^{1,1}$ diffeomorphism from $[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$ onto $[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$ satisfying:

$$\left\|{}^{(\mathfrak{n})}\widetilde{\Lambda} - \mathrm{I}\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \lesssim \varepsilon^{1/2}, \qquad \left\|{}^{(\mathfrak{n})}\widetilde{\Lambda}^{-1} - \mathrm{I}\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \lesssim \varepsilon^{1/2}, \tag{16.3}$$

where $\mathrm{I}(\tau,u,x^2,x^3) \stackrel{\text{def}}{=} (\tau,u,x^2,x^3)$ is the identity map on adapted rough coordinate space and ${}^{(\mathfrak{n})}\widetilde{\Lambda}^{-1}$ is the inverse function of ${}^{(\mathfrak{n})}\widetilde{\Lambda}$. In particular, for each fixed $(\tau,u)\in[\tau,\tau_{\text{Boot}}]\times[-U_1,U_2]$, the map $(x^2,x^3)\mapsto\left({}^{(\mathfrak{n})}\widetilde{\Lambda}^2(\tau,u,x^2,x^3),{}^{(\mathfrak{n})}\widetilde{\Lambda}^3(\tau,u,x^2,x^3)\right)$ is a $C^{1,1}$ diffeomorphism from ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$ onto ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$.

Proof. (16.2) is a trivial consequence of (16.1) the identities ${}^{(\mathfrak{n})}\widetilde{L}{}^{(\mathfrak{n})}\tau = 1$ and ${}^{(\mathfrak{n})}\widetilde{L}u = 0$.

To prove (16.3), we first note that the functions $\left({}^{(\mathfrak{n})}\widetilde{\Lambda}^2 - x^2, {}^{(\mathfrak{n})}\widetilde{\Lambda}^3 - x^3\right)$ solve the transport system:

$$\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}[{}^{(\mathfrak{n})}\widetilde{\Lambda}^A - x^A](\tau,u,x^2,x^3) = {}^{(\mathfrak{n})}\widetilde{L}^A\left(\tau,u,{}^{(\mathfrak{n})}\widetilde{\Lambda}^2,{}^{(\mathfrak{n})}\widetilde{\Lambda}^3\right), \qquad (A=2,3) \tag{16.4}$$

with vanishing data at rough time τ_0. Next, we use definition (6.3), Lemma 5.8, the bootstrap assumptions, Lemma 15.5, and Lemma 15.6 to deduce that:

$$\left\|\left({}^{(\mathfrak{n})}\widetilde{L}^2,{}^{(\mathfrak{n})}\widetilde{L}^3\right)\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \lesssim \varepsilon^{1/2}. \tag{16.5}$$

Hence, commuting (16.4) up to one time with the adapted rough coordinate partial derivatives, using (16.5), and integrating with respect to rough time, we find that for $\tau\in[\tau_0,\tau_{\text{Boot}}]$, we have:

$$\begin{aligned}\max_{A=2,3}\left\|{}^{(\mathfrak{n})}\widetilde{\Lambda}^A - x^A\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau]\times[-U_1,U_2]\times\mathbb{T}^2)} &\leq C\varepsilon^{1/2} \\ &\quad + C\varepsilon^{1/2}\int_{\tau_0}^{\tau}\max_{A=2,3}\left\|{}^{(\mathfrak{n})}\widetilde{\Lambda}^A - x^A\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau']\times[-U_1,U_2]\times\mathbb{T}^2)}\,d\tau'.\end{aligned} \tag{16.6}$$

From (16.6) and Grönwall's inequality, we find that $\max_{A=2,3}\left\|{}^{(\mathfrak{n})}\widetilde{\Lambda}^A - x^A\right\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C\varepsilon^{1/2}$. From this bound and (16.2), we conclude the first bound stated in (16.3). From this bound and the inverse function theorem, we conclude that ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ is a $C^{1,1}_{\text{rough}}$ diffeomorphism from $[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$ onto itself (${}^{(\mathfrak{n})}\widetilde{\Lambda}$ is a global diffeomorphism since it is close to the identity map). The second bound in (16.3) follows from differentiating the identity ${}^{(\mathfrak{n})}\widetilde{\Lambda}^{-1}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda} = \mathrm{I}$ and using the first bound. □

16.2. Estimate for $\det\widetilde{g\!\!/}$ and Minkowski's integral inequality.

In the next lemma, we control the factor $\det\widetilde{g\!\!/}(\tau,u,x^2,x^3)$ featured in the area form $d\varpi_{\widetilde{g\!\!/}}$ (see (8.8)) on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. We then prove a Minkowski's integral inequality-type estimate.

Lemma 16.2 (Estimate for $\det\widetilde{\mathscr{g}}$ and Minkowski's integral inequality)**.** *Recall that $\widetilde{\mathscr{g}}$ is the first fundamental form of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ (see Def. 6.2) and that ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ is the τ_0-normalized flow map of ${}^{(\mathfrak{n})}\widetilde{L}$ from Lemma 16.1. Then for every $\tau_1,\tau_2\in[\tau_0,\tau_{\text{Boot}}]$ and every $(u,x^2,x^3)\in[-U_1,U_2]\times\mathbb{T}^2$, the following estimates hold, where $\det\widetilde{\mathscr{g}}$ is evaluated relative to the adapted rough coordinates via the formula* (6.11)*:*

$$\det\widetilde{\mathscr{g}}(\tau_2,u,x^2,x^3)=\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\det\widetilde{\mathscr{g}}(\tau_1,u,x^2,x^3)=1+\mathcal{O}_{\blacklozenge}(\mathring{\alpha}), \tag{16.7a}$$

$$\det\widetilde{\mathscr{g}}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_2,u,x^2,x^3)=\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\det\widetilde{\mathscr{g}}(\tau_1,u,x^2,x^3)=1+\mathcal{O}_{\blacklozenge}(\mathring{\alpha}). \tag{16.7b}$$

Moreover, for any scalar function F on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, the function $\varphi(\tau,u,x^2,x^3)$ defined by:

$$\varphi(\tau,u,x^2,x^3)\overset{def}{=}\int_{\tau'=\tau_0}^{\tau}F(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \tag{16.8}$$

satisfies the following estimate, valid for $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$:

$$\|\varphi\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}\leq\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\int_{\tau'=\tau_0}^{\tau}\|F\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)}\mathrm{d}\tau'. \tag{16.9}$$

Proof. First, using the expression of the rough metric components $\widetilde{\mathscr{g}}\left(\frac{\widetilde{\partial}}{\partial x^A},\frac{\widetilde{\partial}}{\partial x^B}\right)$ in (6.11) and the identity $\det\mathscr{g}=\frac{1}{c^2(X^1)^2}$ stated in (3.31c), we compute that:

$$\det\widetilde{\mathscr{g}}=\left(\frac{L^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}\tau}\right)^2\frac{1}{c^2(X^1)^2}. \tag{16.10}$$

Using (16.10), our assumptions on the data from Sect. 11.2.1, and (10.9a), and recalling that $c(\rho=0,s=0)=1$ and $X^1=-1+X^1_{(\text{Small})}$, we deduce that $\det\widetilde{\mathscr{g}}(\tau_0,u,x^2,x^3)=1+\mathcal{O}_{\blacklozenge}(\mathring{\alpha})$. Moreover, using (16.10), Lemma 5.5, (5.13a), Prop. 9.1, Lemma 15.5, and the bootstrap assumptions, we deduce that $\frac{\widetilde{\partial}}{\partial\tau}\det\widetilde{\mathscr{g}}=\mathcal{O}(\varepsilon^{1/2})$ which, in view of the mean value theorem, yields that for any $(\tau,u,x^2,x^3)\in[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$, we have $\det\widetilde{\mathscr{g}}(\tau,u,x^2,x^3)=\det\widetilde{\mathscr{g}}(\tau_0,u,x^2,x^3)+\mathcal{O}(\varepsilon^{1/2})=\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\det\widetilde{\mathscr{g}}(\tau_0,u,x^2,x^3)$. In total, these estimates imply (16.7a).

To prove (16.7b), we first note that definition (6.3) and the estimates cited above yield the following estimate relative to the adapted rough coordinates: $\frac{\widetilde{\partial}}{\partial\tau}\left\{\det\widetilde{\mathscr{g}}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\right\}=[{}^{(\mathfrak{n})}\widetilde{L}\det\widetilde{\mathscr{g}}]\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)=\mathcal{O}(\varepsilon^{1/2})$. Using this estimate, arguing as in the previous paragraph, and using the initial condition ${}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_0,u,x^2,x^3)=(\tau_0,u,x^2,x^3)$, we deduce the following estimate for every $(\tau,u,x^2,x^3)\in[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$: $\det\widetilde{\mathscr{g}}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)=\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\det\widetilde{\mathscr{g}}(\tau_0,u,x^2,x^3)$. Combining this estimate with (16.7a), we conclude (16.7b).

The inequality (16.9) follows from the following estimates relative to the adapted rough coordinates, which rely on (16.7a) and Minkowski's inequality for integrals:

$$\begin{aligned}
&\|\varphi\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}\\
&=\int_{u'=-U_1}^{u}\int_{(x^2,x^3)\in\mathbb{T}^2}\left(\int_{\tau'=\tau_0}^{\tau}F(\tau',u',x^2,x^3)\,\mathrm{d}\tau'\right)^2\sqrt{\det\widetilde{\mathscr{g}}(\tau,u',x^2,x^3)}\,\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}u'\\
&\leq\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\int_{u'=-U_1}^{u}\int_{(x^2,x^3)\in\mathbb{T}^2}\left(\int_{\tau'=\tau_0}^{\tau}F(\tau',u',x^2,x^3)[\det\widetilde{\mathscr{g}}(\tau',u',x^2,x^3)]^{1/4}\,\mathrm{d}\tau'\right)^2\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}u'\\
&\leq\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\left\{\int_{\tau'=\tau_0}^{\tau}\left[\int_{u'=-U_1}^{u}\int_{(x^2,x^3)\in\mathbb{T}^2}\left(F(\tau',u',x^2,x^3)\right)^2\sqrt{\det\widetilde{\mathscr{g}}(\tau',u',x^2,x^3)}\,\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}u'\right]^{1/2}\mathrm{d}\tau'\right\}^2\\
&=\left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\left\{\int_{\tau'=\tau_0}^{\tau}\|F\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)}\mathrm{d}\tau'\right\}^2.
\end{aligned} \tag{16.11}$$

□

16.3. **Estimates for solutions to ${}^{(\mathfrak{n})}\widetilde{L}f = F$.** In the next lemma, we derive the simple transport equation estimates that we use to control solutions to ${}^{(\mathfrak{n})}\widetilde{L}f = F$.

Lemma 16.3 (Pointwise, L^2, and L^∞ estimates tied to the integral curves of ${}^{(\mathfrak{n})}\widetilde{L}$). *Let f be a function of the adapted rough coordinates on $[\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2] \times \mathbb{T}^2$, let ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ be the τ_0-normalized flow map of ${}^{(\mathfrak{n})}\widetilde{L}$ from Lemma 16.1. Then relative to the adapted rough coordinates, the following identity holds for any $\tau_0 \leq \tau_1 \leq \tau_2 < \tau_{\text{Boot}}$ and any $(u, x^2, x^3) \in [-U_1, U_2] \times \mathbb{T}^2$:*

$$f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_2, u, x^2, x^3) = f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_1, u, x^2, x^3) + \int_{\tau'=\tau_1}^{\tau_2} ({}^{(\mathfrak{n})}\widetilde{L}f) \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\, d\tau'. \tag{16.12}$$

Moreover, relative to the adapted rough coordinates, we have the following estimate ***for the critically important factor*** *G_{LL}^0:*

$$\left| G_{LL}^0 \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_2, u, x^2, x^3) - G_{LL}^0 \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_1, u, x^2, x^3) \right| \lesssim \varepsilon^{1/2} |\tau_2 - \tau_1|. \tag{16.13}$$

In addition, we have the following L^∞ and L^2 estimates:

$$\|f\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u}\right)} \leq \|f\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_1,u}\right)} + \mathfrak{m}_0 \sup_{\tau' \in [\tau_1, \tau_2]} \|{}^{(\mathfrak{n})}\widetilde{L}f\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u}\right)}, \tag{16.14a}$$

$$\operatorname*{ess\,sup}_{(x^2,x^3)\in\mathbb{T}^2} \left| f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_2, u, x^2, x^3) - f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_1, u, x^2, x^3) \right| \leq \mathfrak{m}_0 \sup_{\tau' \in [\tau_1, \tau_2]} \|{}^{(\mathfrak{n})}\widetilde{L}f\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u}\right)}, \tag{16.14b}$$

$$\|f\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[-U_1,u]}\right)} \leq \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} \|f\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_1}^{[-U_1,u]}\right)} + \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} \int_{\tau'=\tau_1}^{\tau_2} \|{}^{(\mathfrak{n})}\widetilde{L}f\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} d\tau'. \tag{16.15}$$

Furthermore, if f is any function of the adapted rough coordinates on $[\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2] \times \mathbb{T}^2$, then for every $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, the following estimate holds:

$$\|f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} = \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} \|f\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}. \tag{16.16}$$

Finally, let F be a scalar function of the adapted rough coordinates on $[\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2] \times \mathbb{T}^2$, and let φ be the function of the adapted rough coordinates defined by:

$$\varphi(\tau, u, x^2, x^3) \stackrel{\text{def}}{=} \int_{\tau'=\tau_0}^{\tau} F \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\, d\tau'. \tag{16.17}$$

Then for every $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, the following estimate holds:

$$\|\varphi\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \leq \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} \int_{\tau'=\tau_0}^{\tau} \|F\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} d\tau'. \tag{16.18}$$

Proof. (16.12) follows from (16.1) and the fundamental theorem of calculus.

To prove (16.13), we first note that (6.3), Prop. 9.1, (15.12b), and the bootstrap assumptions imply that $\left|{}^{(\mathfrak{n})}\widetilde{L}G_{LL}^0\right| \lesssim \varepsilon^{1/2}$. Hence, using this bound and applying (16.12) with G_{LL}^0 in the role of f, we conclude (16.13).

We now prove (16.14a). First, from (16.12), we deduce:

$$\left| f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_2, u, x^2, x^3) \right| \leq \left| f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_1, u, x^2, x^3) \right| + \int_{\tau'=\tau_1}^{\tau_2} \left| ({}^{(\mathfrak{n})}\widetilde{L}f) \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3) \right| d\tau'. \tag{16.19}$$

We now take the essential supremum norm of both sides of (16.19) over $(x^2, x^3) \in \mathbb{T}^2$ and use Lemma 16.1 to deduce $\|f\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u}\right)} \leq \|f\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_1,u}\right)} + \int_{\tau'=\tau_1}^{\tau_2} \|{}^{(\mathfrak{n})}\widetilde{L}f\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u}\right)} d\tau'$. From this bound and the fact that $|\tau_2 - \tau_1| \leq |\tau_0| = \mathfrak{m}_0$, we conclude (16.14a). The estimate (16.14b) can be proved via a similar argument, and we omit the details.

To prove (16.15), we first take the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[-U_1,u]}\right)}$ norm of both sides of (16.12). We then use Lemma 16.1 (in particular (16.3)), (16.7b), (16.9), and the standard formula for changing variables in an integral (these estimates allow us in particular to replace all terms $f \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3)$ under the L^2 norms with $f(\tau, u, x^2, x^3)$, up to $1 + \mathcal{O}(\varepsilon^{1/2})$ multiplicative factors) to deduce:

$$\|f\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[-U_1,u]}\right)} \leq \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} \|f(\tau_1, \cdot)\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[-U_1,u]}\right)} + \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} \int_{\tau'=\tau_1}^{\tau_2} \left\|{}^{(\mathfrak{n})}\widetilde{L}f\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} d\tau'. \tag{16.20}$$

We then use (16.7a) to deduce that the first term on RHS (16.20) satisfies:

$$\|f(\tau_1,\cdot)\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[-U_1,u]}\right)} = \left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\|f(\tau_1,\cdot)\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_1}^{[-U_1,u]}\right)}, \tag{16.21}$$

which in total yields (16.15).

The estimate (16.16) follows from (16.3), (16.7b), and the standard formula for changing variables in an integral.

Finally, (16.18) follows from (16.9) and (16.16). □

17. L^∞ estimates and improvement of the auxiliary bootstrap assumptions

We continue to work under the assumptions of Sect. 13.2. In this section, we derive L^∞ estimates for the fluid variables and eikonal function quantities that in particular yield improvements of the auxiliary bootstrap assumptions stated in Sect. 12.3.2.

Proposition 17.1 (L^∞ estimates and improvement of the auxiliary bootstrap assumptions)**.** *Under the parameter-size and initial data assumptions of Sects. 10.2 and 11.2 and the bootstrap assumptions of Sects. 12.2 and 12.3, the following estimates hold for* $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$ *(where we recall that* $\vec{\Psi}$ *and* $\vec{\Psi}_{(\text{Partial})}$ *are defined in Def. 2.8, that* ${}^{(\mathfrak{n})}\widetilde{L}$ *is defined in* (6.3)*, and that in Sect. 8.3, we introduced notation for strings of commutation vectorfields).*

L^∞ estimates for small quantities.

$$\left\| \mathcal{R}_{(+)} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\alpha} + C\varepsilon, \tag{17.1}$$

$$\left\| \vec{\Psi}_{(\mathrm{Partial})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \tag{17.2}$$

$$\begin{aligned} &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq N_{\mathrm{top}}-12}\breve{X}\vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,N_{\mathrm{top}}-11];1}\vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq 4}\breve{X}\breve{X}\vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,6];2}\vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq 2}\breve{X}\breve{X}\breve{X}\vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,5];3}\vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\breve{X}\breve{X}\breve{X}\breve{X}\vec{\Psi} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \end{aligned} \tag{17.3}$$

$$\begin{aligned} &\left\| \mathcal{Z}^{\leq N_{\mathrm{top}}-11;1}(\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| \mathcal{Z}^{\leq 6;2}(\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| \mathcal{Z}^{\leq 5;3}(\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| \breve{X}\breve{X}\breve{X}\breve{X}(\Omega, S) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \end{aligned} \tag{17.4}$$

$$\begin{aligned} &\left\| \mathcal{Z}^{\leq N_{\mathrm{top}}-12;1}(\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| \mathcal{Z}^{\leq 6;2}(\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| \mathcal{Z}^{\leq 5;3}(\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| \breve{X}\breve{X}\breve{X}\breve{X}(\mathcal{C}, \mathcal{D}) \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \end{aligned} \tag{17.5}$$

$$\begin{aligned} &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{[1,N_{\mathrm{top}}-12]}\mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{P}_*^{[1,N_{\mathrm{top}}-12]}\mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}_*^{[1,5];1}\mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_{**}^{[1,5];1}\mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}_*^{[1,4];2}\mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_{**}^{[1,4];2}\mu \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \end{aligned} \tag{17.6}$$

$$\left\| L^1_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\alpha} + C\varepsilon, \tag{17.7}$$

$$\left\| L^A_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \tag{17.8}$$

$$\begin{aligned} &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq N_{\mathrm{top}}-11}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{P}^{[1,N_{\mathrm{top}}-11]}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}^{[1,N_{\mathrm{top}}-12];1}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,N_{\mathrm{top}}-12];1}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}^{[1,5];2}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,5];2}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \\ &\left\| {}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}^{[1,4];3}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \left\| \mathcal{Z}_*^{[1,4];3}L^i_{(\mathrm{Small})} \right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon. \end{aligned} \tag{17.9}$$

L^∞ ***estimates tied to pure transversal derivatives.***

$$\left\|\breve{X}^M\mathcal{R}_{(+)}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\delta} + C\varepsilon, \qquad 1 \leq M \leq 4, \tag{17.10}$$

$$\left\|\breve{X}^M\vec{\Psi}_{(\text{Partial})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \qquad 1 \leq M \leq 4, \tag{17.11}$$

$$\left\|\breve{X}^M\mu\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \left\|\breve{X}^M\left\{c^{-1}\right\}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \frac{3}{2\mathring{\delta}_*}\left\|\breve{X}^M\left\{c^{-1}(c^{-1}c_{;\rho}+1)\breve{X}\mathcal{R}_{(+)}\right\}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + C\varepsilon, \quad 0 \leq M \leq 3, \tag{17.12}$$

$$\left\|{}^{(\mathfrak{n})}\widetilde{L}\breve{X}^M\mu\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \frac{1}{\delta_*}\left\|\breve{X}^M\left\{c^{-1}(c^{-1}c_{;\rho}+1)\breve{X}\mathcal{R}_{(+)}\right\}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + C\varepsilon, \qquad 0 \leq M \leq 3, \tag{17.13}$$

$$\left\|\breve{X}^M L^1_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \mathring{\delta} + C\varepsilon, \qquad 1 \leq M \leq 3, \tag{17.14}$$

$$\left\|\breve{X}^M L^A_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon, \qquad 1 \leq M \leq 3. \tag{17.15}$$

Proof. We refer to Sect. 13.2 for various results that we will silently use throughout the analysis. We sometimes silently use the assumptions on the initial data stated in Sect. 11.2, the parameter-relations (10.9a), and the estimate $\frac{1}{L^{(\mathfrak{n})}\tau} \approx 1$ implied by (**BA** $L^{(\mathfrak{n})}\tau$). We also recall that ${}^{(\mathfrak{n})}\widetilde{L}$ is the null vectorfield defined in (6.3). Finally, we remark that the order in which we prove the estimates is important.

Proof of (17.1)–(17.2)**:** We use (16.14a) with $\tau_1 \overset{\text{def}}{=} \tau_0$ and $\tau_2 \overset{\text{def}}{=} \tau$, the bootstrap assumptions (**BA** L^∞ **FUND**), the data-estimate (11.8a), and (10.9a) to conclude that:

$$\|\mathcal{R}_{(+)}\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \|\mathcal{R}_{(+)}\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \mathfrak{m}_0 \sup_{\tau'\in[\tau_0,\tau]}\|{}^{(\mathfrak{n})}\widetilde{L}\mathcal{R}_{(+)}\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u})} \leq \mathring{\alpha} + C\varepsilon \tag{17.16}$$

as desired.

The estimate (17.2) follows from a similar argument based on the data-estimate $\|\vec{\Psi}_{(\text{Partial})}\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} \leq \mathring{\epsilon}$ stated in (11.8c).

Proof of (17.7)–(17.8)**:** We first use Prop. 9.1 to write the transport equation (3.45) for L^i schematically as:

$$LL^i_{(\text{Small})} = \mathrm{f}(\gamma)\cdot\mathcal{P}\vec{\Psi}. \tag{17.17}$$

From (17.17) and the bootstrap assumptions, we find that $\|{}^{(\mathfrak{n})}\widetilde{L}L^i\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} = \mathcal{O}(\varepsilon)$. From this bound, the initial data assumptions, and the same arguments we used to prove (17.1), we conclude (17.7)–(17.8).

Proof of (17.9) **for** $\left\|{}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq N_{\text{top}}-11}L^i_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ **and** $\left\|\mathcal{P}^{[1,N_{\text{top}}-11]}L^i_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$: We commute (17.17) with $\mathcal{P}^N$ for $N \leq N_{\text{top}} - 11$ to obtain:

$$L\mathcal{P}^N L^i = [L,\mathcal{P}^N]L^i + \mathcal{P}^N\left\{\mathrm{f}(\gamma)\cdot\mathcal{P}\vec{\Psi}\right\}. \tag{17.18}$$

From (17.18), the commutator estimate (13.7a), and the bootstrap assumptions, we find that $\left\|L\mathcal{P}^{\leq N_{\text{top}}-11}L^i_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$ and that the same bound holds with ${}^{(\mathfrak{n})}\widetilde{L}$ in place of L. The remainder of the proof relies on the same arguments used above.

Proof of (17.3) **for** $\left\|{}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq N_{\text{top}}-12}\breve{X}\vec{\Psi}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ **and** $\left\|\mathcal{Z}_*^{[1,N_{\text{top}}-11];1}\vec{\Psi}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$: We first commute $\mathcal{P}^{\leq N_{\text{top}}-12}$ through the outermost L operator on LHS (9.18) and use the commutator estimate (13.7a), the bootstrap assumptions, and the estimate (17.9) for $\left\|\mathcal{P}^{[1,N_{\text{top}}-11]}L^i_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ to deduce that $\left\|L\mathcal{P}^{\leq N_{\text{top}}-12}\breve{X}\Psi\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$ and that the same bound holds with ${}^{(\mathfrak{n})}\widetilde{L}$ in place of L. In particular, this yields (17.3) for the first term on the LHS. Also using the same arguments we used to prove (17.1) (including our assumptions on the data), we find that $\left\|\mathcal{P}^{[1,N_{\text{top}}-12]}\breve{X}\Psi\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$. From this bound, the commutator estimate (13.7b), and the bootstrap assumptions, we deduce that $\left\|\mathcal{Z}_*^{[1,N_{\text{top}}-11];1}\vec{\Psi}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \left\|\mathcal{P}^{[1,N_{\text{top}}-12]}\breve{X}\Psi\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} + \left\|\mathcal{P}^{[1,N_{\text{top}}-12]}\Psi\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$, which yields (17.3) for the second term on the LHS.

Proof of (17.10)–(17.11) **for** $M = 1$: Fix $\Psi \in \vec{\Psi} = (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$. From the bounds proved in the previous paragraph and the bootstrap assumptions, we find that $\left\|{}^{(\mathfrak{n})}\widetilde{L}\breve{X}\Psi\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$. From this bound and the same arguments we used to prove (17.1), we conclude (17.10) and (17.11) for $M = 1$.

Proof of (17.6) **for** $\left\|{}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{[1,N_{\text{top}}-12]}\mu\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ **and** $\left\|\mathcal{P}_*^{[1,N_{\text{top}}-12]}\mu\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$: We begin by using Lemmas 3.22 and 9.1 to write the transport equation (3.44) for μ as:

$$L\mu = -\frac{1}{2}c^{-1}(c^{-1}c_{;\rho} + 1)\breve{X}\mathcal{R}_{(+)} + \mathrm{f}(\gamma)\cdot\breve{X}\vec{\Psi}_{(\text{Partial})} + \mathrm{f}(\underline{\gamma})\cdot\mathcal{P}\vec{\Psi} + \mathrm{f}(\underline{\gamma})\cdot S, \tag{17.19}$$

where the first product on RHS (17.19) is written exactly (for use later on) and the remaining ones schematically. Commuting (17.19) with $\mathcal{P}^N$ for $1 \leq N \leq N_{\text{top}} - 12$ and using (13.7a), the bootstrap assumptions, and the already proven bound $\left\|\mathcal{P}^{[1,N_{\text{top}}-12]}\breve{X}\Psi\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$, we find that $\left\|L\mathcal{P}^{[1,N_{\text{top}}-12]}\mu\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$ and that the same bound holds with ${}^{(\mathfrak{n})}\widetilde{L}$ in place of L. In particular, this yields (17.6) for the first term on the LHS. In the case that $\mathcal{P}^N = \mathcal{P}_*^N$, we can combine this bound with the initial data assumptions and the same arguments we used to prove (17.1) in order to conclude $\left\|\mathcal{P}_*^{[1,N_{\text{top}}-12]}\mu\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$, which yields (17.6) for the second term on the LHS.

Proof of (17.12)–(17.13) **in the case** $M = 0$: The bootstrap assumptions and the already proven bound $\left\|\breve{X}\vec{\Psi}_{(\text{Partial})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$ imply that the last three products on RHS (17.19), namely $\mathrm{f}(\gamma)\cdot\breve{X}\vec{\Psi}_{(\text{Partial})}$, $\mathrm{f}(\underline{\gamma})\cdot\mathcal{P}\vec{\Psi}$, and $\mathrm{f}(\underline{\gamma})\cdot S$, are bounded in the norm $\|\cdot\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ by $\lesssim \varepsilon$. Moreover, the bootstrap assumptions and the already proven bound $\left\|{}^{(\mathfrak{n})}\widetilde{L}\breve{X}\Psi\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$ imply that $\left\|{}^{(\mathfrak{n})}\widetilde{L}\left\{\frac{1}{2}c^{-1}(c^{-1}c_{;\rho} + 1)\breve{X}\mathcal{R}_{(+)}\right\}\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$. From this bound and (16.14a), we deduce that the first product on RHS (17.19) satisfies the following bound: $\left\|\frac{1}{2}c^{-1}(c^{-1}c_{;\rho} + 1)\breve{X}\mathcal{R}_{(+)}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \left\|\frac{1}{2}c^{-1}(c^{-1}c_{;\rho} + 1)\breve{X}\mathcal{R}_{(+)}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \varepsilon$. Using all these estimates to control the terms on RHS (17.19), we find that $\|L\mu\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \left\|\frac{1}{2}c^{-1}(c^{-1}c_{;\rho} + 1)\breve{X}\mathcal{R}_{(+)}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \varepsilon$. From this bound and (15.12b), we conclude (17.13) in the case $M = 0$. From this bound and the same arguments we used to prove (17.1), we find that $\|\mu\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq \|\mu\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \mathfrak{m}_0\frac{1}{\delta_*}\left\|\frac{1}{2}c^{-1}(c^{-1}c_{;\rho} + 1)\breve{X}\mathcal{R}_{(+)}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + C\varepsilon$, which, in view of our data-assumption for $\|\mu\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})}$ (and assuming $\mathfrak{m}_0 \leq 1$), yields (17.12) in the case $M = 0$.

Proof of (17.9) **for** $\left\|{}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}^{[1,N_{\text{top}}-12];1}L^i_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ **and** $\left\|\mathcal{Z}_*^{[1,N_{\text{top}}-12];1}L^i_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$: We commute (17.17) with operators of the form $\mathcal{Z}^{N;1}$ for $1 \leq N \leq N_{\text{top}} - 12$ to obtain:

$$L\mathcal{Z}^{N;1}L^i = [L, \mathcal{Z}^{N;1}]L^i + \mathcal{Z}^{N;1}\left\{\mathrm{f}(\gamma)\cdot\mathcal{P}\vec{\Psi}\right\}. \tag{17.20}$$

From (17.20), the commutator estimate (13.7b), the bootstrap assumptions, and the already proven bounds $\left\|\mathcal{P}^{[1,N_{\text{top}}-11]}L^i_{(\text{Small})}\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$ and $\left\|\mathcal{Z}_*^{[1,N_{\text{top}}-11];1}\vec{\Psi}\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \varepsilon$, we find that $\left\|L\mathcal{Z}_*^{[1,N_{\text{top}}-12];1}L^i_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$ and that the same bound holds with ${}^{(\mathfrak{n})}\widetilde{L}$ in place of L. In particular, this yields (17.9) for $\left\|{}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}^{[1,N_{\text{top}}-12];1}L^i_{(\text{Small})}\right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}$. The remainder of the proof relies on the same arguments used above.

Proof of (17.14)–(17.15) **in the case** $M = 1$: The bounds proved in the previous paragraph and the bootstrap assumptions imply that $\left\|L\breve{X}L^1_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$ and $\left\|L\breve{X}L^A_{(\text{Small})}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$, and that the same bound holds with ${}^{(\mathfrak{n})}\widetilde{L}$ in place of L. Also using the same arguments we used to prove (17.1) (including our assumptions on the data), we conclude (17.14)–(17.15) in the case $M = 1$.

Proof of (17.4) **for** $\left\|\mathcal{Z}^{\leq N_{\text{top}}-11;1}(\Omega,S)\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ **and** (17.5) **for** $\left\|\mathcal{Z}^{\leq N_{\text{top}}-12;1}(\mathcal{C},\mathcal{D})\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$: We apply $\mathcal{P}^{\leq N_{\text{top}}-12}$ to (9.6a)–(9.6b) and use the bootstrap assumptions (including (**BA** L^∞ **FUND**)) to deduce that $\left\|\mathcal{P}^{\leq N_{\text{top}}-12}\breve{X}(\Omega,S)\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \varepsilon$. From this bound, the commutator estimate (13.7b), and the bootstrap assumptions, we conclude (17.4) for the first term on the LHS.

The estimate (17.5) for $\left\|\mathcal{Z}^{\leq N_{\text{top}}-12;1}(\mathcal{C},\mathcal{D})\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$ follows from a similar argument based on applying $\mathcal{P}^{\leq N_{\text{top}}-13}$ to (9.9).

Outline of the remainder of the proof: The remaining estimates in the proposition can be derived by commuting the equations with one additional $\breve{X}$ derivative followed by elements of $\mathscr{P}$, using the above arguments and the estimates already proved, and then repeating the process, adding one additional $\breve{X}$ derivative each time. The estimates need to be derived in the same order as above. The commutator terms can be handled with the help of Prop. 13.5. The bootstrap assumptions guarantee that all terms that need to be controlled have sufficient $L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})$-regularity. We omit the tedious, but straightforward details, noting only that the estimates can be proved in the following order: ${}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq 4}\breve{X}\breve{X}\vec{\Psi}$, $\mathcal{Z}_*^{[1,6];2}\vec{\Psi}$, $\breve{X}\breve{X}\mathcal{R}_{(+)}$, $\breve{X}\breve{X}\vec{\Psi}_{(\text{Partial})}$, ${}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}_*^{[1,5];1}\mu$, $\mathcal{Z}_{**}^{[1,5];1}\mu$, ${}^{(\mathfrak{n})}\widetilde{L}\breve{X}\mu$, $\breve{X}\mu$, ${}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}^{[1,5];2}L^i_{(\text{Small})}$, $\mathcal{Z}_*^{[1,5];2}L^i_{(\text{Small})}$, $\breve{X}\breve{X}L^1_{(\text{Small})}$, $\breve{X}\breve{X}L^A_{(\text{Small})}$, $\mathcal{Z}^{\leq 6;2}(\Omega,S)$, $\mathcal{Z}^{\leq 6;2}(\mathcal{C},\mathcal{D})$, ${}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^{\leq 2}\breve{X}\breve{X}\breve{X}\vec{\Psi}$, $\mathcal{Z}_*^{[1,5];3}\vec{\Psi}$, $\breve{X}\breve{X}\breve{X}\mathcal{R}_{(+)}$, $\breve{X}\breve{X}\breve{X}\vec{\Psi}_{(\text{Partial})}$, ${}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}_*^{[1,4];2}\mu$, $\mathcal{Z}_{**}^{[1,4];2}\mu$, ${}^{(\mathfrak{n})}\widetilde{L}\breve{X}\breve{X}\mu$, $\breve{X}\breve{X}\mu$, ${}^{(\mathfrak{n})}\widetilde{L}\mathcal{Z}^{[1,4];3}L^i_{(\text{Small})}$, $\mathcal{Z}_*^{[1,4];3}L^i_{(\text{Small})}$, $\breve{X}\breve{X}\breve{X}L^1_{(\text{Small})}$, $\breve{X}\breve{X}\breve{X}L^A_{(\text{Small})}$, $\mathcal{Z}^{\leq 5;3}(\Omega,S)$, $\mathcal{Z}^{\leq 5;3}(\mathcal{C},\mathcal{D})$, ${}^{(\mathfrak{n})}\widetilde{L}\breve{X}\breve{X}\breve{X}\breve{X}\vec{\Psi}$, $\breve{X}\breve{X}\breve{X}\breve{X}\mathcal{R}_{(+)}$, $\breve{X}\breve{X}\breve{X}\breve{X}\vec{\Psi}_{(\text{Partial})}$, ${}^{(\mathfrak{n})}\widetilde{L}\breve{X}\breve{X}\breve{X}\mu$, $\breve{X}\breve{X}\breve{X}\mu$, $\breve{X}\breve{X}\breve{X}\breve{X}(\Omega,S)$, $\breve{X}\breve{X}\breve{X}\breve{X}(\mathcal{C},\mathcal{D})$.

□

The following corollary is an immediate consequence of the fact that we have improved the auxiliary bootstrap assumptions.

Corollary 17.2 ($\mathring{\alpha}^{1/2}$ and $\varepsilon^{1/2}$ can be replaced by $C\mathring{\alpha}$ and $C\varepsilon$). *All prior inequalities whose RHS feature an explicit factor of $\mathring{\alpha}^{1/2}, \varepsilon^{1/2}$ remain true with $\mathring{\alpha}^{1/2}$, $\varepsilon^{1/2}$ respectively replaced by $C\mathring{\alpha}$, $C\varepsilon$.*

18. Sharp control of μ, properties of Υ, and pointwise estimates tied to the rough acoustic geometry

We continue to work under the assumptions of Sect. 13.2. In this section, we derive sharp control of μ and its derivatives, as well as strict improvements of the bootstrap assumptions from Sects. 12.2.1–12.2.2. Because our L^2-type energies feature μ weights, these sharp estimates will play a fundamental role in our proof of the energy estimates. Next, in Prop. 18.4, we derive homeomorphism and diffeomorphism properties of the change of variables map Υ from geometric coordinate to Cartesian coordinates, which are crucial for understanding the structure of the singular boundary. The results of Prop. 18.4 also yield strict improvements of the bootstrap assumptions for Υ stated in Sect. 12.2.3. In Sect. 18.3, we derive improvements of the bootstrap assumptions for the size of (t,x^1) on the rough hypersurfaces. Finally, in Lemma 18.6, we derive pointwise estimates for various geometric quantities that are tied to the rough acoustic geometry. We will use these pointwise estimates throughout the paper, notably in Sect. 23, when we derive pointwise estimates for the error terms in the elliptic-hyperbolic identities that we use to control the top-order derivatives of Ω and S.

18.1. Sharp control of μ and its derivatives.

Proposition 18.1 (Sharp control of μ and its derivatives). *The following estimates hold.*

***Minima of** μ **occur precisely along** $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$*. *For each fixed* $\tau \in [\tau_0,\tau_{\text{Boot}}] = [-\mathfrak{m}_0,-\mathfrak{m}_{\text{Boot}}]$, *we have:*

$$\min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}} \mu = -\tau, \tag{18.1}$$

and the minimum value of $-\tau$ *in* (18.1) *is achieved by* μ *precisely on the* μ*-adapted torus* $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ *defined in* (4.3c). *In particular,* $\mathfrak{m}_{\text{Boot}} = \inf_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}} \mu$, *and the infimum is not achieved.*

μ ***is large when** $|u| \geq U_{☢}$*. *The following lower bound holds, where* $\mathfrak{m}_1 > 0$ *is the constant appearing in* (11.20)*:*

$$\min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]} \setminus {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \mu \geq \frac{\mathfrak{m}_1}{2}. \tag{18.2}$$

Location of $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}$ ***and*** $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$. *With* $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}$ *and* $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ *denoting the sets defined in* (4.3b)–(4.3c)*, we have:*

$$\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}, \tag{18.3a}$$

$$\text{for each } \tau \in [\tau_0,\tau_{\text{Boot}}], \qquad \breve{\mathbf{T}}_{-\tau,-\mathfrak{n}} \subset {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}. \tag{18.3b}$$

Moreover, with ${}^{(\mathfrak{n})}\Phi^{-1}$ *denoting the inverse function of the function* ${}^{(\mathfrak{n})}\Phi$ *defined in* (5.4a)*, we have:*

$$\text{For each } \mathfrak{m} \in [\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0], \qquad {}^{(\mathfrak{n})}\Phi^{-1}\left(\{\mathfrak{m}\}\times\{-\mathfrak{n}\}\times\mathbb{T}^2\right) \subset \{-\mathfrak{m}\}\times\left[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}\right]\times\mathbb{T}^2. \tag{18.4}$$

Transversal convexity of μ ***and its consequences***. *The following estimates hold, where* ${}^{(\mathfrak{n})}\breve{W}$ *is the vectorfield from Def. 4.1:*

$$\begin{aligned}
\frac{M_2}{2} &\leq \inf_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \Big\{ {}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\breve{W}\mu,\ {}^{(\mathfrak{n})}\breve{W}\breve{X}\mu,\ \breve{X}\breve{X}\mu,\ \breve{X}\breve{X}\mu - \frac{(\breve{X}\mu)L\breve{X}\mu}{L\mu}, \\
&\qquad\qquad \breve{X}\breve{X}\mu + \frac{\mathfrak{n}L\breve{X}\mu}{L\mu},\ \frac{\partial}{\partial u}\breve{X}\mu,\ \frac{\partial}{\partial u}\breve{X}\mu - \frac{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu}{\frac{\partial}{\partial t}\mu},\ \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\mu \Big\} \\
&\leq \sup_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \Big\{ {}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\breve{W}\mu,\ {}^{(\mathfrak{n})}\breve{W}\breve{X}\mu,\ \breve{X}\breve{X}\mu,\ \breve{X}\breve{X}\mu - \frac{(\breve{X}\mu)L\breve{X}\mu}{L\mu}, \\
&\qquad\qquad \breve{X}\breve{X}\mu + \frac{\mathfrak{n}L\breve{X}\mu}{L\mu},\ \frac{\partial}{\partial u}\breve{X}\mu,\ \frac{\partial}{\partial u}\breve{X}\mu - \frac{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu}{\frac{\partial}{\partial t}\mu},\ \frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\mu \Big\} \\
&\leq \frac{2}{M_2}.
\end{aligned} \tag{18.5}$$

$$\min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]} \setminus \mathcal{M}^{(\mathfrak{n})}_{[\tau_0,\tau_{\text{Boot}}],[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}} |\breve{X}\mu + \mathfrak{n}| \geq \frac{M_2 U_{☢}}{8}. \tag{18.6}$$

Moreover, the following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}$*, where* ${}^{(\mathfrak{n})}\breve{R}$ *is the vectorfield defined in* (6.6)*:*

$$|{}^{(\mathfrak{n})}\breve{W}\mu|,\ |{}^{(\mathfrak{n})}\breve{R}\mu| \leq C\sqrt{\mu}. \tag{18.7}$$

Rough control of null and almost null derivatives of μ ***in the interesting region***. *The following estimates hold, where* ${}^{(\mathfrak{n})}\widetilde{L}$ *is the vectorfield defined in* (6.3)*:*

$$-\frac{9}{8}\mathring{\delta}_* \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} L\mu \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} L\mu \leq -\frac{7}{8}\mathring{\delta}_*, \tag{18.8a}$$

$$-\frac{9}{8}\mathring{\delta}_* \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \frac{\partial}{\partial t}\mu \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \frac{\partial}{\partial t}\mu \leq -\frac{7}{8}\mathring{\delta}_*, \tag{18.8b}$$

$$-\frac{3}{2} \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} {}^{(\mathfrak{n})}\widetilde{L}\mu \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} {}^{(\mathfrak{n})}\widetilde{L}\mu \leq -\frac{2}{3}. \tag{18.8c}$$

Sharp control of $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau$ ***and*** $L^{(\mathfrak{n})}\tau$. *The following estimates hold:*

$$\frac{7}{9}\mathring{\delta}_* \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}} \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}} \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau \leq \frac{9}{7}\mathring{\delta}_*, \tag{18.9a}$$

$$\frac{7}{9}\mathring{\delta}_* \leq \min_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}} L^{(\mathfrak{n})}\tau \leq \max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}} L^{(\mathfrak{n})}\tau \leq \frac{9}{7}\mathring{\delta}_*. \tag{18.9b}$$

Estimates for μ ***along the flow map of*** $^{(\mathfrak{n})}\widetilde{\Lambda}$. *Let* $^{(\mathfrak{n})}\widetilde{\Lambda}$ *denote the* τ_0*-normalized flow map of* $^{(\mathfrak{n})}\widetilde{L}$ *from Lemma 16.1. Then the following estimates hold:*

$$\sup_{\substack{|u|\leq U_{☢} \\ \tau_0\leq\tau\leq\tau+\Delta\tau\leq\tau_{\text{Boot}} \\ (x^2,x^3)\in\mathbb{T}^2}} \frac{\mu\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau+\Delta,u,x^2,x^3)}{\mu\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)} \leq 1, \tag{18.10a}$$

$$\sup_{\substack{u\in[-U_1,U_2]\backslash[-U_{☢},U_{☢}] \\ \tau_0\leq\tau\leq\tau+\Delta\tau\leq\tau_{\text{Boot}} \\ (x^2,x^3)\in\mathbb{T}^2}} \frac{\mu\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau+\Delta,u,x^2,x^3)}{\mu\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)} \leq C. \tag{18.10b}$$

A lower bound tied to the blowup in the interesting region. *The following lower bounds holds:*

$$\min_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}} \mu|X\mathcal{R}_{(+)}| \geq \frac{\mathring{\delta}_*}{|\bar{c}_{;\rho}+1|}, \tag{18.11}$$

where $\bar{c}_{;\rho} \overset{def}{=} c_{;\rho}(\rho=0,s=0)$ *is* $c_{;\rho}$ *evaluated at the trivial solution,* $\bar{c}_{;\rho}+1$ *is a* <u>*non-zero*</u> *constant by assumption, and the vectorfield* X *has Euclidean length satisfying* $\sqrt{\sum_{a=1}^3 (X^a)^2} = 1+\mathcal{O}(\mathring{\alpha})$.

Especially sharp control in a small neighborhood. *Recall that* $^{(\mathfrak{n})}\iota_{\Delta u}$ *is the flow map of* $^{(\mathfrak{n})}\breve{W}$ *(see Lemma 14.2). There exists a constant* ΔU *with* $0<\Delta U<\frac{1}{2}U_{☢}$*, a neighborhood (in the geometric coordinate topology of* $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$*)* $^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}$ *of* $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}$ *of the form:*

$$^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]} = {}^{(\mathfrak{n})}\iota_{(-\Delta U,\Delta U)}\left(\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}\right) \overset{def}{=} \bigcup_{\Delta u'\in(-\Delta U,\Delta U)} {}^{(\mathfrak{n})}\iota_{\Delta u'}\left(\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}\right) \tag{18.12}$$

such that:

$$^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}, \tag{18.13}$$

and a constant $\mu_2>0$ *defined by:*

$$\mu_2 \overset{def}{=} \min\left\{\frac{M_2}{4}(\Delta U)^2, \frac{\mathfrak{m}_1}{2}\right\} \tag{18.14}$$

(where $\mathfrak{m}_1$ *is as in* (11.20) *and* M_2 *is as in* (18.5)*) such that the following estimates hold:*

$$-1.01 \leq \min_{^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}} {}^{(\mathfrak{n})}\widetilde{L}\mu \leq \max_{^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}} {}^{(\mathfrak{n})}\widetilde{L}\mu \leq -.99, \tag{18.15}$$

$$\mu_2 \leq \min_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}\backslash{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}} \mu. \tag{18.16}$$

Proof.

Remark 18.2 (Silent use of Lemma 15.6)**.** Throughout this proof, we sometimes silently use the continuous extension properties shown in Lemma 15.6. In particular, these properties allow us to extend various results that were proved prior to the lemma on the half-open rough time interval $[\tau_0,\tau_{\text{Boot}})$ to the closed rough time interval $[\tau_0,\tau_{\text{Boot}}]$.

Proof of (18.5): To prove (18.5) for $^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\breve{W}\mu$, we use first use (4.2), (6.3), Lemmas 5.5 and 15.6, and the estimate $\frac{1}{L^{(\mathfrak{n})}\tau}\approx 1$ (see (15.12b)) to deduce that for $(\tau,u)\in[\tau_0,\tau_{\text{Boot}}]\times[-U_{☢},U_{☢}]$, we have $\|{}^{(\mathfrak{n})}\widetilde{L}{}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\breve{W}\mu\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \leq C$. From this bound, (16.14b), and the data-assumption (11.18), we deduce that in the region $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}$, we have $M_2 - C\mathfrak{m}_0 \leq {}^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\breve{W}\mu \leq \frac{1}{M_2}+C\mathfrak{m}_0$, which, for $\mathfrak{m}_0$ sufficiently small, implies (18.5) for $^{(\mathfrak{n})}\breve{W}{}^{(\mathfrak{n})}\breve{W}\mu$ (recall that, as we highlighted in Sect. 10.3, the constants C can be chosen to be independent of $\mathfrak{m}_0$). The remaining estimates in (18.5) follow from a nearly identical argument, where we use the identity (5.15) when deriving the estimates involving $\frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\mu$.

Proof of (18.2)**,** (18.6)**,** (18.8a)**, and** (18.8b): To prove (18.6), we first argue as in the proof of (18.5) to deduce that $|{}^{(\mathfrak{n})}\widetilde{L}(\breve{X}\mu+\mathfrak{n})|\leq C$. From this bound, the estimate (16.14b), and the data-assumption (11.19b), we find that in the spacetime region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]} \backslash \mathcal{M}^{(\mathfrak{n})}_{[\tau_0,\tau_{\text{Boot}}],[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}$, we have $|\breve{X}\mu + \mathfrak{n}| \geq \frac{M_2 U_{☢}}{4} - C\mathfrak{m}_0$. Assuming $\mathfrak{m}_0$ is sufficiently small, we conclude (18.6).

The estimates (18.2) and (18.8a) follow from similar arguments based on the data-assumptions (11.20) and (11.22) and the estimates $|{}^{(\mathfrak{n})}\widetilde{L}\mu| \leq C$ and $|{}^{(\mathfrak{n})}\widetilde{L}L\mu| \leq C\varepsilon$, which follow from (15.12b) and the estimates of Prop. 17.1. We also conclude (18.8b) by combining these arguments with the relation $\frac{\partial}{\partial t}\mu = L\mu + \mathcal{O}(\varepsilon)$, which follows from the identity $\frac{\partial}{\partial t}\mu = L\mu - L^A \frac{\partial}{\partial x^A}\mu$, Lemma 5.5, Prop. 9.1, and the estimates of Prop. 17.1.

Proof of (18.10b): (18.10b) is a simple consequence of (18.2), (16.2), and the bound $\max_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}} \mu \lesssim 1$ implied by the third item in Lemma 15.6.

Proof of (18.9a)–(18.9b): These estimates follow from the proof of (15.12a)–(15.12b), except we now use the bound (18.8a) in place of the bootstrap assumption (**BA** $L\mu$ neg) used in the proof of (15.12a)–(15.12b).

Proof of (18.8c) **and** (18.10a): (18.8c) follows from definition (6.3) and the estimates (18.8a) and (18.9b). (18.10a) then follows from (18.8c), which implies that μ decreases along the integral curves of ${}^{(\mathfrak{n})}\widetilde{L}$ when $|u| \leq U_{☢}$.

Proof of (18.11): We use (3.44), (3.46), Prop. 9.1, the estimates of Prop. 17.1, and (18.8a) to deduce that in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}$, we have the estimate $\frac{1}{2}c^{-1}(c^{-1}c_{;\rho}+1)|\breve{X}\mathcal{R}_{(+)}| \geq \frac{7}{8}\mathring{\delta}_* + \mathcal{O}(\varepsilon) \geq \frac{3}{4}\mathring{\delta}_*$. Also Taylor expanding $c^{-1}(c^{-1}c_{;\rho}+1)$ around the background solution $\vec{\Psi} = 0$ and using (2.4) and (2.5), we deduce the estimate $c^{-1}(c^{-1}c_{;\rho}+1)|\breve{X}\mathcal{R}_{(+)}| = \{1 + \mathcal{O}(\mathring{\alpha})\}|\bar{c}_{;\rho} + 1||\breve{X}\mathcal{R}_{(+)}|$. Combining these two estimates, we conclude (18.11).

The fact that $\sqrt{\sum_{a=1}^{3}(X^a)^2} = 1 + \mathcal{O}(\mathring{\alpha})$ follows from (3.13), (9.3e), and the estimates of Prop. 17.1.

Proof of (18.3a)–(18.3b) **and** (18.4): Since $|\breve{X}\mu + \mathfrak{n}| = 0$ along $\breve{\mathbb{X}}^{(-\mathfrak{n})}$, (18.3a) follows directly from (18.6).

(18.3b) then follows from (18.3a) (since $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}} \subset \breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}}]}_{-\mathfrak{n}}$).

(18.4) then follows from the equality in (15.32), the definition (5.2) of the change of variables map ${}^{(\mathfrak{n})}\mathscr{T}$, and (18.3b).

Proof of (18.1) **and the fact that** $\min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}} \mu$ **occurs in** $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$: Since the construction of the rough time function ${}^{(\mathfrak{n})}\tau$ is such that $\mu|_{\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}}]}_{-\mathfrak{n}}} = -{}^{(\mathfrak{n})}\tau|_{\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}}]}_{-\mathfrak{n}}}$, it follows from (18.2), (18.3a), and (11.21) that for each fixed $\tau \in [\tau_0,\tau_{\text{Boot}}] = [-\mathfrak{m}_0,-\mathfrak{m}_{\text{Boot}}]$, $\min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}} \mu$ is achieved only in the subset ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}$, which is interior to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$. Hence, since ${}^{(\mathfrak{n})}\breve{W}$ is tangent to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$, it must be that ${}^{(\mathfrak{n})}\breve{W}\mu = 0$ at the minima. Considering also the definition of $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}}]}_{-\mathfrak{n}}$ and the fact that ${}^{(\mathfrak{n})}\breve{W}\mu = \breve{X}\mu + \mathfrak{n}$ in ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}$, we see that the minima of μ in ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ must belong to $\breve{\mathbb{X}}^{[\tau_0,\tau_{\text{Boot}}]}_{-\mathfrak{n}} \cap {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}$. Hence, we conclude that $\min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}} \mu = -\tau$, which is (18.1). Moreover, also using Lemma 15.7, we conclude that the torus $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}} = {}^{(\mathfrak{n})}\mathscr{T}^{-1} \circ {}^{(\mathfrak{n})}\Phi^{-1}\left(\{-\tau\} \times \{-\mathfrak{n}\} \times \mathbb{T}^2\right)$ is exactly the set of points within ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ where μ achieves its minimum value of $-\tau$.

Proof of (18.7): First, we use (4.2), (6.5), (6.6), Lemma 5.5, Lemma 5.8, Prop. 9.1, the estimates of Lemma 15.5, Prop. 17.1, and Cor. 17.2, and (18.8a) to deduce that $|{}^{(\mathfrak{n})}\breve{W}\mu| \leq C$ and ${}^{(\mathfrak{n})}\breve{R}\mu = \breve{W}\mu + \mu\mathcal{O}(\varepsilon) = \breve{W}\mu + \mathcal{O}(\varepsilon)$, where we clarify that we will use the relation ${}^{(\mathfrak{n})}\breve{R}\mu = \breve{W}\mu + \mu\mathcal{O}(\varepsilon)$ in the next paragraph. From these bounds and (18.2), we see that the bounds stated in (18.7) hold in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]} \backslash {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}$.

It remains for us to prove the desired bounds in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{☢},U_{☢}]}$. To this end, we assume that $\tau \in [\tau_0,\tau_{\text{Boot}}]$ and $q \in {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_{☢},U_{☢}]}$. By Lemma 14.2, there is a unique integral curve of ${}^{(\mathfrak{n})}\breve{W}$ that joins q to a point $q_0 \in \breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ on the primal torus $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ (along which $\mu \equiv -\tau$), which, in view of the already proven result (18.3b), is contained in ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}$. Let $\iota = \iota(u')$ denote this integral curve, parameterized by the eikonal function, and let u_0 and u respectively denote the eikonal function values corresponding to q_0 and q. In particular, $\iota(u) = q$, $\iota(u_0) = q_0$, $\mu \circ \iota(u_0) = -\tau$, and by (4.2), $\breve{W}\mu \circ \iota(u_0) = 0$. Using (18.5), the mean value theorem, and Taylor's theorem, we see that $|\breve{W}\mu \circ \iota(u)| \leq \frac{2}{M_2}|u - u_0|$ and $\mu \circ \iota(u) \geq -\tau + \frac{M_2}{4}(u-u_0)^2 \geq \frac{M_2}{4}(u-u_0)^2$. Combining the above results, we find that at q, we have $|\breve{W}\mu| \leq \frac{2}{M_2}\sqrt{\frac{4}{M_2}}\sqrt{\mu} \leq \frac{4}{M_2^{3/2}}\sqrt{\mu}$, which yields (18.7) for the first term on the LHS. From this estimate, the bound ${}^{(\mathfrak{n})}\breve{R}\mu = \breve{W}\mu + \mu\mathcal{O}(\varepsilon)$ noted in the previous paragraph, and the estimate $\mu \leq C$ implied by (17.13), we conclude the desired estimate (18.7) for the second term on the LHS.

Proof of (18.15): Recalling that ${}^{(\mathfrak{n})}\iota_{\Delta u}$ is the flow map of ${}^{(\mathfrak{n})}\breve{W}$, for real numbers Δu small and positive, we consider the set ${}^{(\mathfrak{n})}\iota_{(-\Delta u,\Delta u)}\left(\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}\right) \stackrel{\text{def}}{=} \bigcup_{\Delta u' \in (-\Delta u,\Delta u)} {}^{(\mathfrak{n})}\iota_{\Delta u'}\left(\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}\right)$, which by Lemma 14.2 and (18.3a) is a neighborhood of $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}$ in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{\clubsuit},U_{\clubsuit}]}$. Next, using (4.2), Lemma 5.5, (6.3), Prop. 9.1, the estimates of Lemma 15.5 and Prop. 17.1, and (18.8a), we deduce that $|{}^{(\mathfrak{n})}\breve{W}\,{}^{(\mathfrak{n})}\widetilde{L}\mu| \leq C$. Moreover, since Lemma 15.3 and definition (6.3) imply that ${}^{(\mathfrak{n})}\widetilde{L}\mu|_{\breve{\mathbb{X}}_{-\mathfrak{n}}} = -1$, we can use the mean value theorem to deduce that in ${}^{(\mathfrak{n})}\iota_{(-\Delta u,\Delta u)}\left(\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}\right)$, the following estimates hold: $-1 - C\Delta u \leq {}^{(\mathfrak{n})}\widetilde{L}\mu \circ \iota(u) \leq -1 + C\Delta u$. Choosing and fixing a value of Δu, which we denote by ΔU, to be sufficiently small (and positive) such that $C\Delta U < .01$, we arrive at (18.15) with ${}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]} \stackrel{\text{def}}{=} {}^{(\mathfrak{n})}\iota_{(-\Delta U,\Delta U)}\left(\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}\right)$.

Proof of (18.16): Recall that $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]} = \bigcup_{\mathfrak{m} \in [-\tau_{\text{Boot}},-\tau_0]} \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ (see (15.44)) and that for $\tau \in [\tau_0,\tau_{\text{Boot}}]$, we have $\mu|_{\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}} = -\tau$. For $\tau \in [\tau_0,\tau_{\text{Boot}}]$ and $q_0 \in \breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$, the arguments we used in the proof of (18.7) imply that if $|\Delta u|$ is small enough such that ${}^{(\mathfrak{n})}\iota_{\Delta u}(q_0) \in {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_{\clubsuit},U_{\clubsuit}]}$, then we have $\frac{M_2}{4}(\Delta u)^2 \leq -\tau + \frac{M_2}{4}(\Delta u)^2 \leq \mu \circ {}^{(\mathfrak{n})}\iota_{\Delta u}(q_0)$. In particular, by (15.44) and (18.3a)–(18.3b), if $|\Delta U| \leq \frac{1}{2}U_{\clubsuit}$, then $\dfrac{M_2}{4}(\Delta U)^2 \leq \min\limits_{{}^{(\mathfrak{n})}\iota_{(-\Delta U,\Delta U)}\left(\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\text{Boot}}]}\right)} \mu$. From these observations, the estimate (18.2), and definition (18.14), we see that (18.16) holds with ΔU defined to be the small constant fixed in the proof of (18.15).

□

18.2. **Homeomorphism and diffeomorphism properties of Υ.** Our main goal in this section is to reveal the homeomorphism and diffeomorphism properties of the change of variables map $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$. We start with the following monotonicity lemma, which plays an important role in controlling Υ.

Lemma 18.3 (Monotonicity of x^1). *The following identity holds, where $\phi = \phi(u)$ is the cut-off from Def. 4.1:*

$$\begin{aligned}\frac{\widetilde{\partial}}{\widetilde{\partial} u}x^1 = \mu\left\{X^1 + \frac{X^A X^A}{X^1} + \frac{X^A(\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau)L^B X^B}{(\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau)X^1} + \frac{X^A \frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}L^1\right\} \\ + \phi\frac{\mathfrak{n}}{L\mu}\left\{L^1 + \frac{L^A X^A}{X^1} + \frac{L^A(\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau)L^B X^B}{(\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau)X^1} + \frac{L^A \frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}L^1\right\}.\end{aligned} \tag{18.17}$$

Moreover, the following estimate holds on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$:

$$\frac{\widetilde{\partial}}{\widetilde{\partial} u}x^1 = -\mu\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})\} + \phi\frac{\mathfrak{n}}{L\mu}\{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})\}. \tag{18.18}$$

Finally, for every fixed $(\tau,x^2,x^3) \in [\tau_0,\tau_{\text{Boot}}] \times \mathbb{T}^2$, the map $u \to x^1(\tau,u,x^2,x^3)$ is strictly decreasing on $[-U_1,U_2]$.

Proof. (18.17) follows from (5.15) and (5.6).

(18.18) then follows from (18.17), Prop. 9.1, the estimates of Lemma 15.4, Prop. 17.1, and Cor. 17.2, and (10.9a), which in particular imply that $X^1 = -1 + X^1_{(\text{Small})} = -1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})$, $L^1 = 1 + L^1_{(\text{Small})} = 1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})$, and $X^A, L^A = \mathcal{O}(\varepsilon) = \mathcal{O}_{\blacklozenge}(\mathring{\alpha})$.

To prove the monotonicity of x^1, we note that (18.18) and Prop. 18.1 imply that $\frac{\widetilde{\partial}}{\widetilde{\partial} u}x^1 < 0$, except in the case $\tau_{\text{Boot}} = \mathfrak{n} = 0$, where $\frac{\widetilde{\partial}}{\widetilde{\partial} u}x^1$ vanishes precisely along the torus $\breve{\mathbf{T}}_{0,0}$, which is contained in ${}^{(0)}\widetilde{\Sigma}_0^{[-\frac{1}{2}U_{\clubsuit},\frac{1}{2}U_{\clubsuit}]}$. Moreover, (15.40) implies that $\frac{\widetilde{\partial}}{\widetilde{\partial} u}\breve{X}\mu|_{\breve{\mathbf{T}}_{0,0}} > 0$. Thus, since (15.37) implies that ${}^{(\mathfrak{n})}\mathscr{T}(\breve{\mathbf{T}}_{0,0})$ (i.e., the image of the crease in adapted rough coordinate space $\mathbb{R}_\tau \times \mathbb{R}_u \times \mathbb{T}^2$) is a graph over $\mathbb{T}^2$, we see that every integral curve of $\frac{\widetilde{\partial}}{\widetilde{\partial} u}$ in ${}^{(0)}\widetilde{\Sigma}_0^{[-U_1,U_2]}$ intersects $\breve{\mathbf{T}}_{0,0}$ in precisely one point. In total, we have shown that for every fixed $(\tau,x^2,x^3) \in [\tau_0,\tau_{\text{Boot}}] \times \mathbb{T}^2$, the map $u \to x^1(\tau,u,x^2,x^3)$ on the domain $[-U_1,U_2]$ has a negative derivative, except at possibly a single point in $[-\frac{1}{2}U_{\clubsuit},\frac{1}{2}U_{\clubsuit}]$. From this fact, we conclude that the map is strictly decreasing, as desired.

□

Proposition 18.4 (Homeomorphism and diffeomorphism properties of Υ and the embedded tori $\Upsilon\left(\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\right)$). *The change of variables map $\Upsilon(t,u,x^2,x^3)=(t,x^1,x^2,x^3)$ is **injective** on the compact set ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ and satisfies:*

$$\|\Upsilon\|_{C^{3,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]})} \leq C. \tag{18.19}$$

In particular, Υ is a homeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ onto its image.

Moreover, with $d_{\text{geo}}\Upsilon$ denoting the Jacobian matrix of Υ, we have:

$$\det d_{\text{geo}}\Upsilon = \mu\frac{c^2}{X^1} \approx -\mu. \tag{18.20}$$

In addition, if $\tau_{\text{Boot}}<0$, then Υ is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ onto its image.

Finally, let $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ be the μ-adapted torus defined in (4.3c), and let $\Upsilon\left(\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\right)$ be the image in Cartesian coordinate space of $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ under Υ. Then for $\mathfrak{m}\in[\mathfrak{m}_{\text{Boot}},\mathfrak{m}_0]=[-\tau_{\text{Boot}},-\tau_0]$, $\Upsilon\left(\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\right)$ is a $C^{1,1}$ embedded sub-manifold of Cartesian space that is diffeomorphic to $\mathbb{T}^2$. More precisely, with $\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)$ and $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}$ denoting the functions on $\mathbb{T}^2$ from (15.43)*, the map ${}^{(\mathfrak{n})}\mathfrak{I}$ defined by:*

$$^{(\mathfrak{n})}\mathfrak{I}(x^2,x^3) \stackrel{\text{def}}{=} \Upsilon\circ\left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right) \tag{18.21}$$

is a $C^{1,1}$ embedding, i.e., a $C^{1,1}$ diffeomorphism from $\mathbb{T}^2$ onto $\Upsilon\left(\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\right)$.

Proof. We already proved the bound (18.19) in Lemma 15.6.

Next, we use (5.6), Prop. 9.1, and Prop. 17.1 to compute that $\frac{\partial\Upsilon(t,u,x^2,x^3)}{\partial(t,u,x^2,x^3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ L^1+* & \mu\frac{c^2}{X^1} & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where here and in the rest of the proof, "$*$" denotes any quantity that is pointwise bounded in magnitude by $\mathcal{O}(\mathring{\alpha})$. Thus, $\det\frac{\partial\Upsilon(t,u,x^2,x^3)}{\partial(t,u,x^2,x^3)}=\mu\frac{c^2}{X^1}$, and therefore, using (3.26a), Prop. 9.1, and Prop. 17.1, we compute that $\det\frac{\partial\Upsilon(t,u,x^2,x^3)}{\partial(t,u,x^2,x^3)} = -\{1+\mathcal{O}(\mathring{\alpha})\}\mu$, which yields (18.20).

If $\tau_{\text{Boot}}<0$, then by Prop. 18.1, μ is uniformly positive on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$, and from (18.20) and the inverse function theorem, we see that Υ is a local diffeomorphism on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$. Thus, to complete the proof, we need to show that Υ is injective on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$, even if $\tau_{\text{Boot}}=0$. We will achieve this by proving the injectivity of the map $({}^{(\mathfrak{n})}\tau,u,x^2,x^3)\to({}^{(\mathfrak{n})}\tau,x^1,x^2,x^3)$ on the domain $[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$, and then the injectivity of the map $({}^{(\mathfrak{n})}\tau,x^1,x^2,x^3)\to(t,x^1,x^2,x^3)$; since the composition of two injective functions is injective, this would finish the proof.

To proceed, we first note that the injectivity of the map $({}^{(\mathfrak{n})}\tau,u,x^2,x^3)\to({}^{(\mathfrak{n})}\tau,x^1,x^2,x^3)$ on $[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2$ follows from the monotonicity of the map $u\to x^1(\tau,u,x^2,x^3)$ guaranteed by Lemma 18.3. For use below, we also note that by (3.10), (3.26b), (5.13a), Lemma 5.5, (15.20), and the estimates of Prop. 17.1, we have:

$$\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}x^1 \approx \frac{\partial}{\partial t}x^1 = Lx^1 - L^A\frac{\partial}{\partial x^A}x^1 = 1 + L_{(\text{Small})} - L^A\frac{\partial}{\partial x^A}x^1 \approx 1. \tag{18.22}$$

Now for each fixed $(x^2,x^3)\in\mathbb{T}^2$, let ${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}$ denote the image of the set $[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\{(x^2,x^3)\}$ under the map $({}^{(\mathfrak{n})}\tau,u,x^2,x^3)\to({}^{(\mathfrak{n})}\tau,x^1,x^2,x^3)$. The arguments given above, including the monotonicity guaranteed by (18.22), imply that for each fixed $(u,x^2,x^3)\in[-U_1,U_2]\times\mathbb{T}^2$, the map $\tau\to x^1(\tau,u,x^2,x^3)$ is strictly increasing on $[\tau_0,\tau_{\text{Boot}}]$, and that for each fixed $(\tau,x^2,x^3)\in[\tau_0,\tau_{\text{Boot}}]\times\mathbb{T}^2$, the map $u\to x^1(\tau,u,x^2,x^3)$ is strictly decreasing on $[-U_1,U_2]$. It follows that there exist scalar functions $\tau\to{}^{(\mathfrak{n})}a_{x^2,x^3}(\tau)$ and $\tau\to{}^{(\mathfrak{n})}b_{x^2,x^3}(\tau)$ on $[\tau_0,\tau_{\text{Boot}}]$ such that ${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}=\{(\tau,x^1,x^2,x^3)\mid\tau\in[\tau_0,\tau_{\text{Boot}}],{}^{(\mathfrak{n})}a_{x^2,x^3}(\tau)\leq x^1\leq{}^{(\mathfrak{n})}b_{x^2,x^3}(\tau)\}$, where ${}^{(\mathfrak{n})}a_{x^2,x^3}(\cdot)$ and ${}^{(\mathfrak{n})}b_{x^2,x^3}(\cdot)$ are C^1 functions of τ such that ${}^{(\mathfrak{n})}a_{x^2,x^3}(\tau)<{}^{(\mathfrak{n})}b_{x^2,x^3}(\tau)$ and $\frac{d}{d\tau}{}^{(\mathfrak{n})}a_{x^2,x^3}$, $\frac{d}{d\tau}{}^{(\mathfrak{n})}b_{x^2,x^3}\approx 1$.

To complete the proof, it remains for us to show that the map $({}^{(\mathfrak{n})}\tau,x^1,x^2,x^3)\to(t,x^1,x^2,x^3)$ is injective. Let ${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}$ be the set from the previous paragraph. It suffices to show that for each fixed $(x^2,x^3)\in\mathbb{T}^2$, any two distinct points in ${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}$ with the same x^1 coordinate must be mapped to distinct points under the map $({}^{(\mathfrak{n})}\tau,x^1,x^2,x^3)\to(t,x^1,x^2,x^3)$. The structure of ${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}$ revealed in the previous paragraph shows that for any two distinct points in

${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}$ with the same x^1 coordinate, the straight line segment joining them (along which x^1, x^2, x^3 are constant and ${}^{(\text{Interesting})}\tau$ varies) is contained in ${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}$ (this can be thought of as the vertical convexity of ${}^{(\mathfrak{n})}\mathcal{I}_{x^2,x^3}$). Hence, to complete the proof, it suffices for us to show that the partial derivative of t with respect to ${}^{(\mathfrak{n})}\tau$ in the coordinate system $({}^{(\mathfrak{n})}\tau, x^1, x^2, x^3)$ is positive, except possibly when ${}^{(\mathfrak{n})}\tau = 0$. To proceed, we note that the partial derivative of interest is equal to $\frac{1}{\partial_t {}^{(\mathfrak{n})}\tau}$, where ∂_t is the Cartesian partial derivative. We now use (3.10), (3.13), (3.26a), (4.2), (4.4), (5.9a), Lemma 5.5, Prop. 9.1, and the estimates of Lemma 15.5, and Prop. 17.1, (18.8a), and (18.9b) to compute that $\partial_t = L + (1+*)X + *Y_{(2)} + *Y_{(3)}$ and that $\frac{1}{\partial_t {}^{(\mathfrak{n})}\tau} \approx \frac{1}{1+\frac{\mathfrak{n}\phi}{\mu}} = \frac{\mu}{\mu + \mathfrak{n}\phi}$, where ϕ is the cut-off function from Def. 4.1. We now recall that by (18.1), μ can vanish only when ${}^{(\mathfrak{n})}\tau = 0$. It follows that $\frac{1}{\partial_t {}^{(\mathfrak{n})}\tau} > 0$ except possibly when ${}^{(\mathfrak{n})}\tau = 0$. We have therefore shown that the map $({}^{(\mathfrak{n})}\tau, x^1, x^2, x^3) \to (t, x^1, x^2, x^3)$ is injective, as desired.

Finally, since the map ${}^{(\mathfrak{n})}\mathrm{I}$ from (18.21) is the composition of the injective $C^{3,1}$ map Υ with the $C^{1,1}$ embedding of $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ given by (15.43), it follows that ${}^{(\mathfrak{n})}\mathrm{I}$ is a $C^{1,1}$ injection from $\mathbb{T}^2$ into Cartesian coordinate space. Moreover, its differential $d_{(x^2,x^3)} {}^{(\mathfrak{n})}\mathrm{I}$ is a 4×2 matrix whose lower 2×2 block is the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. That is, $d_{(x^2,x^3)} {}^{(\mathfrak{n})}\mathrm{I}$ is full rank, and therefore ${}^{(\mathfrak{n})}\mathrm{I}$ is a $C^{1,1}$ embedding. This concludes the proof of the proposition.

□

18.3. **Control of the size of t and x^1.** In the next lemma, we derive improvements of the bootstrap assumptions of Sect. 12.2.5.

Lemma 18.5 (Control of the size of t and x^1 in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$). *The following estimates hold for $\tau \in [\tau_0, \tau_{\text{Boot}}]$:*

$$\frac{1}{3\mathring{\delta}_*} \leq \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}} t \leq \sup_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}} t \leq \frac{3}{\mathring{\delta}_*}, \tag{18.23a}$$

$$-U_2 + \frac{1}{3\mathring{\delta}_*} \leq \min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}} x^1 \leq \sup_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}} x^1 \leq U_1 + \frac{3}{\mathring{\delta}_*}. \tag{18.23b}$$

Proof. We first prove (18.23a). From (3.21), (6.3), and (**BA** $L{}^{(\mathfrak{n})}\tau$), we have ${}^{(\mathfrak{n})}\widetilde{L}t = \frac{1}{L^{(\mathfrak{n})}\tau} Lt = \frac{1}{L^{(\mathfrak{n})}\tau} \approx 1$. Hence, recalling that ${}^{(\mathfrak{n})}\widetilde{L}u = 0$ and ${}^{(\mathfrak{n})}\widetilde{L}\tau = 1$, we can integrate along the integral curves of ${}^{(\mathfrak{n})}\widetilde{L}$ starting from any point in ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$ and use the data-assumption (11.17a) and the identity $|\tau_0| = \mathfrak{m}_0$ to deduce that in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$, we have $\frac{1}{2\mathring{\delta}_*} + \frac{1}{C}\mathfrak{m}_0 \leq t \leq \frac{2}{\mathring{\delta}_*} + C\mathfrak{m}_0$, which, for $\mathfrak{m}_0$ sufficiently small, implies (18.23a).

To prove (18.23b), we first use (3.21), (6.3), (**BA** $L{}^{(\mathfrak{n})}\tau$), and (**AUX** $L^1_{(\text{Small})}$ **SMALL**) to deduce that we have ${}^{(\mathfrak{n})}\widetilde{L}x^1 = \frac{1}{L^{(\mathfrak{n})}\tau} Lx^1 = \frac{1}{L^{(\mathfrak{n})}\tau}(1 + L^1_{(\text{Small})}) \approx 1$. We now argue as in the proof of (18.23a) using the data-assumption (11.17b), thereby arriving at (18.23b).

□

18.4. **Pointwise estimates tied to the rough acoustic geometry.** Equation (21.63) below provides the main elliptic-hyperbolic integral identity that we will use to control the top-order derivatives of Ω and S. In Prop. 23.4, we derive pointwise estimates for the error integrands on RHS (21.63). Some of these error integrands (e.g., the term $\mathfrak{E}_{(\text{Lower-order})}[V, V]$ defined in (21.47)) depend on geometric quantities that are tied to the rough acoustic geometry. In Lemma 18.6, we derive pointwise estimates for these quantities. This serves as a preliminary step for our proof of Prop. 23.4. Moreover, some of the estimates from Lemma 18.6 have other uses. For example, we use the estimate (18.31) for the deformation tensor of ${}^{(\mathfrak{n})}\widetilde{L}$ in our proof of the fundamental theorem of calculus-type estimate (20.5) on rough tori.

Lemma 18.6 (Pointwise estimates tied to the rough acoustic geometry). *Let $\mathbf{1}_{\{\mu < -\phi \frac{\mathfrak{n}}{L\mu}\}}$ be the characteristic function of the set $\left\{(t, u, x^2, x^3) \mid \mu(t, u, x^2, x^3) < -\phi(u) \frac{\mathfrak{n}}{L\mu(t,u,x^2,x^3)}\right\}$ (where ϕ is the cut-off function introduced in Def. 4.1) and similarly for $\mathbf{1}_{\{\mu \geq -\phi \frac{\mathfrak{n}}{L\mu}\}}$. Let ${}^{({}^{(\mathfrak{n})}\widetilde{L})}\boldsymbol{\pi}$ be the deformation tensor of the vectorfield ${}^{(\mathfrak{n})}\widetilde{L}$ defined in (6.3), let ${}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}$ be the deformation tensor of the vectorfield ${}^{(\mathfrak{n})}\breve{R}$ defined in (6.6), let $\mathrm{tr}_{\widetilde{g\!\!/}} {}^{({}^{(\mathfrak{n})}\widetilde{L})}\boldsymbol{\pi}$ and $\mathrm{tr}_{\widetilde{g\!\!/}} {}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}$ respectively denote their traces with respect to $\widetilde{g\!\!/}$, and let $\widetilde{\mathrm{div\!\!/}}\, {}^{(\mathfrak{n})}U$ be the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-divergence (see Def. 6.13) of the vectorfield ${}^{(\mathfrak{n})}U$ defined in (6.5). Then the following pointwise* *estimates hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:

$$\sum_{\alpha=0,1,2,3} |{}^{(\mathfrak{n})}\breve{R}^{\alpha}| \lesssim -\mathbf{1}_{\{\mu<-\phi\frac{\mathfrak{n}}{L\mu}\}}\phi\frac{\mathfrak{n}}{L\mu} + \mathbf{1}_{\{\mu\geq-\phi\frac{\mathfrak{n}}{L\mu}\}}\mu, \tag{18.24}$$

$$\sum_{\alpha=0,1,2,3} |{}^{(\mathfrak{n})}U^{\alpha}| \lesssim \varepsilon, \tag{18.25}$$

$$|\mathcal{P}^{\leq 1}\,{}^{(\mathfrak{n})}U\mu| \lesssim \varepsilon^2, \tag{18.26}$$

$$|\mathcal{P}^{\leq 1}\,{}^{(\mathfrak{n})}r|, |\breve{X}\,{}^{(\mathfrak{n})}r| \lesssim \varepsilon^2, \tag{18.27}$$

$$|{}^{(\mathfrak{n})}\breve{R}L\mu| \lesssim 1, \tag{18.28}$$

$$|{}^{(\mathfrak{n})}UL\mu| \lesssim \varepsilon^2, \tag{18.29}$$

$$|\widetilde{\mathrm{d}\!\!/\!\mathrm{iv}}\,{}^{(\mathfrak{n})}U| \lesssim \varepsilon, \tag{18.30}$$

$$|\mathrm{tr}_{\widetilde{g\!\!/}}{}^{({}^{(\mathfrak{n})}\widetilde{L})}\boldsymbol{\pi}| \lesssim 1, \tag{18.31}$$

$$|\mathrm{tr}_{\widetilde{g\!\!/}}{}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}| \lesssim 1. \tag{18.32}$$

Proof. **Proof of** (18.25)–(18.27)**:** To prove (18.27), we first use (7.7), (3.31b), Lemma 5.5, and Prop. 9.1 to deduce that:

$${}^{(\mathfrak{n})}r = \frac{1}{(L\,{}^{(\mathfrak{n})}\tau)^2}(g\!\!/^{-1})^{AB}\left(\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau\right)\frac{\partial}{\partial x^B}{}^{(\mathfrak{n})}\tau = \frac{1}{(L\,{}^{(\mathfrak{n})}\tau)^2}\mathrm{f}(\gamma)\cdot\left(\frac{\partial}{\partial x^2}{}^{(\mathfrak{n})}\tau, \frac{\partial}{\partial x^3}{}^{(\mathfrak{n})}\tau\right)\cdot\left(\frac{\partial}{\partial x^2}{}^{(\mathfrak{n})}\tau, \frac{\partial}{\partial x^3}{}^{(\mathfrak{n})}\tau\right), \tag{18.33}$$

where the expression on RHS (18.33) is depicted schematically. From (18.33), (15.24), (18.9b), the bootstrap assumptions, and Cor. 17.2, we deduce that $|{}^{(\mathfrak{n})}r| \lesssim \varepsilon^2$ as desired. To prove that $|\mathcal{P}\,{}^{(\mathfrak{n})}r|, |\breve{X}\,{}^{(\mathfrak{n})}r| \lesssim \varepsilon^2$, we differentiate (18.33) with elements $P \in \{L, Y_{(2)}, Y_{(3)}\}$ and $\breve{X}$ and apply a similar argument, where we use Lemma 5.5 to express the vectorfield derivatives in terms of geometric coordinate partial derivatives when they fall on ${}^{(\mathfrak{n})}\tau$.

The estimates (18.25) and (18.26) follow from similar arguments based on the identity (7.8).

Proof of (18.24)**:** Since (**BA** $L\mu$ neg) implies that $-L\mu \approx 1$ on the support of ϕ, the estimate (18.24) follows from the decomposition ${}^{(\mathfrak{n})}\breve{R}^{\alpha} = \mu X^{\alpha} + \phi\frac{\mathfrak{n}}{L\mu}L^{\alpha} - \mu\,{}^{(\mathfrak{n})}U^{\alpha}$ (see (6.6)), Prop. 9.1, the bootstrap assumptions (in particular (**BA** $L\mu$ neg)), and (18.25).

Proof of (18.28)–(18.29)**:** These estimates follow from the decompositions (6.6) and (7.8) and the arguments we used in the proofs of (18.25)–(18.27).

Proof of (18.30)**:** First, computing relative to the coordinates (x^2, x^3) on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, we deduce that:

$$\begin{aligned}\widetilde{\mathrm{d}\!\!/\!\mathrm{iv}}\,{}^{(\mathfrak{n})}U &= \frac{\widetilde{\partial}}{\widetilde{\partial} x^A}{}^{(\mathfrak{n})}U^A \\ &+ \frac{1}{2}{}^{(\mathfrak{n})}U^A(\widetilde{g\!\!/}^{-1})(\mathrm{d}x^B, \mathrm{d}x^C)\left\{\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\widetilde{g\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^C}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial} x^C}\widetilde{g\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right) - \frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\widetilde{g\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^C}\right)\right\}.\end{aligned} \tag{18.34}$$

Next, we use (6.5) to write ${}^{(\mathfrak{n})}U^A = (\widetilde{g\!\!/}^{-1})(\mathrm{d}x^A, \mathrm{d}x^B)\frac{\frac{\partial}{\partial x^B}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}$. We then use (5.8c)–(5.8d), (5.13c), and Prop. 9.1 to schematically express $\frac{\widetilde{\partial}}{\widetilde{\partial} x^A} = \mathrm{f}\left(\gamma, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}, \mathcal{Y}\,{}^{(\mathfrak{n})}\tau\right)Y_{(2)} + \mathrm{f}\left(\gamma, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}, \mathcal{Y}\,{}^{(\mathfrak{n})}\tau\right)Y_{(3)} + \mathrm{f}\left(\gamma, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau},\right)\cdot(\mathcal{Y}\,{}^{(\mathfrak{n})}\tau)\cdot L$, and we also use (3.31a), (3.31b) (6.11), (6.14), (6.17), and Prop. 9.1 to deduce the schematic identities $\widetilde{g\!\!/}\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right) = \mathrm{f}\left(\gamma, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}, \mathcal{Y}\,{}^{(\mathfrak{n})}\tau\right)$ and $(\widetilde{g\!\!/}^{-1})(\mathrm{d}x^A, \mathrm{d}x^B) = \mathrm{f}\left(\gamma, \frac{1}{L\,{}^{(\mathfrak{n})}\tau}, \mathcal{Y}\,{}^{(\mathfrak{n})}\tau\right)$. From these identities, we deduce the following schematic identity:

$$\widetilde{\mathrm{d}\!\!/\!\mathrm{iv}}\,{}^{(\mathfrak{n})}U = \mathrm{f}\left(\mathcal{P}^{\leq 1}\gamma, \frac{1}{L\,{}^{(\mathfrak{n})}\tau}, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}, \mathcal{P}^{\leq 2}\,{}^{(\mathfrak{n})}\tau\right)\cdot\mathcal{Y}^{[1,2]}\,{}^{(\mathfrak{n})}\tau. \tag{18.35}$$

From (18.35), Prop. 9.1, the estimates (15.22), (15.24), and (18.9a)–(18.9b) for the rough time function, the bootstrap assumptions, and Cor. 17.2, we arrive at the desired estimate (18.30).

Proof of (18.32)**:** First, using (6.12) and (9.16), we compute that relative to the coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$, we have $\mathrm{tr}_{\widetilde{g}}{}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi} = \widetilde{g}^{-1}\left(\mathrm{d}x^A, \mathrm{d}x^B\right){}^{(\mathfrak{n})}\breve{R}\widetilde{g}\left(\frac{\widetilde{\partial}}{\partial x^A}, \frac{\widetilde{\partial}}{\partial x^B}\right) + 2\frac{\widetilde{\partial}}{\partial x^A}{}^{(\mathfrak{n})}\breve{R}^A$. Considering this identity, arguing as in the proof of (18.35), and also using (5.7b) and (6.6), we deduce the following schematic identity:

$$\mathrm{tr}_{\widetilde{g}}{}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi} = \mathrm{f}\left(\mathcal{P}^{\leq 2}\underline{\gamma}, \mathcal{Z}\gamma, \frac{1}{L^{(\mathfrak{n})}\tau}, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}, \phi\frac{\mathfrak{n}}{L\mu}, \mathcal{P}^{\leq 2(\mathfrak{n})}\tau, \breve{X}\mathcal{P}^{(\mathfrak{n})}\tau\right). \tag{18.36}$$

From (18.36), Lemmas 5.5 and 9.1, the estimates (15.22) and (18.9a)–(18.9b) for the rough time function, and the bootstrap assumptions (note in particular that (**BA** $L\mu$ neg) implies that $-L\mu \approx 1$ on the support of ϕ), we arrive at the desired estimate (18.32).

Proof of (18.31)**:** Considering definition (6.3) and arguing as in the proof of (18.36), we find that:

$$\mathrm{tr}_{\widetilde{g}}{}^{({}^{(\mathfrak{n})}\widetilde{L})}\boldsymbol{\pi} = \mathrm{f}\left(\mathcal{P}^{\leq 1}\gamma, \frac{1}{L^{(\mathfrak{n})}\tau}, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}, \mathcal{P}^{\leq 2(\mathfrak{n})}\tau\right). \tag{18.37}$$

From (18.37), Lemmas 5.5 and 9.1, the estimates (15.11a) and (18.9a)–(18.9b) for the rough time function, and the bootstrap assumptions, we arrive at the desired estimate (18.31).

□

19. Modified quantities for controlling the acoustic geometry

We continue to work under the assumptions of Sect. 13.2. In this section, we construct "modified" versions of the eikonal function quantity $\mathrm{tr}_{g\!\!/}\chi$, and we derive the transport equations that they satisfy. There are two kinds of modified quantities: "partially modified" and "fully modified." The partially modified quantities, when combined with integration by parts, will allow us to avoid uncontrollably large error integrals in the wave equation energy identities. The fully modified quantities, when combined with elliptic estimates on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, will allow us to control the top-order $Y_{(A)}$-derivatives of $\mathrm{tr}_{g\!\!/}\chi$ without losing derivatives. The basic ideas behind the modified quantities originated in the works [24, 26, 45].

19.1. **Decompositions of $\mathbf{Ric}_{LL}$.** In the next lemma, we provide two key decompositions of $\mathbf{Ric}_{LL}$, where $\mathbf{Ric}$ is the Ricci curvature of the acoustical metric $\mathbf{g}$. The decompositions are a crucial ingredient in our derivation of the transport equations satisfied by the modified quantities; see the proofs of Props. 19.4 and 19.5.

Lemma 19.1 (The key identities verified by $\mathbf{Ric}_{LL}$)**.** *Assume that the entries of $\vec{\Psi} = (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$ solve the geometric wave equations* (2.22a)–(2.22d)*. Then the following identity holds, where $\mathbf{Ric}$ is the Ricci curvature of $\mathbf{g}$:*

$$\begin{aligned}\mu\mathbf{Ric}_{LL} = L&\left\{-\vec{G}_{LL}\diamond\breve{X}\vec{\Psi} - \frac{1}{2}\mu\mathrm{tr}_{g\!\!/}\vec{G\!\!/}\diamond L\vec{\Psi} - \frac{1}{2}\mu\vec{G}_{LL}\diamond L\vec{\Psi} + \mu\vec{G\!\!/}_L^{\#}\diamond\cdot d\!\!/\vec{\Psi}\right\}\\ &+ \mathfrak{A},\end{aligned} \tag{19.1}$$

where $\mathfrak{A}$ has the following schematic structure:

$$\mathfrak{A} = \mu\mathrm{f}(\vec{\Psi})\cdot(\mathcal{C},\mathcal{D}) + \mathrm{f}(\underline{\gamma}, \mathcal{Z}\vec{\Psi})\cdot\mathcal{P}\vec{\Psi} + \mathrm{f}(\underline{\gamma}, \Omega, S, \mathcal{Z}\vec{\Psi})\cdot(\Omega, S). \tag{19.2}$$

Moreover, <u>without</u> the assumption that the geometric wave equations (2.22a)–(2.22d) *are satisfied, the following identity holds:*

$$\begin{aligned}\mathbf{Ric}_{LL} = \frac{L\mu}{\mu}\mathrm{tr}_{g\!\!/}\chi + L&\left\{-\frac{1}{2}\mathrm{tr}_{g\!\!/}\vec{G\!\!/}\diamond L\vec{\Psi} + \vec{G\!\!/}_L^{\#}\diamond\cdot d\!\!/\vec{\Psi}\right\}\\ -\frac{1}{2}&\vec{G}_{LL}\diamond \Delta\!\!\!/\vec{\Psi} + \mathfrak{B},\end{aligned} \tag{19.3}$$

where $\mathfrak{B}$ has the following schematic structure:

$$\mathfrak{B} = \mathrm{f}(\gamma)\cdot\mathcal{P}\vec{\Psi}\cdot\mathcal{P}\gamma. \tag{19.4}$$

Sketch of a proof. In [50, Lemma 6.1], in the case of two spatial dimensions, analogs of (19.1) and (19.3) were derived. Analogous identities were also derived in [69, Corollary 11.4] in the case of quasilinear wave equations in three space dimensions. The proof of (19.3) given in [69, Corollary 11.4] is based on first writing $\mathbf{Ric}_{LL}$ relative to the Cartesian coordinates, then writing all derivatives of $\vec{\Psi}$ in terms of derivatives with respect to elements of $\{L, X\}$ and $\ell_{t,u}$-tangent differentiations, and then finally expressing (with the help of Lemma 3.21) all of the principal terms (i.e., the terms that depend on the second derivatives of $\vec{\Psi}$) as a perfect L derivative up to lower order terms, except for the term $-\frac{1}{2}\vec{G}_{LL}\diamond\not\Delta\vec{\Psi}$ on the last line of RHS (19.3). The detailed proof given in [69, Corollary 11.4] goes through nearly verbatim, except we have used Prop. 9.1 to simplify our schematic presentation of the term $\mathfrak{B}$.

Similarly, (19.1) can be proved using the same arguments given in [69, Corollary 11.4], but the new feature of the present work is the structure of the terms on RHS (19.1). To see how these terms arise, we explain how the proof of (19.1) is connected to the identity (19.3). To pass from (19.3) to (19.1), one uses the identity (3.51a), the wave equations (2.22a)–(2.22d), Lemma 3.21, and Prop. 9.1 to express the product of μ and the term $-\frac{1}{2}\vec{G}_{LL}\diamond\not\Delta\vec{\Psi}$ on RHS (19.3) as follows:

$$
\begin{aligned}
-\frac{1}{2}\mu\vec{G}_{LL}\diamond\not\Delta\vec{\Psi} = -\frac{1}{2}L\left\{\vec{G}_{LL}\diamond(\mu L\vec{\Psi}+2\breve{X}\vec{\Psi})\right\} - \frac{1}{2}\mathrm{tr}_{\not g}\chi\vec{G}_{LL}\diamond\breve{X}\vec{\Psi} \\
+\mathrm{f}(\gamma)\cdot\mathrm{Inhom}+\mathrm{f}(\underline{\gamma},\mathcal{Z}\vec{\Psi})\cdot\mathcal{P}\vec{\Psi}.
\end{aligned} \tag{19.5}
$$

In (19.5), "Inhom" denotes the inhomogeneous terms $\mu\times$RHS (2.22a)–(2.22d). In a detailed proof (see [69, Corollary 11.4]), one finds that the term $-\frac{1}{2}\mathrm{tr}_{\not g}\chi\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}$ on RHS (19.5) is canceled (and hence does not appear in (19.1)–(19.2)) by part of the first product on RHS (19.3), where one uses (3.44) to substitute for the factor $L\mu$ on RHS (19.3). Next, decomposing $\mu\times$RHS (2.22a)–(2.22d) using (9.7b), (9.8a), and (9.8b), we find that the term $\mathrm{f}(\gamma)\cdot$Inhom on RHS (19.5) can be schematically expressed as follows: $\mathrm{f}(\gamma)\cdot\mathrm{Inhom}=\mu\mathrm{f}(\vec{\Psi})\cdot(\mathcal{C},\mathcal{D})+\mathrm{f}(\gamma,\mathcal{Z}\vec{\Psi})\cdot\mathcal{P}\vec{\Psi}+\mathrm{f}(\underline{\gamma},\Omega,S,\mathcal{Z}\vec{\Psi})\cdot(\Omega,S)$. Placing these terms on RHS (19.2), and also incorporating the last term on RHS (19.5) into RHS (19.2), we arrive at (19.1)–(19.2). □

19.2. **Definition of the modified quantities.** We are now ready to define the modified quantities. The definitions are motivated by the structure of the terms in Lemma 19.1; this will become clear in the proofs of Props. 19.4 and 19.5.

Definition 19.2 (Modified versions of the $\mathcal{P}_u$-tangential derivatives of $\mathrm{tr}_{\not g}\chi$)**.** Let $N=N_{\mathrm{top}}$, and let $\mathcal{P}^N\in\mathfrak{P}^{(N)}$, where $\mathfrak{P}^{(N)}$ is the set of order N $\mathcal{P}_u$-tangential commutator operators from Def. 8.10. We define the *fully modified quantity* ${}^{(\mathcal{P}^N)}\mathscr{X}$ as follows:

$$
{}^{(\mathcal{P}^N)}\mathscr{X} \overset{\mathrm{def}}{=} \mu\mathcal{P}^N\mathrm{tr}_{\not g}\chi+\mathcal{P}^N\mathfrak{X}, \tag{19.6a}
$$

$$
\mathfrak{X} \overset{\mathrm{def}}{=} -\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}-\frac{1}{2}\mu\mathrm{tr}_{\not g}\vec{\not G}\diamond L\vec{\Psi}-\frac{1}{2}\mu\vec{G}_{LL}\diamond L\vec{\Psi}+\mu\vec{\not G}_L^{\#}\diamond\cdot\not d\vec{\Psi}. \tag{19.6b}
$$

Moreover, with $N=N_{\mathrm{top}}-1$ and $\mathcal{P}^N\in\mathfrak{P}^{(N)}$, we define the *partially modified quantity* ${}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}}$ as follows:

$$
{}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}} \overset{\mathrm{def}}{=} \mathcal{P}^N\mathrm{tr}_{\not g}\chi+{}^{(\mathcal{P}^N)}\widetilde{\mathfrak{X}}, \tag{19.7a}
$$

$$
{}^{(\mathcal{P}^N)}\widetilde{\mathfrak{X}} \overset{\mathrm{def}}{=} -\frac{1}{2}\mathrm{tr}_{\not g}\vec{\not G}\diamond L\mathcal{P}^N\vec{\Psi}+\vec{\not G}_L^{\#}\diamond\cdot\not d\mathcal{P}^N\vec{\Psi}. \tag{19.7b}
$$

Finally, we define the following "0^{th}-order" version of (19.7b):

$$
\widetilde{\mathfrak{X}} \overset{\mathrm{def}}{=} -\frac{1}{2}\mathrm{tr}_{\not g}\vec{\not G}\diamond L\vec{\Psi}+\vec{\not G}_L^{\#}\diamond\cdot\not d\vec{\Psi}. \tag{19.8}
$$

19.3. **Transport equations for the modified quantities.** In this section, we derive transport equations for the fully modified quantities from Def. 19.2. We start with the following lemma, which provides the transport equation satisfied by $\mathrm{tr}_{\not g}\chi$. This transport equation is an analog of the well-known *Raychaudhuri equation* in General Relativity [63].

Lemma 19.3 (Raychaudhuri-type transport equation for $\mathrm{tr}_{\not g}\chi$)**.** $\mathrm{tr}_{\not g}\chi$ *obeys the following transport equation:*

$$
\mu L\mathrm{tr}_{\not g}\chi=(L\mu)\mathrm{tr}_{\not g}\chi-\mu\mathbf{Ric}_{LL}-\mu|\chi|_{\not g}^2. \tag{19.9}
$$

Proof. The same proof of [69, (11.23)] holds in the current setting. □

Proposition 19.4 (Transport equation satisfied by ${}^{(\mathcal{P}^N)}\mathscr{X}$)**.** *Assume that* $\vec{\Psi}=(\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s)$ *solve the geometric wave equations* (2.22a)–(2.22d). *Let* $N=N_{\mathrm{top}}$, *and let* $\mathcal{P}^N\in\mathfrak{P}^{(N)}$, *where* $\mathfrak{P}^{(N)}$ *is the set of order* N $\mathcal{P}_u$*-tangential commutator operators from Def. 8.10. Let* ${}^{(\mathcal{P}^N)}\mathscr{X}$ *be the fully modified quantity defined in* (19.6a), *let* $\mathfrak{X}$ *be as defined in*

(19.6b), *and let* $\mathfrak{A}$ *be the term on RHS* (19.1). *Then* ${}^{(\mathcal{P}^N)}\mathscr{X}$ *obeys the following transport equation, where* $\mathcal{P}^N$ *is the same differential operator every time it appears in* (19.10)*:*

$$\begin{aligned} L^{(\mathcal{P}^N)}\mathscr{X} - \left(2\frac{L\mu}{\mu}\right)^{(\mathcal{P}^N)}\mathscr{X} = & -\left(2\frac{L\mu}{\mu}\right)\mathcal{P}^N\mathfrak{X} + \mu[L,\mathcal{P}^N]\mathrm{tr}_{\not{g}}\chi \\ & + [L,\mathcal{P}^N]\mathfrak{X} + [\mu,\mathcal{P}^N]L\mathrm{tr}_{\not{g}}\chi + [\mathcal{P}^N, L\mu]\mathrm{tr}_{\not{g}}\chi \\ & - \mathcal{P}^N\left(\mu|\chi|^2_{\not{g}}\right) - \mathcal{P}^N\mathfrak{A}. \end{aligned} \tag{19.10}$$

Sketch of a proof. First, we use (19.1) to substitute for the product $\mu\mathbf{Ric}_{LL}$ in (19.9). We then differentiate the resulting equation with $\mathcal{P}^N$ and carry out tedious but straightforward commutations. Also taking into account definition (19.6a), we conclude (19.10). We refer to the proof of [73, Proposition 6.2] for more details. □

Proposition 19.5 (Transport equation satisfied by ${}^{(\mathcal{P}^{N-1})}\widetilde{\mathscr{X}}$)**.** *Let* $N = N_{\mathrm{top}}$, *and let* $\mathcal{P}^{N-1} \in \mathfrak{P}^{(N-1)}$, *where* $\mathfrak{P}^{(N-1)}$ *is the set of order* $N-1$ $\mathcal{P}_u$*-tangential commutator operators from Def. 8.10. Let* ${}^{(\mathcal{P}^{N-1})}\widetilde{\mathscr{X}}$ *be the corresponding partially modified quantity defined in* (19.7a), *let* ${}^{(\mathcal{P}^{N-1})}\widetilde{\mathfrak{X}}$ *be the term defined in* (19.7b), *let* $\widetilde{\mathfrak{X}}$ *be the term defined in* (19.8), *and let* $\mathfrak{B}$ *be the term on RHS* (19.3). *Then* ${}^{(\mathcal{P}^{N-1})}\widetilde{\mathscr{X}}$ *obeys the following transport equation, where* $\mathcal{P}^{N-1}$ *is the same differential operator every time it appears in* (19.11)*:*

$$L^{(\mathcal{P}^{N-1})}\widetilde{\mathscr{X}} = \frac{1}{2}\vec{G}_{LL} \diamond \not{\Delta}\mathcal{P}^{N-1}\vec{\Psi} + {}^{(\mathcal{P}^{N-1})}\mathfrak{B}, \tag{19.11}$$

where:

$$\begin{aligned} {}^{(\mathcal{P}^{N-1})}\mathfrak{B} \overset{def}{=} & -\mathcal{P}^{N-1}\mathfrak{B} - \mathcal{P}^{N-1}\left(|\chi|^2_{\not{g}}\right) \\ & + \frac{1}{2}[\mathcal{P}^{N-1}, \vec{G}_{LL}] \diamond \not{\Delta}\vec{\Psi} + \frac{1}{2}\vec{G}_{LL} \diamond [\mathcal{P}^{N-1}, \not{\Delta}]\vec{\Psi} \\ & + [L,\mathcal{P}^{N-1}]\mathrm{tr}_{\not{g}}\chi + [L,\mathcal{P}^{N-1}]\widetilde{\mathfrak{X}} + L\left\{{}^{(\mathcal{P}^{N-1})}\widetilde{\mathfrak{X}} - \mathcal{P}^{N-1}\widetilde{\mathfrak{X}}\right\}. \end{aligned} \tag{19.12}$$

Proof. Substituting the identity (19.3) into (19.9), dividing the resulting equation by μ, and appealing to the definition (19.8) of $\widetilde{\mathfrak{X}}$, we deduce that:

$$L\left(\mathrm{tr}_{\not{g}}\chi + \widetilde{\mathfrak{X}}\right) = \frac{1}{2}\vec{G}_{LL} \diamond \not{\Delta}\vec{\Psi} - |\chi|^2_{\not{g}} - \mathfrak{B}. \tag{19.13}$$

The transport equation (19.11) then follows from differentiating (19.13) with $\mathcal{P}^{N-1}$, carrying out straightforward commutations, and accounting for definitions (19.7a)–(19.8).

□

20. Basic ingredients in the L^2 analysis

We continue to work under the assumptions of Sect. 13.2. In this section, we establish some preliminary ingredients that we will use when we derive energy estimates. In Sect. 21, we will derive one more crucial ingredient: elliptic-hyperbolic integral identities that we use to control the top-order derivatives of the specific vorticity and entropy gradient. In Sect. 20.1, we derive some differential and integral identities involving the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. In Sect. 20.2, we establish some basic Sobolev embedding estimates and fundamental theorem of calculus-type estimates on the rough tori. In Sect. 20.3, we use the identities from Sect. 20.1 to prove various integration by parts identities that will play a key role in our energy estimates. Next, in Sect. 20.4, we use the vectorfield multiplier method to construct the building block L^2-based energies and null-fluxes that we will use to control the wave-variables $\vec{\Psi}$. We also construct a companion set of building block energies and null-fluxes that we will use to control the transport-variables Ω, S, $\mathcal{C}$, and $\mathcal{D}$. Furthermore, in Prop. 20.9, we establish the fundamental energy-null-flux integral identities that we exploit in our L^2 analysis. In Sect. 20.5, we use the building block energies and null fluxes to define the quantities that we will use to control the solution in L^2. Of particular interest are the spacetime integrals $\mathbb{K}_N(\tau,u)$ and $\mathbb{K}_N^{(\mathrm{Partial})}(\tau,u)$ defined in (20.43b) and (20.44b) respectively. These spacetime integrals appear on the left-hand side of our energy identity (20.26), and they are fundamental for controlling error integrals that involve the quantities $\not{d}\mathcal{P}^N\Psi$. Finally, in Sect. 20.6, we exhibit the key coerciveness properties of our L^2-controlling quantities with respect to the L^2 norms from Sect. 8.2.3.

20.1. **Differential and integral identities involving $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$.** The following lemma, though standard, plays an important role in our proof of the energy–null-flux identities for the wave-variables (see Prop. 20.9). Moreover, the identity (20.2) plays a crucial role in our proof that one of the rough tori error integrals (specifically, the second term on LHS (21.14)) in our elliptic-hyperbolic identities for the vorticity and entropy has a favorable sign.

Lemma 20.1 (Differential and integral identities involving $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$). *Let f be a scalar function on $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, let $^{(\mathfrak{n})}\widetilde{L}$ be the vectorfield defined in* (6.3)*, and let $^{(\mathfrak{n})}\breve{R}$ be the vectorfield defined in* (6.6)*. For $Z \in \{^{(\mathfrak{n})}\widetilde{L}, {}^{(\mathfrak{n})}\breve{R}\}$, let $\text{tr}_{\widetilde{g}}{}^{(Z)}\boldsymbol{\pi}$ be the $\widetilde{g}$-trace (see Def. 6.10) of the deformation tensor $^{(Z)}\boldsymbol{\pi}$ of Z. Then the following integral identities hold for $\tau \in [\tau_0, \tau_{\text{Boot}})$ and $-U_1 \leq u_1 \leq u_2 \leq U_2$:*

$$\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}\left(\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} f \,\mathrm{d}\varpi_{\widetilde{g}}\right) = \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \left\{^{(\mathfrak{n})}\widetilde{L}f + \frac{1}{2} f \,\text{tr}_{\widetilde{g}}{}^{(^{(\mathfrak{n})}\widetilde{L})}\boldsymbol{\pi}\right\} \mathrm{d}\varpi_{\widetilde{g}}, \tag{20.1a}$$

$$\frac{\widetilde{\partial}}{\widetilde{\partial}u}\left(\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} f \,\mathrm{d}\varpi_{\widetilde{g}}\right) = \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \left\{^{(\mathfrak{n})}\breve{R}f + \frac{1}{2} f \,\text{tr}_{\widetilde{g}}{}^{(^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}\right\} \mathrm{d}\varpi_{\widetilde{g}}. \tag{20.1b}$$

Moreover, for $u_1 \leq u_2$, we have:

$$\int_{^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[u_1,u_2]}} \left\{^{(\mathfrak{n})}\breve{R}f + \frac{1}{2} f \,\text{tr}_{\widetilde{g}}{}^{(^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}\right\} \mathrm{d}\underline{\varpi} = \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u_2}} f \,\mathrm{d}\varpi_{\widetilde{g}} - \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u_1}} f \,\mathrm{d}\varpi_{\widetilde{g}}. \tag{20.2}$$

Proof. We prove only (20.1b) and (20.2) because (20.1a) was proved[59] in [4, Lemma 6.3]. Throughout this proof, we use the same abuse of notation highlighted in in Remark 8.4, for example by identifying $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ with its image $\{\tau\} \times \{u\} \times \mathbb{T}^2$ in adapted rough coordinate space under the map $^{(\mathfrak{n})}\mathscr{T}$. To proceed, we let $\Phi_{(\Delta u)} = \Phi_{(\Delta u)}(\tau, u, x^2, x^3)$ be the flow map of $^{(\mathfrak{n})}\breve{R}$ relative to the adapted rough coordinates. More precisely, for each fixed $(\tau, u, x^2, x^3) \in [\tau_0, \tau_{\text{Boot}}] \times [-U_1, U_2] \times \mathbb{T}^2$, $\Phi_{(\Delta u)}(\tau, u, x^2, x^3)$ solves the ODE system $\frac{\partial}{\partial \Delta u}\Phi_{(\Delta u)}(\tau, u, x^2, x^3) = {}^{(\mathfrak{n})}\breve{R} \circ \Phi_{(\Delta u)}(\tau, u, x^2, x^3)$, with the initial condition $\Phi_{(0)}(\tau, u, x^2, x^3) = (\tau, u, x^2, x^3)$. Since $^{(\mathfrak{n})}\breve{R}\tau = 0$ and $^{(\mathfrak{n})}\breve{R}u = 1$, and since (4.2), (6.6), (7.8), Prop. 9.1, Lemma 15.5, and Lemma 15.6 imply that $\|^{(\mathfrak{n})}\breve{R}\|_{C^{1,1}_{\text{rough}}([\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C$, it follows that if $\tau \in [\tau_0, \tau_{\text{Boot}}]$, $u \in (-U_1, U_2)$, and $|\Delta u|$ is sufficiently small (depending on u), then $\Phi_{(\Delta u)}$ restricts to diffeomorphism from the rough torus $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ onto the rough torus $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u+\Delta u} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$. Hence, in view of (8.8), (8.12a), and the standard formula for change of variables in an integral over $\mathbb{T}^2$, we deduce the following identity, where $\Phi^*_{(\Delta u)}$ denotes pullback by the restriction of $\Phi_{(\Delta u)}$ to $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ (in particular, $[\Phi^*_{(\Delta u)}f](x^2, x^3) = f \circ \Phi_{(\Delta u)}(\tau, u, x^2, x^3)$), and throughout this proof, determinants are taken relative to the coordinates (x^2, x^3) on the rough tori:

$$\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u+\Delta u}} f \,\mathrm{d}\varpi_{\widetilde{g}} = \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} [\Phi^*_{(\Delta u)}f] \mathrm{d}\varpi_{\Phi^*_{(\Delta u)}\widetilde{g}} = \int_{\mathbb{T}^2} f \circ \Phi_{(\Delta u)}(\tau, u, x^2, x^3)\sqrt{\det[\Phi^*_{(\Delta u)}\widetilde{g}]}\,\mathrm{d}x^2\mathrm{d}x^3. \tag{20.3}$$

Next, using that $^{(\mathfrak{n})}\breve{R}$ is the infinitesimal generator of the flow map $\Phi_{(\Delta u)}$, and using (9.16) and the differentiation identity $\frac{d}{d\Delta u}\ln\left(\det M_{(\Delta u)}\right) = \left(M^{-1}_{(\Delta u)}\right)^{AB}\frac{d}{d\Delta u}(M_{(\Delta u)})_{AB}$ (which holds for invertible matrices $M_{(\Delta u)}$ with entries that are functions of Δu), we note the following differentiation identities: $\frac{\partial}{\partial \Delta u}|_{\Delta u=0}\Phi^*_{(\Delta u)}f = {}^{(\mathfrak{n})}\breve{R}f$ and $\frac{\partial}{\partial \Delta u}|_{\Delta u=0}\sqrt{\det[\Phi^*_{(\Delta u)}\widetilde{g}]} = \frac{1}{2}\sqrt{\det\widetilde{g}}\,\text{tr}_{\widetilde{g}}{}^{(^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}$. Using these identities, we differentiate (20.3) under the integral on the RHS to obtain:

$$\frac{\widetilde{\partial}}{\widetilde{\partial}u}\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} f \,\mathrm{d}\varpi_{\widetilde{g}} = \frac{\partial}{\partial \Delta u}\bigg|_{\Delta u=0}\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u+\Delta u}} f \,\mathrm{d}\varpi_{\widetilde{g}} = \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \left\{^{(\mathfrak{n})}\breve{R}f + \frac{1}{2} f \,\text{tr}_{\widetilde{g}}{}^{(^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}\right\} \mathrm{d}\varpi_{\widetilde{g}}. \tag{20.4}$$

We have therefore proved (20.1b).

(20.2) then follows from integrating (20.1b) with respect to u and using the fundamental theorem of calculus, (8.9), and (8.12c). □

[59]Since $^{(\mathfrak{n})}\widetilde{L}$ is $\mathbf{g}$-orthogonal to $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and $^{(\mathfrak{n})}\widetilde{L}\tau = 1$, the vectorfield $^{(\mathfrak{n})}\widetilde{L}$ agrees with the vectorfield denoted by "$\underline{\breve{H}}$" in [4, Lemma 6.3].

20.2. **Sobolev embedding and fundamental theorem of calculus-type estimates on the rough tori.**

Lemma 20.2 (Sobolev embedding and fundamental theorem of calculus-type estimates on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$)**.** *Let f be a scalar function on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$. Then the following estimates hold for $(\tau,u)\in[\tau_0,\tau_{\mathrm{Boot}})\times[-U_1,U_2]$:*

$$\|\mathcal{P}^N f\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \|\mathcal{P}^N f\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \int_{{}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau]}} \frac{1}{L^{(\mathfrak{n})}\tau}|L\mathcal{P}^N f|^2\,\mathrm{d}\overline{\varpi}, \tag{20.5}$$

$$\|f\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \|\mathcal{P}^{\leq 2} f\|_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \tag{20.6a}$$

$$\|f\|^2_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \|\mathcal{P}^{\leq 2} f\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u})} + \int_{{}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau]}} \frac{1}{L^{(\mathfrak{n})}\tau}|L\mathcal{P}^{\leq 2} f|^2\,\mathrm{d}\overline{\varpi}. \tag{20.6b}$$

Proof. Using (20.1a) with f^2 in place of f, (6.3), the pointwise estimate (18.31), and Young's inequality, we deduce $\left|\frac{\widetilde{\partial}}{\partial\tau}\|\mathcal{P}^N f\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}\right| \leq \left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}L\mathcal{P}^N f\right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} + C\|\mathcal{P}^N f\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$. Integrating this inequality with respect to τ, applying Grönwall's inequality, and also using the identity $\int_{\tau'=\tau_0}^{\tau}\left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}\mathcal{P}^N f\right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u})}\mathrm{d}\tau' = \int_{{}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau]}}\frac{1}{L^{(\mathfrak{n})}\tau}|L\mathcal{P}^N f|^2\,\mathrm{d}\overline{\varpi}$, we conclude the inequality (20.5).

We now prove (20.6a)–(20.6b). We begin with the following estimate, which holds at fixed (τ,u) by virtue of the standard Sobolev embedding result $H^2(\mathbb{T}^2)\hookrightarrow L^\infty(\mathbb{T}^2)$:

$$\|f\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \lesssim \sum_{I+J\leq 2}\left\{\int_{\mathbb{T}^2}\left|\left(\frac{\widetilde{\partial}}{\widetilde{\partial}x^2}\right)^I\left(\frac{\widetilde{\partial}}{\widetilde{\partial}x^3}\right)^J f(\tau,u,x^2,x^3)\right|^2\mathrm{d}x^2\mathrm{d}x^3\right\}^{1/2}. \tag{20.7}$$

Using (5.13c), Lemma 15.4 to estimate derivatives of τ in (5.13c), the identities (5.8c)–(5.8d), Prop. 9.1, the bootstrap assumptions, and the area form element comparison estimate (16.7a), we deduce, in view of definitions (8.8) and (8.12a), that RHS (20.7) $\lesssim \|\mathcal{P}^{\leq 2}f\|_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$. We have therefore proved (20.6a). The estimate (20.6b) then then follows from (20.6a) and (20.5) with $N=0,1,2$.

□

20.3. **Integration by parts identities.** In this section, we establish several integration by parts identities that we will exploit in our top-order energy estimates. We start with the following lemma, which provides a useful expression for the covariant divergence of a spacetime vectorfield.

Lemma 20.3 (Covariant divergence identity for spacetime vectorfields)**.** *Let $\mathscr{J}$ be a spacetime vectorfield. Consider the decomposition $\mu\mathscr{J} = -\mu\mathscr{J}_L L - \mathscr{J}_{\check{X}}L - \mathscr{J}_L\check{X} + \mu{\not\Pi}\mathscr{J}$ afforded by Lemma 3.9, where $\mathscr{J}_L = \mathbf{g}(\mathscr{J},L)$, $\mathscr{J}_{\check{X}} = \mathbf{g}(\mathscr{J},\check{X})$, and ${\not\Pi}\mathscr{J}$ is the $\ell_{t,u}$-projection of $\mathscr{J}$ (see Def. 3.3). Then the following identity holds:*

$$\mu\mathbf{D}_\alpha\mathscr{J}^\alpha = -L(\mu\mathscr{J}_L) - L(\mathscr{J}_{\check{X}}) - \check{X}(\mathscr{J}_L) + \mathrm{d}\!\!\not{\mathrm{i}}\mathrm{v}(\mu{\not\Pi}\mathscr{J}) - \mu\mathrm{tr}_{\not{g}}\not{k}\,\mathscr{J}_L - \mathrm{tr}_{\not{g}}\chi\,\mathscr{J}_{\check{X}}. \tag{20.8}$$

Proof. The same proof of [73, Lemma 4.3] holds with minor modifications to account for the third space dimension. □

The following lemma provides some preliminary integration by parts identities.

Lemma 20.4 (Preliminary integration by parts identities)**.** *Let $(\tau,u)\in[\tau_0,\tau_{\mathrm{Boot}})\times[-U_1,U_2]$, and let υ and ζ be scalar functions on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}$. Then the following integration by parts identities hold, where in* (20.9b)*, $A=2,3$, and*

$\mathrm{tr}_{\not{g}}{}^{(Y_{(A)})}\not{\pi}$ *denotes the* $\not{g}$*-trace of the* $\ell_{t,u}$*-projection of the deformation tensor of* $Y_{(A)}$*:*

$$\begin{aligned}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} ({}^{(\mathfrak{n})}\widetilde{L}\upsilon)\zeta\,\mathrm{d}\varpi &= -\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \upsilon({}^{(\mathfrak{n})}\widetilde{L}\zeta)\,\mathrm{d}\varpi \\ &\quad -\frac{1}{2}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \mathrm{tr}_{\widetilde{\not{g}}}{}^{({}^{(\mathfrak{n})}\widetilde{L})}\boldsymbol{\pi}\upsilon\zeta\,\mathrm{d}\varpi \\ &\quad + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \upsilon\zeta\,\mathrm{d}\underline{\varpi} - \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{u}} \upsilon\zeta\,\mathrm{d}\underline{\varpi},\end{aligned} \tag{20.9a}$$

$$\begin{aligned}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}(Y_{(A)}\upsilon)\zeta\,\mathrm{d}\varpi &= -\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\upsilon(Y_{(A)}\zeta)\,\mathrm{d}\varpi \\ &\quad -\frac{1}{2}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\mathrm{tr}_{\not{g}}{}^{(Y_{(A)})}\not{\pi}\upsilon\zeta\,\mathrm{d}\varpi \\ &\quad + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}(Y_{(A)}{}^{(\mathfrak{n})}\tau)\upsilon\zeta\,\mathrm{d}\underline{\varpi} - \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{u}} \frac{1}{L^{(\mathfrak{n})}\tau}(Y_{(A)}{}^{(\mathfrak{n})}\tau)\upsilon\zeta\,\mathrm{d}\underline{\varpi}.\end{aligned} \tag{20.9b}$$

Proof. The identity (20.9a) follows from setting $f \stackrel{\text{def}}{=} \upsilon\zeta$ in (20.1a), then integrating the resulting identity with respect to τ and then with respect to u, then using Fubini's theorem to switch the order of τ and u integrations, and appealing to the definitions of the forms in Def. 8.3.

We now prove (20.9b). Since $Y_{(A)}$ is $\mathcal{P}_u$-tangent, we can expand it in terms of the adapted rough coordinate partial derivative vectorfields as follows:

$$Y_{(A)} = (Y_{(A)}{}^{(\mathfrak{n})}\tau)\frac{\widetilde{\partial}}{\widetilde{\partial}\tau} + YvfA^B\frac{\widetilde{\partial}}{\widetilde{\partial}x^B}. \tag{20.10}$$

Next, using (20.10) and the fact that relative to the adapted rough coordinates, we have $\sqrt{|\det \mathbf{g}|} = \frac{\mu}{L^{(\mathfrak{n})}\tau}\sqrt{\det\widetilde{\not{g}}}$ (see (8.13b)), we can expand the product of the covariant divergence of the vectorfield $\frac{1}{\mu}\upsilon\zeta Y_{(A)}$ and $\sqrt{|\det \mathbf{g}|}$ as follows (again, relative to the adapted rough coordinates):

$$\sqrt{|\det \mathbf{g}|}\mathbf{D}_\alpha\left(\frac{1}{\mu}\upsilon\zeta Y_{(A)}\right)^\alpha = \frac{\widetilde{\partial}}{\widetilde{\partial}\tau}\left(\frac{1}{L^{(\mathfrak{n})}\tau}(Y_{(A)}{}^{(\mathfrak{n})}\tau)\upsilon\zeta\sqrt{\det\widetilde{\not{g}}}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial}x^B}\left(\frac{1}{L^{(\mathfrak{n})}\tau}Y_{(A)}^B\upsilon\zeta\sqrt{\det\widetilde{\not{g}}}\right). \tag{20.11}$$

Through a straightforward Leibniz-rule-based expansion, we also have:

$$\sqrt{|\det \mathbf{g}|}\mathbf{D}_\alpha\left(\frac{1}{\mu}\upsilon\zeta Y_{(A)}\right)^\alpha = \left\{\frac{1}{L^{(\mathfrak{n})}\tau}(Y_{(A)}\upsilon)\zeta + \frac{1}{L^{(\mathfrak{n})}\tau}\upsilon(Y_{(A)}\zeta) - \frac{1}{L^{(\mathfrak{n})}\tau}\frac{Y_{(A)}\mu}{\mu}\upsilon\zeta + \frac{1}{L^{(\mathfrak{n})}\tau}\upsilon\zeta\left(\mathbf{D}_\alpha Y_{(A)}^\alpha\right)\right\}\sqrt{\det\widetilde{\not{g}}}. \tag{20.12}$$

Next, we use (20.8), the fact that $Y_{(A)}$ is $\ell_{t,u}$-tangent, and Lemma 3.9 to express the term $\mathbf{D}_\alpha Y_{(A)}^\alpha$ on RHS (20.12) as follows, where $\mathrm{tr}_{\not{g}}{}^{(Y_{(A)})}\not{\pi}$ is the $\not{g}$-trace of the $\ell_{t,u}$-projection of the deformation tensor of $Y_{(A)}$:

$$\mathbf{D}_\alpha Y_{(A)}^\alpha = \frac{1}{\mu}Y_{(A)}\mu + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\, Y_{(A)} = \frac{1}{\mu}Y_{(A)}\mu + \frac{1}{2}\mathrm{tr}_{\not{g}}{}^{(Y_{(A)})}\not{\pi}. \tag{20.13}$$

Using (20.13) to substitute for the factor $\mathbf{D}_\alpha Y_{(A)}^\alpha$ on RHS (20.12) and noting that the factor $\frac{1}{\mu}Y_{(A)}\mu$ on RHS (20.13) leads to the cancellation of the product $-\frac{1}{L^{(\mathfrak{n})}\tau}\frac{Y_{(A)}\mu}{\mu}\upsilon\zeta$ in (20.12), we deduce that:

$$\sqrt{|\det \mathbf{g}|}\mathbf{D}_\alpha\left(\frac{1}{\mu}\upsilon\zeta Y_{(A)}\right)^\alpha = \left\{\frac{1}{L^{(\mathfrak{n})}\tau}(Y_{(A)}\upsilon)\zeta + \frac{1}{L^{(\mathfrak{n})}\tau}\upsilon(Y_{(A)}\zeta) + \frac{1}{2}\frac{1}{L^{(\mathfrak{n})}\tau}\upsilon\zeta\mathrm{tr}_{\not{g}}{}^{(Y_{(A)})}\not{\pi}\right\}\sqrt{\det\widetilde{\not{g}}}. \tag{20.14}$$

Next, we integrate RHS (20.11) over $[\tau_0,\tau)\times[-U_1,u]\times\mathbb{T}^2$ with respect to $\mathrm{d}x^2\,\mathrm{d}x^3\,\mathrm{d}u'\,\mathrm{d}\tau'$, equate it to the integral of RHS (20.14), and use Fubini's theorem. In view of definitions (8.8) and (8.11), we see that the spacetime integrals of the terms on RHS (20.14) appear in (20.9b). The integral of the last term on RHS (20.11) vanishes because $\mathbb{T}^2$ is a closed manifold. Finally, we note that the integral of the first term on RHS (20.11) yields, in view of (8.8), (8.12c), and the fundamental theorem of calculus, the two hypersurface integrals in (20.9b).

□

We are now ready to establish the integration by parts identity that we will use to control top-order wave equation error terms that are tied to the partially modified quantities defined in (19.7a).

Lemma 20.5 (The key integration by parts identity tied to the partially modified quantities)**.** *Let $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, and let φ and η be scalar functions on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}$. Let $N = N_{\text{top}}$, and let $\mathcal{P}^N \in \mathfrak{P}^{(N)}$, where $\mathfrak{P}^{(N)}$ is the set of order N $\mathcal{P}_u$-tangential commutator operators from Def. 8.10. Let $Y_{(A)} \in \mathscr{Y}$. Then the following integration by parts identity holds:*

$$
\begin{aligned}
&\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} (1+2\mu)(\breve{X}\varphi)({}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^N\varphi)Y_{(A)}\eta\,\mathrm{d}\varpi \\
&= \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} (1+2\mu)(\breve{X}\varphi)(Y_{(A)}\mathcal{P}^N\varphi){}^{(\mathfrak{n})}\widetilde{L}\eta\,\mathrm{d}\varpi \\
&\quad - \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} (1+2\mu)(\breve{X}\varphi)(Y_{(A)}\mathcal{P}^N\varphi)\eta\,\mathrm{d}\underline{\varpi} + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} (Y_{(A)}{}^{(\mathfrak{n})}\tau)(1+2\mu)(\breve{X}\varphi)({}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^N\varphi)\eta\,\mathrm{d}\underline{\varpi} \\
&\quad + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{u}} (1+2\mu)(\breve{X}\varphi)(Y_{(A)}\mathcal{P}^N\varphi)\eta\,\mathrm{d}\underline{\varpi} - \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{u}} (Y_{(A)}{}^{(\mathfrak{n})}\tau)(1+2\mu)(\breve{X}\varphi)({}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^N\varphi)\eta\,\mathrm{d}\underline{\varpi} \\
&\quad + \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \mathrm{Error}[\varphi;\eta;\mathcal{P}^N;Y_{(A)}]\,\mathrm{d}\varpi,
\end{aligned}
\tag{20.15}
$$

where:

$$
\begin{aligned}
\mathrm{Error}[\varphi;\eta;\mathcal{P}^N;Y_{(A)}] &\overset{\text{def}}{=} \frac{1}{L^{(\mathfrak{n})}\tau}(1+2\mu)(\breve{X}\varphi)([L,Y_{(A)}]\mathcal{P}^N\varphi)\eta \\
&\quad + 2\frac{1}{L^{(\mathfrak{n})}\tau}(L\mu)(\breve{X}\varphi)(Y_{(A)}\mathcal{P}^N\varphi)\eta + \frac{1}{L^{(\mathfrak{n})}\tau}(1+2\mu)(L\breve{X}\varphi)(Y_{(A)}\mathcal{P}^N\varphi)\eta \\
&\quad - 2\frac{1}{L^{(\mathfrak{n})}\tau}(Y_{(A)}\mu)(\breve{X}\varphi)(L\mathcal{P}^N\varphi)\eta - \frac{1}{L^{(\mathfrak{n})}\tau}(1+2\mu)(Y_{(A)}\breve{X}\varphi)(L\mathcal{P}^N\varphi)\eta \\
&\quad - \frac{1}{2}\frac{1}{L^{(\mathfrak{n})}\tau}\mathrm{tr}_{\not{g}}{}^{(Y_{(A)})}\not{\pi}(1+2\mu)(\breve{X}\varphi)(L\mathcal{P}^N\varphi)\eta \\
&\quad + \frac{1}{2}\mathrm{tr}_{\widetilde{\not{g}}}{}^{({}^{(\mathfrak{n})}\widetilde{L})}\boldsymbol{\pi}(1+2\mu)(\breve{X}\varphi)(Y_{(A)}\mathcal{P}^N\varphi)\eta.
\end{aligned}
\tag{20.16}
$$

Proof. We will summarize the tedious but straightforward calculations that yield (20.15). First, we use definition (6.3) and the integration by parts identity (20.9b) with $\upsilon \overset{\text{def}}{=} \eta$ and $\zeta \overset{\text{def}}{=} (1+2\mu)(\breve{X}\varphi)L\mathcal{P}^N\varphi$ to remove the $Y_{(A)}$ operator from η on LHS (20.15). This produces the main integral $-\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}(1+2\mu)(\breve{X}\varphi)(Y_{(A)}L\mathcal{P}^N\varphi)\eta\,\mathrm{d}\varpi$ as well as many error integrals. We then commute $Y_{(A)}$ and L and appeal to definition (6.3) to rewrite the main integral as follows:

$$
\begin{aligned}
&- \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} (1+2\mu)(\breve{X}\varphi)({}^{(\mathfrak{n})}\widetilde{L}Y_{(A)}\mathcal{P}^N\varphi)\eta\,\mathrm{d}\varpi \\
&+ \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}(1+2\mu)(\breve{X}\varphi)([L,Y_{(A)}]\mathcal{P}^N\varphi)\eta\,\mathrm{d}\varpi.
\end{aligned}
\tag{20.17}
$$

Finally, we use the integration by parts identity (20.9a) $\upsilon \overset{\text{def}}{=} Y_{(A)}\mathcal{P}^N\varphi$ and $\zeta \overset{\text{def}}{=} (1+2\mu)(\breve{X}\varphi)\eta$ to remove the factor of ${}^{(\mathfrak{n})}\widetilde{L}$ from the factor ${}^{(\mathfrak{n})}\widetilde{L}Y_{(A)}\mathcal{P}^N\varphi$ in the first integral in (20.17). □

20.4. **Fundamental energy identity.** In this section, we derive the fundamental energy-null-flux integral identities that we will use to derive hyperbolic L^2-type estimates for the wave- and transport-variables.

20.4.1. *Energy-momentum tensor, energy currents, and the multiplier vectorfield for the wave-variables.* To derive energy identities for the wave-variables $\vec{\Psi}$, which solve the quasilinear wave equations (2.22a)–(2.22d), we rely on the well-known multiplier method, which we now introduce.

Let f be a scalar function (in our applications, f will be some vectorfield derivative of one of the wave-variables). We define the *energy-momentum tensor* associated to f to be the following symmetric type $\binom{0}{2}$ tensorfield, where we recall that $\mathbf{D}$ is the Levi-Civita connection of the acoustical metric $\mathbf{g}$:

$$\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\alpha\beta}[f] \stackrel{\text{def}}{=} (\mathbf{D}_\alpha f)\mathbf{D}_\beta f - \frac{1}{2}\mathbf{g}_{\alpha\beta}(\mathbf{g}^{-1})^{\kappa\lambda}(\mathbf{D}_\kappa f)\mathbf{D}_\lambda f. \tag{20.18}$$

Given any scalar function f and any *multiplier vectorfield* Z, we define the corresponding *energy current* vectorfield as follows:

$$^{(Z)}\mathbf{J}^\alpha[f] \stackrel{\text{def}}{=} \mathbf{Q}^{\alpha\beta}[f]Z_\beta. \tag{20.19}$$

Recall that the deformation tensor of Z is the following symmetric type $\binom{0}{2}$ tensorfield:

$$^{(Z)}\boldsymbol{\pi}_{\alpha\beta} \stackrel{\text{def}}{=} \mathbf{D}_\alpha Z_\beta + \mathbf{D}_\beta Z_\alpha. \tag{20.20}$$

The starting point for our derivation of our L^2-type integral identities for solutions of covariant wave equations is the following well-known identity, which follows easily from the definitions, the Leibniz rule, and the fact that $\mathbf{D}_\alpha\mathbf{D}_\beta f = \mathbf{D}_\beta\mathbf{D}_\alpha f$ for scalar functions f:

$$\mathbf{D}_\alpha {}^{(Z)}\mathbf{J}^\alpha[f] = (\square_{\mathbf{g}(\vec{\Psi})} f)Zf + \frac{1}{2}\mathbf{Q}^{\alpha\beta\,(Z)}\boldsymbol{\pi}_{\alpha\beta}. \tag{20.21}$$

In order to obtain wave equation energy estimates that are sufficient to allow us to track the solution up to the singular boundary, we use the multiplier vectorfield from the next definition.

Definition 20.6 (The $\mathbf{g}$-timelike multiplier vectorfield $\breve{T}$)**.** We define $\breve{T}$ to be the following vectorfield:

$$\breve{T} \stackrel{\text{def}}{=} (1 + 2\upmu)L + 2\breve{X}. \tag{20.22}$$

Simple calculations based on Lemma 3.9 imply that $\mathbf{g}(\breve{T}, \breve{T}) = -4\upmu(1 + \upmu)$. Hence, $\breve{T}$ is $\mathbf{g}$-timelike whenever $\upmu > 0$. This property is important because it leads to coercive energy identities.

20.4.2. *Building block energies, null-fluxes, and spacetime integrals.* We are now ready to define our building block energies and null-fluxes for the fluid variables. We also define spacetime integrals that play a crucial role in our energy estimates; see Def. 20.8.

Definition 20.7 (Energies and null-fluxes for the fluid variables)**.** Let f be a scalar function on ${}^{(\mathfrak{n})}\mathcal{M}_{[\uptau_0,\uptau_{\text{Boot}}),[-U_1,U_2]}$. Recall that ${}^{(\mathfrak{n})}\hat{N}$ denotes the future-directed $\mathbf{g}$-timelike unit normal of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\uptau^{[-U_1,u]}$, that $\breve{T}$ denotes the multiplier vectorfield defined in (20.22), that the area forms $d\underline{\varpi}$ and $d\overline{\varpi}$ are defined in Def. 8.3, and that $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ is as in (6.20a). For $(\uptau, u) \in [\uptau_0, \uptau_{\text{Boot}}) \times [-U_1, U_2]$, we respectively define the *wave energy* and the *null-flux* associated to f as follows:

$$\mathbb{E}_{(\text{Wave})}[f](\uptau, u) \stackrel{\text{def}}{=} \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\uptau^{[-U_1,u]}} \mathbf{Q}[f](\breve{T}, {}^{(\mathfrak{n})}\hat{N})|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}\, d\underline{\varpi}, \tag{20.23a}$$

$$\mathbb{F}_{(\text{Wave})}[f](\uptau, u) \stackrel{\text{def}}{=} \int_{{}^{(\mathfrak{n})}\mathcal{P}_u^{[\uptau_0,\uptau)}} \frac{1}{L^{(\mathfrak{n})}\uptau}\mathbf{Q}[f](\breve{T}, L)\, d\overline{\varpi}. \tag{20.23b}$$

If f is a scalar function on ${}^{(\mathfrak{n})}\mathcal{M}_{[\uptau_0,\uptau_{\text{Boot}}),[-U_1,U_2]}$ and $(\uptau, u) \in [\uptau_0, \uptau_{\text{Boot}}) \times [-U_1, U_2]$, then we respectively define the *transport energy* and the null-flux associated to f as follows, where ϕ is the cut-off function introduced in Def. 4.1:

$$\mathbb{E}_{(\text{Transport})}[f](\uptau, u) \stackrel{\text{def}}{=} \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\uptau^{[-U_1,u]}} \left(\upmu - \phi\frac{\mathfrak{n}}{L\upmu}\right) f^2\, d\underline{\varpi}, \tag{20.24a}$$

$$\mathbb{F}_{(\text{Transport})}[f](\uptau, u) \stackrel{\text{def}}{=} \int_{{}^{(\mathfrak{n})}\mathcal{P}_u^{[\uptau_0,\uptau)}} \frac{1}{L^{(\mathfrak{n})}\uptau} f^2\, d\overline{\varpi}. \tag{20.24b}$$

The integrals appearing in the next definition yield spacetime L^2-control of $\not{d} f$ *without degenerate* $\upmu$ *weights.* These integrals arise from favorable terms in the wave equation energy identities (see Prop. 20.9). They are of crucial importance for our energy estimates.

Definition 20.8 (Key coercive spacetime integrals). If f is a scalar function on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ and $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$, then we define the following spacetime integral, where $\not{d}f$ is as in Def. 3.10, $\mathbf{1}_{[-U_{\clubsuit},U_{\clubsuit}]}=\mathbf{1}_{[-U_{\clubsuit},U_{\clubsuit}]}(u')$ denotes the characteristic function of the interval $[-U_{\clubsuit},U_{\clubsuit}]$, $\phi=\phi(u')$ is the cut-off function from Def. 4.1, the vectorfield ${}^{(\mathfrak{n})}\widetilde{L}$ is defined in (6.3), and the volume form $\mathrm{d}\varpi=\mathrm{d}\varpi(\tau',u',x^2,x^3)$ is defined in (8.11):

$$\mathbb{K}[f](\tau,u)\overset{\text{def}}{=}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\left\{-\frac{1}{2}\mathbf{1}_{[-U_{\clubsuit},U_{\clubsuit}]}({}^{(\mathfrak{n})}\widetilde{L}\mu)+\frac{1}{L^{(\mathfrak{n})}\tau}\mathfrak{n}\phi\right\}|\not{d}f|^2_{\not{g}}\,\mathrm{d}\varpi. \tag{20.25}$$

20.4.3. *The fundamental energy–null-flux integral identities.* In the next proposition, we provide the fundamental energy–null-flux integral identities that form the foundation of our hyperbolic energy estimates for the fluid variables.

Proposition 20.9 (Fundamental energy–null-flux identities).
Wave equation energy–null-flux identity. *Suppose that on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, the scalar function f is a solution to the inhomogeneous covariant wave equation $\mu\square_{\mathbf{g}}f=\mathfrak{G}$. Let $\mathbb{E}_{(\text{Wave})}[f](\tau,u)$ and $\mathbb{F}_{(\text{Wave})}[f](\tau,u)$ be as defined in Def. 20.7, and let $\mathbb{K}[f](\tau,u)$ be as defined in* (20.25). *Then for $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$, the following identity holds:*

$$\begin{aligned}\mathbb{E}_{(\text{Wave})}[f](\tau,u)+\mathbb{F}_{(\text{Wave})}[f](\tau,u)+\mathbb{K}[f](\tau,u)&=\mathbb{E}_{(\text{Wave})}[f](\tau_0,u)+\mathbb{F}_{(\text{Wave})}[f](\tau,-U_1)\\&\quad-\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)Lf+2\breve{X}f\right\}\mathfrak{G}\,\mathrm{d}\varpi\\&\quad+\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}{}^{(\breve{T})}\mathfrak{B}[f]\,\mathrm{d}\varpi.\end{aligned} \tag{20.26}$$

The error term ${}^{(\breve{T})}\mathfrak{B}[f]$ appearing in the last integral on RHS (20.26) *can be decomposed as follows, where $\mathbf{1}_{[-U_{\clubsuit},U_{\clubsuit}]^c}=\mathbf{1}_{[-U_{\clubsuit},U_{\clubsuit}]^c}(u')$ denotes the characteristic function of $[-U_{\clubsuit},U_{\clubsuit}]^c=(-\infty,-U_{\clubsuit})\cup(U_{\clubsuit},\infty)$:*

$${}^{(\breve{T})}\mathfrak{B}[f]\overset{\text{def}}{=}\frac{1}{2}\mathbf{1}_{[-U_{\clubsuit},U_{\clubsuit}]^c}(L\mu)|\not{d}f|^2_{\not{g}}+\sum_{i=1}^{6}{}^{(\breve{T})}\mathfrak{B}_{(i)}[f], \tag{20.27}$$

where:

$${}^{(\breve{T})}\mathfrak{B}_{(1)}[f]\overset{\text{def}}{=}(Lf)^2\left\{-\frac{1}{2}L\mu+\breve{X}\mu-\frac{1}{2}\mu\mathrm{tr}_{\not{g}}\chi-\mu^2\mathrm{tr}_{\not{g}}\not{k}^{(\text{Tan}-\vec{\Psi})}-\mu\mathrm{tr}_{\not{g}}\not{k}^{(\text{Trans}-\vec{\Psi})}\right\}, \tag{20.28a}$$

$${}^{(\breve{T})}\mathfrak{B}_{(2)}[f]\overset{\text{def}}{=}-(Lf)(\breve{X}f)\left\{\mathrm{tr}_{\not{g}}\chi+2\mu\mathrm{tr}_{\not{g}}\not{k}^{(\text{Tan}-\vec{\Psi})}+2\mathrm{tr}_{\not{g}}\not{k}^{(\text{Trans}-\vec{\Psi})}\right\}, \tag{20.28b}$$

$${}^{(\breve{T})}\mathfrak{B}_{(3)}[f]\overset{\text{def}}{=}|\not{d}f|^2_{\not{g}}\left\{{}^{(\mathfrak{n})}\breve{R}\mu+\mu{}^{(\mathfrak{n})}U\mu+2\mu L\mu+\frac{1}{2}\mu\mathrm{tr}_{\not{g}}\chi+\mu^2\mathrm{tr}_{\not{g}}\not{k}^{(\text{Tan}-\vec{\Psi})}+\mu\mathrm{tr}_{\not{g}}\not{k}^{(\text{Trans}-\vec{\Psi})}\right\}, \tag{20.28c}$$

$${}^{(\breve{T})}\mathfrak{B}_{(4)}[f]\overset{\text{def}}{=}(Lf)(\not{d}^{\#}f)\cdot\left\{(1-2\mu)\not{d}\mu+2\mu\zeta^{(\text{Tan}-\vec{\Psi})}+2\zeta^{(\text{Trans}-\vec{\Psi})}\right\}, \tag{20.28d}$$

$${}^{(\breve{T})}\mathfrak{B}_{(5)}[f]\overset{\text{def}}{=}-2(\breve{X}f)(\not{d}^{\#}f)\cdot\left\{\not{d}\mu+2\mu\zeta^{(\text{Tan}-\vec{\Psi})}+2\zeta^{(\text{Trans}-\vec{\Psi})}\right\}, \tag{20.28e}$$

$${}^{(\breve{T})}\mathfrak{B}_{(6)}[f]\overset{\text{def}}{=}-\mu\not{d}^{\#}f\otimes\not{d}^{\#}f\cdot\left\{\chi+2\mu\not{k}^{(\text{Tan}-\vec{\Psi})}+2\not{k}^{(\text{Trans}-\vec{\Psi})}\right\}. \tag{20.28f}$$

In (20.28a)–(20.28f) *and below in* (20.29), *the vectorfields ${}^{(\mathfrak{n})}\breve{R}$ and ${}^{(\mathfrak{n})}U$ are as in Def. 6.4 and the $\ell_{t,u}$-tangent tensorfields χ, $\not{k}^{(\text{Tan}-\vec{\Psi})}$, $\not{k}^{(\text{Trans}-\vec{\Psi})}$, $\zeta^{(\text{Tan}-\vec{\Psi})}$, and $\zeta^{(\text{Trans}-\vec{\Psi})}$ are as in Lemma 3.25.*

Transport equation energy–null-flux identity. *Suppose that on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, f is a solution to the inhomogeneous transport equation $\mu\mathbf{B}f=\mathfrak{G}$, and let $\mathbb{E}_{(\text{Transport})}[f](\tau,u)$ and $\mathbb{F}_{(\text{Transport})}[f](\tau,u)$ be as defined in Def. 20.7. Then for $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$, the following identity holds:*

$$\begin{aligned}\mathbb{E}_{(\text{Transport})}[f](\tau,u)+\mathbb{F}_{(\text{Transport})}[f](\tau,u)&=\mathbb{E}_{(\text{Transport})}[f](\tau_0,u)+\mathbb{F}_{(\text{Transport})}[f](\tau,-U_1)\\&\quad+2\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}f\cdot\mathfrak{G}\,\mathrm{d}\varpi\\&\quad+\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}\left\{L\mu+\mu\mathrm{tr}_{\not{g}}\not{k}\right\}f^2\,\mathrm{d}\varpi.\end{aligned} \tag{20.29}$$

Proof.

Proof of (20.26): We will apply the covariant divergence identity (20.21) with $Z \stackrel{\text{def}}{=} \breve{T}$, where $\breve{T}$ is defined in (20.22). Throughout, we often use the abbreviated notation $\mathbf{J} \stackrel{\text{def}}{=} {}^{(\breve{T})}\mathbf{J}[f]$ to denote the energy current defined by (20.19). To proceed, we first write $\mathbf{J}$ in terms of the adapted rough coordinate partial derivative vectorfields as follows:

$$\mathbf{J} = \mathbf{J}^{\tau}\frac{\widetilde{\partial}}{\widetilde{\partial}\tau} + \mathbf{J}^{u}\frac{\widetilde{\partial}}{\widetilde{\partial}u} + \mathbf{J}^{A}\frac{\widetilde{\partial}}{\widetilde{\partial}x^{A}}. \tag{20.30}$$

Next, we claim that the following identities hold, where $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ is as in (6.20a):

$$\mathbf{J}^{\tau} = -\frac{(L^{(\mathfrak{n})}\tau)|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}{\mu}\mathbf{Q}[f](\breve{T}, {}^{(\mathfrak{n})}\hat{N}), \tag{20.31a}$$

$$\mathbf{J}^{u} = -\frac{1}{\mu}\mathbf{Q}[f](\breve{T}, L). \tag{20.31b}$$

To prove (20.31a), we first note that since ${}^{(\mathfrak{n})}\hat{N}$ is $\mathbf{g}$-orthogonal to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}$ while $\frac{\widetilde{\partial}}{\widetilde{\partial}u}$ and $\frac{\widetilde{\partial}}{\widetilde{\partial}x^{A}}$ are tangent to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}$, we have, in view of (20.19) and (20.30):

$$\mathbf{Q}[f](\breve{T}, {}^{(\mathfrak{n})}\hat{N}) = \mathbf{g}\left({}^{(\breve{T})}\mathbf{J}[f], {}^{(\mathfrak{n})}\hat{N}\right) = \mathbf{J}^{\tau}\mathbf{g}\left(\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}, {}^{(\mathfrak{n})}\hat{N}\right). \tag{20.32}$$

Next, we use (6.3)–(6.4) and the fact that $Lu = \frac{\widetilde{\partial}}{\widetilde{\partial}\tau}u = 0$ to deduce the identity $\frac{\widetilde{\partial}}{\widetilde{\partial}\tau} = \frac{1}{L^{(\mathfrak{n})}\tau}L - \frac{1}{L^{(\mathfrak{n})}\tau}L^{C}\frac{\widetilde{\partial}}{\widetilde{\partial}x^{C}}$, and we use Lemma 3.9, (6.7), (6.20d), and (7.9) to deduce the identity $\mathbf{g}(L, {}^{(\mathfrak{n})}\hat{N}) = -\frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}$. Combining these two identities with (20.32) and using that $\mathbf{g}\left(\frac{\widetilde{\partial}}{\widetilde{\partial}x^{C}}, {}^{(\mathfrak{n})}\hat{N}\right) = 0$, we find that:

$$\mathbf{Q}[f](\breve{T}, {}^{(\mathfrak{n})}\hat{N}) = \frac{1}{L^{(\mathfrak{n})}\tau}\mathbf{J}^{\tau}\mathbf{g}(L, {}^{(\mathfrak{n})}\hat{N}) = -\frac{\mu}{(L^{(\mathfrak{n})}\tau)|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\mathbf{J}^{\tau}, \tag{20.33}$$

which yields (20.31a). The identity (20.31b) can be proved using similar arguments based on taking the $\mathbf{g}$-inner product of ${}^{(\breve{T})}\mathbf{J}[f]$ with L and using that L is $\mathbf{g}$-orthogonal to $\mathcal{P}_u$ as well as the identity $\frac{\widetilde{\partial}}{\widetilde{\partial}u} = {}^{(\mathfrak{n})}\breve{R} - {}^{(\mathfrak{n})}\breve{R}^{C}\frac{\widetilde{\partial}}{\widetilde{\partial}x^{C}}$, which follows from Lemma 3.9 and (6.5)–(6.6).

Next, we note the following formula, which follows by combining the standard identity for the divergence of a vectorfield expressed relative to the adapted rough coordinates with the identities (8.13b) and (20.31a)–(20.31b):

$$\begin{aligned}
\sqrt{|\det\mathbf{g}|}\,\mathbf{D}_{\alpha}{}^{(\breve{T})}\mathbf{J}^{\alpha}[f] &= \frac{\widetilde{\partial}}{\widetilde{\partial}\tau}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{\tau}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial}u}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{u}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial}x^{A}}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{A}\right)\\
&= -\frac{\widetilde{\partial}}{\widetilde{\partial}\tau}\left(\mathbf{Q}[f](\breve{T}, {}^{(\mathfrak{n})}\hat{N})|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}\sqrt{\det\widetilde{g\!\!/}}\right) - \frac{\widetilde{\partial}}{\widetilde{\partial}u}\left(\frac{1}{L^{(\mathfrak{n})}\tau}\mathbf{Q}[f](\breve{T}, L)\sqrt{\det\widetilde{g\!\!/}}\right)\\
&\quad + \frac{\widetilde{\partial}}{\widetilde{\partial}x^{A}}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{A}\right).
\end{aligned} \tag{20.34}$$

Integrating (20.34) over ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}$ with respect to $\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}u'\mathrm{d}\tau'$, using (8.14b) to relate the canonical volume form $\mathrm{dvol}_{\mathbf{g}}$ to $\mathrm{d}\varpi$, using (20.21) to substitute for $\mathbf{D}_{\alpha}{}^{(\breve{T})}\mathbf{J}^{\alpha}[f]$ on LHS (20.34), applying Fubini's theorem, and using that the integral of the last term on RHS (20.34) over $\mathbb{T}^2$ vanishes (since $\mathbb{T}^2$ is a closed manifold), we deduce, in view of definitions (20.23a)–(20.23b) and (20.22), the following identity:

$$\begin{aligned}
\mathbb{E}_{(\mathrm{Wave})}[f](\tau,u) + \mathbb{F}_{(\mathrm{Wave})}[f](\tau,u) &= \mathbb{E}_{(\mathrm{Wave})}[f](\tau_0,u) + \mathbb{F}_{(\mathrm{Wave})}[f](\tau,-U_1)\\
&\quad - \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)Lf + 2\breve{X}f\right\}\mathfrak{G}\,\mathrm{d}\varpi\\
&\quad - \frac{1}{2}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\mu\mathbf{Q}^{\alpha\beta}[f]{}^{(\breve{T})}\boldsymbol{\pi}_{\alpha\beta}\,\mathrm{d}\varpi.
\end{aligned} \tag{20.35}$$

Next, we decompose the integrand in the last integral on RHS (20.35), namely $-\frac{1}{2}\frac{1}{L^{(\mathfrak{n})}\tau}\mu(\mathbf{g}^{-1})^{\alpha\delta}(\mathbf{g}^{-1})^{\beta\sigma}\mathbf{Q}_{\delta\sigma}[f]^{(\check{T})}\boldsymbol{\pi}_{\alpha\beta}$, using definitions (20.18), (20.20), and (20.22), the identity $(\mathbf{g}^{-1})^{\alpha\delta} = -L^{\alpha}L^{\delta} - L^{\alpha}X^{\delta} - X^{\alpha}L^{\delta} + (g\!\!\!/^{-1})^{\alpha\delta}$ (see (3.34b)), and the analogous identity for $(\mathbf{g}^{-1})^{\beta\sigma}$. Among the many terms that arise from the expansion are the following two:[60]

$$\frac{1}{2}\frac{1}{L^{(\mathfrak{n})}\tau}(L\mu)|d\!\!/ f|_{g\!\!\!/}^2 + \frac{1}{L^{(\mathfrak{n})}\tau}(\check{X}\mu)|d\!\!/ f|_{g\!\!\!/}^2. \tag{20.36}$$

We now decompose the first product in (20.36) as follows, where we use definition (6.3):

$$\frac{1}{2}\frac{1}{L^{(\mathfrak{n})}\tau}(L\mu)|d\!\!/ f|_{g\!\!\!/}^2 = \frac{1}{2}\mathbf{1}_{[-U_{\maltese},U_{\maltese}]}({}^{(\mathfrak{n})}\widetilde{L}\mu)|d\!\!/ f|_{g\!\!\!/}^2 + \frac{1}{2}\mathbf{1}_{[-U_{\maltese},U_{\maltese}]^c}\frac{1}{L^{(\mathfrak{n})}\tau}(L\mu)|d\!\!/ f|_{g\!\!\!/}^2. \tag{20.37}$$

Since the spacetime integral of $\frac{1}{2}\mathbf{1}_{[-U_{\maltese},U_{\maltese}]}({}^{(\mathfrak{n})}\widetilde{L}\mu)|d\!\!/ f|_{g\!\!\!/}^2$ is found in the *negative* of the coercive spacetime integral defined in (20.25) (i.e., $-\mathbb{K}[f](\tau,u)$), we bring this term to LHS (20.35) as part of $\mathbb{K}[f](\tau,u)$. The remaining term $\frac{1}{2}\mathbf{1}_{[-U_{\maltese},U_{\maltese}]}\frac{1}{L^{(\mathfrak{n})}\tau}(L\mu)|d\!\!/ f|_{g\!\!\!/}^2$ in the decomposition (20.37) is manifestly present on RHS (20.26) as the first term in RHS (20.27). Next, we examine the term $\frac{1}{L^{(\mathfrak{n})}\tau}(\check{X}\mu)|d\!\!/ f|_{g\!\!\!/}^2$ present in (20.36). Using (6.6), we express this term as follows:

$$\frac{1}{L^{(\mathfrak{n})}\tau}(\check{X}\mu)|d\!\!/ f|_{g\!\!\!/}^2 = -\frac{1}{L^{(\mathfrak{n})}\tau}\phi\mathfrak{n}|d\!\!/ f|_{g\!\!\!/}^2 + \frac{1}{L^{(\mathfrak{n})}\tau}\left\{{}^{(\mathfrak{n})}\check{R}\mu + \mu^{(\mathfrak{n})}U\mu\right\}|d\!\!/ f|_{g\!\!\!/}^2. \tag{20.38}$$

We bring the integral of the first product $-\frac{1}{L^{(\mathfrak{n})}\tau}\mathfrak{n}\phi|d\!\!/ f|_{g\!\!\!/}^2$ on RHS (20.38) over to LHS (20.36) as the remaining part of $\mathbb{K}[f](\tau,u)$ (see definition (20.25)). The terms in the last product on RHS (20.38) are manifestly present in the term ${}^{(\check{T})}\mathfrak{B}_{(3)}[f]$ defined in (20.28c). The remaining terms in the decomposition of $-\frac{1}{2}\frac{1}{L^{(\mathfrak{n})}\tau}\mu(\mathbf{g}^{-1})^{\alpha\delta}(\mathbf{g}^{-1})^{\beta\sigma}\mathbf{Q}_{\delta\sigma}[f]^{(\check{T})}\boldsymbol{\pi}_{\alpha\beta}$ can be derived using the same arguments, based on straightforward but tedious calculations, given in the proof of [73, Lemma 3.3]. We remark that in the last two products $-\mu^2\mathrm{tr}_{g\!\!\!/}k\!\!\!/^{(\mathrm{Tan}-\vec{\Psi})} - \mu\mathrm{tr}_{g\!\!\!/}k\!\!\!/^{(\mathrm{Trans}-\vec{\Psi})}$ on RHS (20.28a), we have added a factor of μ that was mistakenly omitted from [73, Equation (3.14a)]; this factor will be negligible in the context of our estimates. We have therefore proved (20.26).

Proof of (20.26): We will apply the divergence theorem to the vectorfield $\mathbf{J} \overset{\mathrm{def}}{=} f^2\mathbf{B}$. We begin by expressing this vectorfield in terms of the rough geometric coordinate vectorfields as follows:

$$\mathbf{J} = \mathbf{J}^{\tau}\frac{\widetilde{\partial}}{\widetilde{\partial}\tau} + \mathbf{J}^{u}\frac{\widetilde{\partial}}{\widetilde{\partial}u} + \mathbf{J}^{A}\frac{\widetilde{\partial}}{\widetilde{\partial}x^A}. \tag{20.39}$$

Next, we claim that the following identities hold:

$$\mathbf{J}^{\tau} = f^2\left(L^{(\mathfrak{n})}\tau - \phi\frac{\mathfrak{n}}{\mu L\mu}L^{(\mathfrak{n})}\tau\right) \tag{20.40a}$$

$$\mathbf{J}^{u} = \frac{1}{\mu}f^2. \tag{20.40b}$$

To prove (20.40a), we note that $\mathbf{J}^{\tau} = \mathbf{J}^{(\mathfrak{n})}\tau = f^2\mathbf{B}^{(\mathfrak{n})}\tau$, and we use the equations $0 = {}^{(\mathfrak{n})}\check{W}^{(\mathfrak{n})}\tau = \check{X}^{(\mathfrak{n})}\tau + \phi\frac{\mathfrak{n}}{L\mu}L^{(\mathfrak{n})}\tau$ (see (4.2) and (4.4)) and $\mathbf{B} = L + X$ (see (3.24)). Similarly, to prove (20.40b), we note that $\mathbf{J}^u = \mathbf{J}u = f^2\mathbf{B}u$, and then we use Lemma 3.9.

Next, using (8.13b), and (20.40a)–(20.40b), we deduce that:

$$\begin{aligned}\sqrt{|\det\mathbf{g}|}\mathbf{D}_{\alpha}\mathbf{J}^{\alpha} = \sqrt{|\det\mathbf{g}|}\mathbf{D}_{\alpha}(f^2\mathbf{B}^{\alpha}) &= \frac{\widetilde{\partial}}{\widetilde{\partial}\tau}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{\tau}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial}u}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{u}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial}x^A}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{A}\right)\\ &= \frac{\widetilde{\partial}}{\widetilde{\partial}\tau}\left(f^2\left\{\mu - \phi\frac{\mathfrak{n}}{L\mu}\right\}\sqrt{\det\widetilde{g\!\!\!/}}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial}u}\left(\frac{1}{L^{(\mathfrak{n})}\tau}f^2\sqrt{\det\widetilde{g\!\!\!/}}\right) + \frac{\widetilde{\partial}}{\widetilde{\partial}x^A}\left(\sqrt{|\det\mathbf{g}|}\mathbf{J}^{A}\right).\end{aligned} \tag{20.41}$$

Next, we note the following covariant divergence identity, which follows from the relation $\mathbf{J} = f^2\mathbf{B}$, the Leibniz rule, Lemma 3.9, and (20.8):

$$\mathbf{D}_{\alpha}\mathbf{J}^{\alpha} = \frac{1}{\mu}(L\mu)f^2 + \mathrm{tr}_{g\!\!\!/}k\!\!\!/ f^2 + 2f\mathbf{B}f. \tag{20.42}$$

[60] The precise origin of these terms is $-\mathbf{Q}[f](L,X)^{(\check{T})}\boldsymbol{\pi}_{L\check{X}}$, which is found in the expansion of $-\frac{1}{2}\mu\mathbf{Q}^{\alpha\beta}[f]^{(\check{T})}\boldsymbol{\pi}_{\alpha\beta}$.

Substituting (20.42) into LHS (20.41) and then integrating both sides of the resulting identity over ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}$ with respect to $\mathrm{d}x^2\,\mathrm{d}x^3\,\mathrm{d}u'\,\mathrm{d}\tau'$, using (8.14b) to relate the canonical volume form $\mathrm{dvol}_{\mathbf{g}}$ to $\mathrm{d}\varpi$, applying Fubini's theorem, and using that the integral of the last term on RHS (20.34) over $\mathbb{T}^2$ vanishes (since $\mathbb{T}^2$ is a closed manifold) we conclude, in view of definitions (20.24a)–(20.24b), the desired identity (20.29).

□

20.5. **The fundamental L^2-controlling quantities.** In this section, we use the building block quantities from Sect. 20.4.2 to construct the "fundamental L^2-controlling quantities" that we use to control $\vec{\Psi}$, Ω, S, $\mathcal{C}$, $\mathcal{D}$, and their derivatives in L^2 in various regions.

Definition 20.10 (The fundamental L^2-controlling quantities)**.** Let $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$. In terms of the energy-null-flux quantities of Def. 20.7, the spacetime integrals of Def. 20.8, and the vectorfield differentiation conventions established in Def. 8.10, we define the following L^2-controlling quantities:

- **Total wave-controlling quantities**.

$$\mathbb{Q}_N(\tau,u) \overset{\text{def}}{=} \max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)} \\ \Psi\in\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}}} \sup_{(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]} \left\{\mathbb{E}_{(\text{Wave})}[\mathcal{P}^N\Psi](\tau',u') + \mathbb{F}_{(\text{Wave})}[\mathcal{P}^N\Psi](\tau',u')\right\}, \tag{20.43a}$$

$$\mathbb{K}_N(\tau,u) \overset{\text{def}}{=} \max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)} \\ \Psi\in\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}}} \mathbb{K}[\mathcal{P}^N\Psi](\tau,u), \tag{20.43b}$$

$$\mathbb{W}_N(\tau,u) \overset{\text{def}}{=} \max\{\mathbb{Q}_N(\tau,u),\,\mathbb{K}_N(\tau,u)\}, \tag{20.43c}$$

- **Partial wave-controlling quantities**.

$$\mathbb{Q}_N^{(\text{Partial})}(\tau,u) \overset{\text{def}}{=} \max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)} \\ \Psi\in\{\mathcal{R}_{(-)},v^2,v^3,s\}}} \sup_{(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]} \left\{\mathbb{E}_{(\text{Wave})}[\mathcal{P}^N\Psi](\tau',u') + \mathbb{F}_{(\text{Wave})}[\mathcal{P}^N\Psi](\tau',u')\right\}, \tag{20.44a}$$

$$\mathbb{K}_N^{(\text{Partial})}(\tau,u) \overset{\text{def}}{=} \max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)} \\ \Psi\in\{\mathcal{R}_{(-)},v^2,v^3,s\}}} \mathbb{K}[\mathcal{P}^N\Psi](\tau,u), \tag{20.44b}$$

$$\mathbb{W}_N^{(\text{Partial})}(\tau,u) \overset{\text{def}}{=} \max\left\{\mathbb{Q}_N^{(\text{Partial})}(\tau,u),\,\mathbb{K}_N^{(\text{Partial})}(\tau,u)\right\}. \tag{20.44c}$$

- **Specific vorticity- and entropy-controlling quantities**.

$$\mathbb{V}_N(\tau,u) \overset{\text{def}}{=} \max_{\mathcal{P}^N\in\mathfrak{P}^{(N)}} \sup_{(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]} \left\{\mathbb{E}_{(\text{Transport})}[\mathcal{P}^N\Omega](\tau',u') + \mathbb{F}_{(\text{Transport})}[\mathcal{P}^N\Omega](\tau',u')\right\}, \tag{20.45a}$$

$$\mathbb{S}_N(\tau,u) \overset{\text{def}}{=} \max_{\mathcal{P}^N\in\mathfrak{P}^{(N)}} \sup_{(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]} \left\{\mathbb{E}_{(\text{Transport})}[\mathcal{P}^N S](\tau',u') + \mathbb{F}_{(\text{Transport})}[\mathcal{P}^N S](\tau',u')\right\}, \tag{20.45b}$$

$$\mathbb{V}_N^{(\text{Rough Tori})}(\tau,u) \overset{\text{def}}{=} \max_{\mathcal{P}^N\in\mathfrak{P}^{(N)}} \left\|\mathcal{P}^N\Omega\right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \tag{20.46a}$$

$$\mathbb{S}_N^{(\text{Rough Tori})}(\tau,u) \overset{\text{def}}{=} \max_{\mathcal{P}^N\in\mathfrak{P}^{(N)}} \left\|\mathcal{P}^N S\right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}. \tag{20.46b}$$

- **Modified fluid variable-controlling quantities**.

$$\mathbb{C}_N(\tau,u) \overset{\text{def}}{=} \max_{\mathcal{P}^N\in\mathfrak{P}^{(N)}} \sup_{(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]} \left\{\mathbb{E}_{(\text{Transport})}[\mathcal{P}^N\mathcal{C}](\tau',u') + \mathbb{F}_{(\text{Transport})}[\mathcal{P}^N\mathcal{C}](\tau',u')\right\}, \tag{20.47a}$$

$$\mathbb{D}_N(\tau,u) \overset{\text{def}}{=} \sup_{(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]} \left\{\mathbb{E}_{(\text{Transport})}[\mathcal{P}^N\mathcal{D}](\tau',u') + \mathbb{F}_{(\text{Transport})}[\mathcal{P}^N\mathcal{D}](\tau',u')\right\}, \tag{20.47b}$$

$$\mathbb{C}_N^{(\text{Rough Tori})}(\tau,u) \stackrel{\text{def}}{=} \max_{\mathcal{P}^N \in \mathfrak{P}^{(N)}} \left\| \mathcal{P}^N \mathcal{C} \right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}, \tag{20.48a}$$

$$\mathbb{D}_N^{(\text{Rough Tori})}(\tau,u) \stackrel{\text{def}}{=} \max_{\mathcal{P}^N \in \mathfrak{P}^{(N)}} \left\| \mathcal{P}^N \mathcal{D} \right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}. \tag{20.48b}$$

Remark 20.11 (Differences between $\mathbb{Q}_N$ and $\mathbb{Q}_N^{(\text{Partial})}$)**.** Although $\mathbb{Q}_N^{(\text{Partial})}$ might seem to be a redundant quantity, it plays an important role in our energy estimates. In particular, when controlling the solution's top-order derivatives, we will exploit that the partial energy $\mathbb{Q}_N^{(\text{Partial})}$ is only weakly influenced by the full energy $\mathbb{Q}_N$. Similar remarks apply to $\mathbb{K}_N^{(\text{Partial})}(\tau,u)$ and $\mathbb{W}_N^{(\text{Partial})}(\tau,u)$.

Definition 20.12 (Summed L^2-controlling quantities)**.** For positive integers $N_1 < N_2$ and non-negative integers N, we define the following summed L^2-controlling quantities:

$$\mathbb{Q}_{[N_1,N_2]}(\tau,u) \stackrel{\text{def}}{=} \sum_{M=N_1}^{N_2} \mathbb{Q}_M(\tau,u), \qquad \mathbb{V}_{\leq N}(\tau,u) = \sum_{M=0}^{N} \mathbb{V}_M(\tau,u), \tag{20.49}$$

and similarly for the other controlling quantities. When $N = 0$, we often omit the subscript, e.g., we write $\mathbb{V}(\tau,u)$ instead of $\mathbb{V}_0(\tau,u)$.

20.6. **The coerciveness of the fundamental L^2-controlling quantities.** In this section, we exhibit the coerciveness properties of the L^2-controlling quantities from Def. 20.10.

20.6.1. *Decomposition of components of the energy-momentum tensor.* We start with the following lemma, which yields identities for various components of the energy-momentum tensor. The components $\mathbf{Q}[f](\mathbf{B},L)$ and $\mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},\check{T})$ are of particular interest since they appear in our energy–null-flux identity (20.26).

Lemma 20.13 (Decomposition of various components of the energy-momentum tensor)**.** *Let f be a scalar function, and let $\mathbf{Q}$ be the corresponding energy-momentum tensor defined in* (20.18)*. Then the following identities hold, where* $|\nabla\!\!\!/ f|^2_{g\!\!\!/} = |d\!\!\!/ f|^2_{g\!\!\!/} = (g\!\!\!/^{-1})^{AB}(\frac{\partial}{\partial x^A} f)\frac{\partial}{\partial x^B} f$:

$$\mathbf{Q}[f](L,L) = (Lf)^2, \tag{20.50a}$$

$$\mathbf{Q}[f](X,L) = -\frac{1}{2}(Lf)^2 + \frac{1}{2}|\nabla\!\!\!/ f|^2_{g\!\!\!/}, \tag{20.50b}$$

$$\mathbf{Q}[f](\mathbf{B},L) = \frac{1}{2}(Lf)^2 + \frac{1}{2}|\nabla\!\!\!/ f|^2_{g\!\!\!/}, \tag{20.50c}$$

$$\mathbf{Q}[f](X,\mathbf{B}) = (Lf)Xf + (Xf)^2, \tag{20.50d}$$

$$\mathbf{Q}[f]\left(L, \frac{\widetilde{\partial}}{\partial x^A}\right) = (Lf)\frac{\widetilde{\partial}}{\partial x^A} f, \tag{20.50e}$$

$$\mathbf{Q}[f]\left(X, \frac{\widetilde{\partial}}{\partial x^A}\right) = (Xf)\frac{\widetilde{\partial}}{\partial x^A} f + \frac{1}{2}\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}(Lf)^2 + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}(Lf)Xf - \frac{1}{2}\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}|\nabla\!\!\!/ f|^2_{g\!\!\!/}, \tag{20.50f}$$

$$\mathbf{Q}[f]\left(\mathbf{B}, \frac{\widetilde{\partial}}{\partial x^A}\right) = (Lf)\frac{\widetilde{\partial}}{\partial x^A} f + (Xf)\frac{\widetilde{\partial}}{\partial x^A} f + \frac{1}{2}\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}(Lf)^2 + \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}(Lf)Xf - \frac{1}{2}\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}|\nabla\!\!\!/ f|^2_{g\!\!\!/}. \tag{20.50g}$$

Moreover, the following identities also hold, where $\breve{T}$ *is the multiplier vectorfield defined in* (20.22)*,* ϕ *is the cut-off function from Def. 4.1,* ${}^{(\mathfrak{n})}\breve{R}$ *is as in* (6.6)*,* ${}^{(\mathfrak{n})}\hat{N}$ *is as in* (6.9)*,* $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ *is as in* (6.20a)*, and* ${}^{(\mathfrak{n})}r$ *is defined by* (6.20b)*:*

$$\mathbf{Q}[f](\breve{T},L) = (1+\mu)(Lf)^2 + \mu|\nabla\!\!\!/ f|^2_{g\!\!\!/}, \tag{20.51a}$$

$$\begin{aligned}\mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},L) &= \frac{\mu(1-{}^{(\mathfrak{n})}r) - \frac{\mathfrak{n}\phi}{L\mu}}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\mathbf{Q}[f](L,L) + \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\mathbf{Q}[f](X,L) \\ &\quad - \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A,\mathrm{d}x^B\right)\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\mathbf{Q}[f]\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^B},L\right),\end{aligned} \tag{20.51b}$$

$$\begin{aligned}\mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},\mathbf{B}) &= \frac{\mu(1-{}^{(\mathfrak{n})}r) - \frac{\mathfrak{n}\phi}{L\mu}}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\mathbf{Q}[f](\mathbf{B},L) + \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\mathbf{Q}[f](\mathbf{B},X) \\ &\quad - \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A,\mathrm{d}x^B\right)\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\mathbf{Q}[f]\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^B},\mathbf{B}\right),\end{aligned} \tag{20.51c}$$

$$\begin{aligned}\mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},\breve{T}) &= \frac{\mu^2(1-{}^{(\mathfrak{n})}r) - \mu\frac{\mathfrak{n}\phi}{L\mu}}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\left\{(Lf)^2 + |\nabla\!\!\!/ f|^2_{g\!\!\!/}\right\} + \frac{2\mu^2}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\left\{(Lf)Xf + (Xf)^2\right\} \\ &\quad + \frac{\mu^2}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\Bigg\{-2(Lf)\frac{\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A,\mathrm{d}x^B\right)\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}f - 2(Xf)\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A,\mathrm{d}x^B\right)\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}f \\ &\qquad\qquad - {}^{(\mathfrak{n})}r(Lf)^2 - 2{}^{(\mathfrak{n})}r(Lf)Xf + {}^{(\mathfrak{n})}r|\nabla\!\!\!/ f|^2_{g\!\!\!/}\Bigg\} \\ &\quad + \frac{\mu\left(\frac{1}{2}-{}^{(\mathfrak{n})}r\right) - \phi\frac{\mathfrak{n}}{L\mu}}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}(Lf)^2 + \frac{\mu}{2|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}|\nabla\!\!\!/ f|^2_{g\!\!\!/} - \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A,\mathrm{d}x^B\right)\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}(Lf)\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}f.\end{aligned} \tag{20.51d}$$

Proof. (20.50a)–(20.50d) are straightforward consequences of the definition (20.18) of $\mathbf{Q}$, Lemma 3.9, and the decompositions of $\mathbf{g}^{-1}$ implied by (3.34b).

(20.50e)–(20.50g) follow from combining similar arguments with the identity (5.13c).

(20.51a) follows from the identity $\breve{T} = L + 2\mu\mathbf{B}$ (which is a consequence of (3.24) and (20.22)), (20.50a), and (20.50c).

To derive (20.51b)–(20.51c), we first decompose ${}^{(\mathfrak{n})}\hat{N}$ as follows using (6.5), (6.6), (6.7), (6.20a), (6.20b), and (6.20d):

$$\begin{aligned}{}^{(\mathfrak{n})}\hat{N} &= \frac{\mu(1-{}^{(\mathfrak{n})}r) - \frac{2\mathfrak{n}\phi}{L\mu}}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}L + \frac{1}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\breve{X} + \frac{\mathfrak{n}\phi}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}L\mu}L - \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A,\mathrm{d}x^B\right)\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau} \\ &= \frac{\mu(1-{}^{(\mathfrak{n})}r) - \phi\frac{\mathfrak{n}}{L\mu}}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}L + \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}X - \frac{\mu}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}}\widetilde{g\!\!\!/}^{-1}\left(\mathrm{d}x^A,\mathrm{d}x^B\right)\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau\frac{\widetilde{\partial}}{\widetilde{\partial} x^B}}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}.\end{aligned} \tag{20.52}$$

(20.51b)–(20.51c) now follow from (20.52) and the linearity of the map $Z \to \mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},Z)$.

Finally, (20.51d) follows from (20.50a)–(20.50g), (20.51b)–(20.51c) and the following identity identity, which follows from the identity $\breve{T} = L + 2\mu\mathbf{B}$ mentioned above: $\mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},\breve{T}) = \mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},L) + 2\mu\mathbf{Q}[f]({}^{(\mathfrak{n})}\hat{N},\mathbf{B})$. □

20.6.2. *Coerciveness estimates on sub-manifolds.* In the next lemma, we exhibit the coerciveness of the L^2-controlling quantities from Def. 20.10. In particular, we exhibit their L^2-coerciveness properties on the sub-manifolds ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}$, ${}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau)}$, and ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. We reveal the coerciveness of the spacetime integrals $\mathbb{K}_N(\tau,u)$ and $\mathbb{K}_N^{(\mathrm{Partial})}(\tau,u)$ in a separate lemma, namely Lemma 20.15.

Lemma 20.14 (The coerciveness on sub-manifolds of the fundamental L^2-controlling quantities)**.** *Let* $\mathbb{Q}_N(\tau,u)$, $\cdots$, $\mathbb{D}_N(\tau,u)$ *be the* L^2*-controlling quantities from Def. 20.10, let* $(\tau,u) \in [\tau_0,\tau_{\mathrm{Boot}}) \times [-U_1,U_2]$*, and let* $(\tau',u') \in [\tau_0,\tau] \times [-U_1,u]$*. Then for* $1 \leq N \leq N_{\mathrm{top}}$*, the following lower bounds hold, where* $\mathfrak{P}^{(N)}$ *is the set of order* N $\mathcal{P}_u$*-tangential* *commutator operators from Def. 8.10:*

$$\mathbb{Q}_N(\tau,u) \geq \max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)}\\ \Psi\in\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}}} \left\{ 0.49\left\|\sqrt{\mu-\frac{2\mathfrak{n}\phi}{L\mu}}L\mathcal{P}^N\Psi\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \right.$$
$$0.49\left\|\sqrt{\mu}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, 0.99\left\|\breve{X}\mathcal{P}^N\Psi\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \tag{20.53}$$
$$\left.\left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}L\mathcal{P}^N\Psi\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}, \left\|\frac{\sqrt{\mu}}{\sqrt{L^{(\mathfrak{n})}\tau}}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}\right\},$$

$$\mathbb{Q}_N^{(\mathrm{Partial})}(\tau,u) \geq \max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)}\\ \Psi\in\{\mathcal{R}_{(-)},v^2,v^3,s\}}} \left\{ 0.49\left\|\sqrt{\mu-\frac{2\mathfrak{n}\phi}{L\mu}}L\mathcal{P}^N\Psi\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \right.$$
$$0.49\left\|\sqrt{\mu}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, 0.99\left\|\breve{X}\mathcal{P}^N\Psi\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \tag{20.54}$$
$$\left.\left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}L\mathcal{P}^N\Psi\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}, \left\|\frac{\sqrt{\mu}}{\sqrt{L^{(\mathfrak{n})}\tau}}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}\right\}.$$

Moreover, for $N \leq N_{\mathrm{top}}$, the following lower bounds hold:

$$\mathbb{V}_N(\tau,u) \geq \max\left\{\left\|\sqrt{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathcal{P}^N\Omega\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}\mathcal{P}^N\Omega\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}\right\}, \tag{20.55a}$$

$$\mathbb{S}_N(\tau,u) \geq \max\left\{\left\|\sqrt{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathcal{P}^N S\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}\mathcal{P}^N S\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}\right\}, \tag{20.55b}$$

$$\mathbb{C}_N(\tau,u) \geq \max\left\{\left\|\sqrt{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathcal{P}^N \mathcal{C}\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}\mathcal{P}^N \mathcal{C}\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}\right\}, \tag{20.56a}$$

$$\mathbb{D}_N(\tau,u) \geq \max\left\{\left\|\sqrt{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathcal{P}^N \mathcal{D}\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}, \left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}\mathcal{P}^N \mathcal{D}\right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau']})}\right\}. \tag{20.56b}$$

In addition, $1 \leq N \leq N_{\mathrm{top}}$, $N' \leq N_{\mathrm{top}}$, and $\Psi \in \{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$, then the following estimates hold:

$$\left\|\mathcal{P}^N\Psi\right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u'})} \leq C\mathring{\epsilon}^2 + C\mathbb{Q}_N(\tau,u), \tag{20.57a}$$

$$\left\|\mathcal{P}^{\leq N'}\Omega\right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u'})} \leq C\mathring{\epsilon}^2 + C\mathbb{V}_{\leq N'+1}(\tau,u), \tag{20.57b}$$

$$\left\|\mathcal{P}^{\leq N'}S\right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u'})} \leq C\mathring{\epsilon}^2 + C\mathbb{S}_{\leq N'+1}(\tau,u), \tag{20.57c}$$

$$\left\|\mathcal{P}^N\Psi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)} \leq C\mathring{\epsilon} + C\int_{\tau''=\tau_0}^{\tau'} \frac{\mathbb{Q}_N^{1/2}(\tau'',u')}{|\tau''|^{1/2}}\,d\tau'' \leq C\mathring{\epsilon} + C\mathbb{Q}_N^{1/2}(\tau,u). \tag{20.58}$$

Finally, if $1 \leq N \leq N_{\mathrm{top}}$, $\Psi \in \{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$, $\mathcal{P}^N \in \mathfrak{P}^{(N)}$, and ε and $\mathfrak{m}_0$ are sufficiently small (where $\mathfrak{m}_0 > 0$ is the parameter introduced in Sect. 10.1), then the following sharpened coerciveness estimates hold whenever $0 \leq \mu < \mathfrak{m}_0$:

$$\mathbb{Q}_N(\tau,u) \geq \mathbb{E}_{(\mathrm{Wave})}[\mathcal{P}^N\Psi](\tau',u') \geq 1.99\left\|\breve{X}\mathcal{P}^N\Psi\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}. \tag{20.59}$$

Proof.
Proof of (20.55a)–(20.56b): These estimates follow directly from the definitions (20.24a)–(20.24b), (20.45a)–(20.45b), and (20.47a)–(20.47b).

Proof of (20.53)–(20.54): We fix any $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, $(\tau', u') \in [\tau_0, \tau] \times [-U_1, u]$, $\Psi \in \{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$, and $\mathcal{P}^N \in \mathfrak{P}^{(N)}$. First, using (20.23b) and (20.51a) with $f \stackrel{\text{def}}{=} \mathcal{P}^N \Psi$, (8.15a), and (20.43a), we find that $\left\| \frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}} L \mathcal{P}^N \Psi \right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau')}_{u'})} \leq \mathbb{Q}_N(\tau, u)$ and $\left\| \frac{\sqrt{\mu}}{\sqrt{L^{(\mathfrak{n})}\tau}} \left| d\!\!/ \mathcal{P}^N \Psi \right|_{g\!\!/} \right\|^2_{L^2({}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau')}_{u'})} \leq \mathbb{Q}_N(\tau, u)$ as desired. Next, we note that the product of $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ and the terms on first line of RHS (20.51d) can be expressed as follows, where $\mathcal{P}^N \Psi$ is in the role of f:

$$
\begin{aligned}
\left\{ \mu^2 \left(\frac{0.001}{1.001} - {}^{(\mathfrak{n})}r \right) - \mu \phi \frac{\mathfrak{n}}{L\mu} \right\} (L \mathcal{P}^N \Psi)^2 + \left\{ \mu^2 (1 - {}^{(\mathfrak{n})}r) - \mu \phi \frac{\mathfrak{n}}{L\mu} \right\} |d\!\!/ \mathcal{P}^N \Psi|^2_{g\!\!/} \\
+ \left(\frac{\mu}{\sqrt{1.001}} L \mathcal{P}^N \Psi + \sqrt{1.001} \breve{X} \mathcal{P}^N \Psi \right)^2 + 0.999 (\breve{X} \Psi)^2.
\end{aligned}
\tag{20.60}
$$

Next, we multiply the terms on the second and third lines of RHS (20.51d) by $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ and use (3.31a)–(3.31b), (5.8c)–(5.8d), (5.13c), (6.11)–(6.13), Prop. 9.1, (13.11a), (15.20), (15.24), the estimates of Prop. 17.1, Cor. 17.2, (18.27), and Young's inequality to bound the magnitude of the resulting terms as follows:

$$
\leq C \varepsilon \mu^2 (L \mathcal{P}^N \Psi)^2 + C \varepsilon \mu^2 |d\!\!/ \mathcal{P}^N \Psi|^2_{g\!\!/} + C \varepsilon (\breve{X} \mathcal{P}^N \Psi)^2. \tag{20.61}
$$

The same reasoning, in conjunction with (18.8a) (which implies that $-L\mu \approx 1$ on the support of ϕ), yields that all the terms in (20.60) are positive definite and that the product of $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ and the last term on RHS (20.51d) can be absorbed by the product of $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ and the two terms $\frac{\mu\left(\frac{1}{2} - {}^{(\mathfrak{n})}r\right) - \phi \frac{\mathfrak{n}}{L\mu}}{|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}} (L \mathcal{P}^N \Psi)^2 + \frac{\mu}{2|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}} |d\!\!/ \mathcal{P}^N \Psi|^2_{g\!\!/}$ on the last line of (20.51d). In total, we see that if ε is small enough, then in the product of $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ and (20.51d), the overall coefficient of $(\breve{X} \mathcal{P}^N \Psi)^2$ can be bounded from below by 0.99, the overall coefficient of $\mu (L \mathcal{P}^N \Psi)^2$ can be bounded from below by $0.49 \left(\mu - \frac{2\mathfrak{n}\phi}{L\mu} \right)$, and the overall coefficient of $\mu |\nabla\!\!\!/ \mathcal{P}^N \Psi|^2_{g\!\!/}$ can be bounded from below by 0.49. From these estimates, (20.23a) with $f \stackrel{\text{def}}{=} \mathcal{P}^N \Psi$, (8.15b), and (20.43a), we deduce that $0.99 \left\| \breve{X} \mathcal{P}^N \Psi \right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}^{[-U_1,u']}_{\tau'}\right)} \leq \mathbb{Q}_N(\tau, u)$, $0.49 \left\| \sqrt{\mu - \frac{2\mathfrak{n}\phi}{L\mu}} L \mathcal{P}^N \Psi \right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}^{[-U_1,u']}_{\tau'}\right)} \leq \mathbb{Q}_N(\tau, u)$, and $0.49 \left\| \sqrt{\mu} \left| d\!\!/ \mathcal{P}^N \Psi \right|_{g\!\!/} \right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}^{[-U_1,u']}_{\tau'}\right)} \leq \mathbb{Q}_N(\tau, u)$. Combining all five of the L^2 bounds along ${}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau')}_{u'}$ and ${}^{(\mathfrak{n})}\widetilde{\Sigma}^{[-U_1,u']}_{\tau'}$ that we derived in this paragraph, we conclude the desired lower bound (20.53).

Taking into account definition (20.44a), we can prove the lower bounds stated in (20.54) via exactly the same argument.

Proof of (20.59): We again fix any $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, $(\tau', u') \in [\tau_0, \tau] \times [-U_1, u]$, $\Psi \in \{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$, and $\mathcal{P}^N \in \mathfrak{P}^{(N)}$. We consider the product of $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}}$ and RHS (20.51d) with $f \stackrel{\text{def}}{=} \mathcal{P}^N \Psi$. The terms in the second braces on the first line of RHS (20.51d) generate the terms $2\mu (L \mathcal{P}^N \Psi) \breve{X} \mathcal{P}^N \Psi + 2 (\breve{X} \mathcal{P}^N \Psi)^2$, which, by Young's inequality, can be pointwise bounded from below by $\geq 1.999 (\breve{X} \mathcal{P}^N \Psi)^2 - 1000 \mu^2 (L \mathcal{P}^N \Psi)^2$. If $\mathfrak{m}_0 \leq 10^{-5}$, then (18.1) implies that on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, we have the pointwise bound $|1000 \mu^2 (L \mathcal{P}^N \Psi)^2| \leq \frac{1}{10} \mu (L \mathcal{P}^N \Psi)^2$. Hence, if ε is sufficiently small, then the same arguments we used in the proof of (20.53) imply that we can absorb the term $-1000 \mu^2 (L \mathcal{P}^N \Psi)^2$ into the terms $\mu \left\{ \left(\frac{1}{2} - {}^{(\mathfrak{n})}r \right) - \phi \frac{\mathfrak{n}}{L\mu} \right\} (L \mathcal{P}^N \Psi)^2$ generated by the first product on the last line of RHS (20.51d). The arguments we used in the proof of (20.53) also imply that all remaining terms on RHS (20.51d) are either positive definite or can be absorbed into the positive definite terms by exploiting the smallness of ε. In total, these arguments yield the pointwise estimate $|{}^{(\mathfrak{n})}\breve{R}|_{\mathbf{g}} \times \text{RHS (20.51d)} \geq 1.99 (\breve{X} \mathcal{P}^N \Psi)^2$. From this estimate and the same arguments we used to prove (20.53), we conclude the desired lower bound (20.59).

Proof of (20.57a)–(20.57c): We again fix any $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, $(\tau', u') \in [\tau_0, \tau] \times [-U_1, u]$, $\Psi \in \{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$, and $\mathcal{P}^N \in \mathfrak{P}^{(N)}$. To prove (20.57a), we first use (20.5) with $f \stackrel{\text{def}}{=} \mathcal{P}^N \Psi$ as well as the already proved coerciveness result (20.53) to deduce that $\left\| \mathcal{P}^N \Psi \right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau',u'})} \leq C \left\| \mathcal{P}^N \Psi \right\|^2_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u'})} + C \mathbb{Q}_N(\tau, u)$. From this estimate and the data-estimate

(11.13a), we conclude (20.57a). The estimates (20.57b)–(20.57c) follow from a nearly identical argument based on the coerciveness results (20.55a)–(20.55b) and the data-estimates (11.13b).

Proof of (20.58): We again fix any $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$, $(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]$, $\Psi\in\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$, and $\mathcal{P}^N\in\mathfrak{P}^{(N)}$. We first use (11.11a) and (16.15) to deduce that:

$$\left\|\mathcal{P}^N\Psi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u']}\right)}\lesssim\mathring{\epsilon}+\int_{\tau''=\tau_0}^{\tau}\left\|{}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^N\Psi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau''}^{[-U_1,u']}\right)}\,\mathrm{d}\tau''. \tag{20.62}$$

Then, using (6.3), (18.1), (18.9b), and (20.53), we bound the time integral on RHS (20.62) by $\lesssim\int_{\tau''=\tau_0}^{\tau'}\frac{\mathbb{Q}_N^{1/2}(\tau'',u')}{|\tau''|^{1/2}}\,\mathrm{d}\tau''$, which yields the first inequality stated in (20.58). The second inequality stated in (20.58) follows from the fact that $\mathbb{Q}_N(\tau,u)$ is increasing in its arguments.

□

20.6.3. *Coerciveness of the spacetime integrals* $\mathbb{K}$. The next lemma complements Lemma 20.14 by exhibiting the coerciveness of the spacetime integrals $\mathbb{K}_N(\tau,u)$ and $\mathbb{K}_N^{(\text{Partial})}(\tau,u)$ defined in (20.43b) and (20.44b) respectively; recall that these spacetime integrals appear on the left-hand side of our energy identity (20.26) for the wave-variables. The key point is that the integrands on RHSs (20.63a)–(20.63b) are quantitatively positive in the region $\{|u|\leq U_{☢}\}$ where the shock can form and in particular, these integrands *do not contain any degenerate factor of* μ. We fundamentally need the non-degenerate coerciveness guaranteed by (20.63a)–(20.63b) in order to control some of the error integrals that arise in our energy estimates.

Lemma 20.15 (The coerciveness of the spacetime integrals $\mathbb{K}$)**.** *Let* $1\leq N\leq N_{\text{top}}$*, let* $\mathbb{K}_N(\tau,u)$ *and* $\mathbb{K}_N^{(\text{Partial})}(\tau,u)$ *be the spacetime integrals defined in* (20.43b) *and* (20.44b) *respectively, and let* $\mathfrak{P}^{(N)}$ *be the set of order* N $\mathcal{P}_u$*-tangential commutator operators from Def. 8.10. Then the following lower bounds hold for* $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$*, where* $\mathbf{1}_{[-U_{☢},U_{☢}]}=\mathbf{1}_{[-U_{☢},U_{☢}]}(u')$ *denotes the characteristic function of the interval* $[-U_{☢},U_{☢}]$ *and* ϕ *is the cut-off from Def. 4.1:*

$$\mathbb{K}_N(\tau,u)\geq\max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)}\\ \Psi\in\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}}}\frac{1}{4}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\left\{\mathbf{1}_{[-U_{☢},U_{☢}]}(u')+4\frac{1}{L^{(\mathfrak{n})}\tau}\mathfrak{n}\phi\right\}\left|\mathrm{d\!\!/}\mathcal{P}^N\Psi\right|_{g\!\!/}^2\,\mathrm{d}\varpi, \tag{20.63a}$$

$$\mathbb{K}_N^{(\text{Partial})}(\tau,u)\geq\max_{\substack{\mathcal{P}^N\in\mathfrak{P}^{(N)}\\ \Psi\in\{\mathcal{R}_{(-)},v^2,v^3,s\}}}\frac{1}{4}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\left\{\mathbf{1}_{[-U_{☢},U_{☢}]}(u')+4\frac{1}{L^{(\mathfrak{n})}\tau}\mathfrak{n}\phi\right\}\left|\mathrm{d\!\!/}\mathcal{P}^N\Psi\right|_{g\!\!/}^2\,\mathrm{d}\varpi. \tag{20.63b}$$

Proof. Let f be a scalar function, and let $\mathbb{K}[f](\tau,u)$ be the corresponding spacetime integral defined in (20.25). Using the estimate (18.8c), we deduce that:

$$\mathbb{K}[f](\tau,u)\geq\frac{1}{4}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}}\left\{\mathbf{1}_{[-U_{☢},U_{☢}]}(u')+4\frac{1}{L^{(\mathfrak{n})}\tau}\mathfrak{n}\phi\right\}|\mathrm{d\!\!/}f|_{g\!\!/}^2\,\mathrm{d}\varpi. \tag{20.64}$$

The desired bounds (20.63a)–(20.63b) now follow from (20.64) and definitions (20.43b) and (20.44b).

□

21. The elliptic-hyperbolic integral identities

We continue to work under the assumptions of Sect. 13.2. In this section, we set up the top-order elliptic-hyperbolic regularity theory for the transport-div-curl system satisfied by Ω and S, i.e., for the top-order derivatives of solutions to equations (2.23a), (2.23c), (2.24a)–(2.24b), and (2.25a)–(2.25b). More precisely, for any Σ_t-tangent vectorfield V, we derive coercive integral identities – featuring error terms – that are localized to spacetime regions of the form ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}$; see Prop. 21.14 for the main identity, which, in view of the coerciveness guaranteed by Lemma 21.9, yields L^2 spacetime control of $\boldsymbol{\partial}V$ (see definition (21.13)) in terms of error terms. In our forthcoming applications, we will apply the identity with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V, and we will use the special structure of the equations of Theorem 2.15 and commutator estimates to control the error terms. Ultimately, this will yield (see Prop. 27.5) spacetime L^2-control over the top-order terms $\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega$ and $\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S$.

The integral identities are adaptations of the framework we developed in [4] to handle the structure of the singular boundary of shock-forming solutions. Unlike in [4], our setup here avoids boundary integrals along the characteristics

$\mathcal{P}_u$. This allows us to avoid error integrals on $\mathcal{P}_u$ that involve the top-order derivatives of μ, which would have been uncontrollable. Our setup also yields error terms whose singularity strength is controllable under the scope of our approach. To achieve these goals, we rely on the following key ingredients:

- New well-constructed *characteristic currents* (see Sect. 21.4), the analysis of which incorporates both the elliptic and the hyperbolic sub-structures in the equations of Theorem 2.15.
- A delicate integration by parts identity to handle some difficult boundary integrals along ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}$, which takes into account the rough acoustic geometry and the precise structure of the equations of Theorem 2.15; see Lemma 21.13 for a differential version of the identity. Ultimately, this leads to an integral identity (see Prop. 21.14) that provides control (see Prop. 27.5) of the spacetime integrals $\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} |\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega|^2 \, d\boldsymbol{\varpi}$ and $\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} |\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S|^2 \, d\boldsymbol{\varpi}$ *as well as* the rough tori integrals $\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} |\mathcal{P}^{N_{\text{top}}}\Omega|^2 \, d\varpi_{\widetilde{g}}$ and $\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} |\mathcal{P}^{N_{\text{top}}}S|^2 \, d\varpi_{\widetilde{g}}$. We stress that the delicate integration by parts mentioned above yields rough tori integrals with *favorable signs*, and that our proof would not have closed if the integrals had the wrong signs. The availability of good signs for these terms is a key aspect of the framework developed in [4].

Throughout this section, we will use the observations provided by Remark 3.19.

21.1. **Basic geometric constructions and definitions.** In this section, we define some basic geometric objects that play a role in our derivation of the localized integral identities.

Definition 21.1 (Projection onto $\mathcal{P}_u$ and $\mathcal{P}_u$-tangency)**.** Let $\underline{L}$ be the $\mathbf{g}$-null vectorfield defined in (7.1) (which, in view of (7.4), is transversal to the characteristics $\mathcal{P}_u$).

1. We define the type $\binom{1}{1}$ projection tensorfield $\overline{\Pi}$ onto the characteristic hypersurfaces $\mathcal{P}_u$ as follows, where δ_β^α denotes the Kronecker delta:

$$\overline{\Pi}_\beta^{\ \alpha} \stackrel{\text{def}}{=} \delta_\beta^\alpha + \frac{1}{2}\underline{L}^\alpha L_\beta. \tag{21.1}$$

2. Given any type $\binom{m}{n}$ spacetime tensorfield ξ, we define its $\mathcal{P}_u$-projection $\overline{\Pi}\xi$, as follows:

$$(\overline{\Pi}\xi)_{\beta_1\cdots\beta_n}^{\alpha_1\cdots\alpha_m} \stackrel{\text{def}}{=} \overline{\Pi}_{\widetilde{\alpha}_1}^{\ \alpha_1}\cdots\overline{\Pi}_{\widetilde{\alpha}_m}^{\ \alpha_m}(\mathbf{g}^{-1})^{\widetilde{\beta}_1\widetilde{\gamma}_1}\mathbf{g}_{\beta_1\gamma_1}\overline{\Pi}_{\widetilde{\beta}_1}^{\ \gamma_1}\cdots(\mathbf{g}^{-1})^{\widetilde{\beta}_n\widetilde{\gamma}_n}\mathbf{g}_{\beta_n\gamma_n}\overline{\Pi}_{\widetilde{\beta}_n}^{\ \gamma_n}\xi_{\widetilde{\gamma}_1\cdots\widetilde{\gamma}_n}^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}. \tag{21.2}$$

3. We say that a spacetime tensorfield ξ is $\mathcal{P}_u$-tangent if $\overline{\Pi}\xi = \xi$.

We refer to Remark 7.2 for comments on the role of the vectorfield $\underline{L}$ in this paper.

Remark 21.2 (Lack of symmetry)**.** The type $\binom{0}{2}$ tensorfield $\mathbf{g}_{\alpha\widetilde{\alpha}}\overline{\Pi}_\beta^{\ \widetilde{\alpha}}$ is not symmetric. This is the reason that in equation (21.2), there are factors of $\mathbf{g}^{-1}$ and $\mathbf{g}$ and we were careful about the placement of the indices on $\overline{\Pi}$ that are contracted against the lower indices of ξ, unlike in equations (3.6a), (3.6b), and (6.27).

Definition 21.3 (Additional geometric tensorfields used in the elliptic-hyperbolic identities)**.** We respectively define $\mathbf{h}$ and $\mathbf{h}^{-1}$ to be the type $\binom{0}{2}$ and $\binom{2}{0}$ tensorfields with the following Cartesian components:

$$\mathbf{h}_{\alpha\beta} \stackrel{\text{def}}{=} \mathbf{g}_{\alpha\beta} + 2\mathbf{B}_\alpha\mathbf{B}_\beta, \tag{21.3a}$$

$$(\mathbf{h}^{-1})^{\alpha\beta} \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\beta} + 2\mathbf{B}^\alpha\mathbf{B}^\beta. \tag{21.3b}$$

By using $\mathbf{h}$ and $\mathbf{h}^{-1}$ to lower and raise the indices on the projection tensorfield $\overline{\Pi}$, we also define the type $\binom{0}{2}$ tensorfield $\overline{e}$ and the type $\binom{2}{0}$ tensorfield $\overline{E}$ as follows:

$$\overline{e}_{\alpha\beta} \stackrel{\text{def}}{=} \mathbf{h}_{\alpha\sigma}\overline{\Pi}_\beta^{\ \sigma}, \tag{21.4}$$

$$\overline{E}^{\alpha\beta} \stackrel{\text{def}}{=} (\mathbf{h}^{-1})^{\alpha\sigma}\overline{\Pi}_\sigma^{\ \beta}. \tag{21.5}$$

In the next lemma, we exhibit some basic properties of the tensorfields from Defs. 21.1 and 21.3. Later on, we will use the positive definiteness of $\mathbf{h}$ and $\mathbf{h}^{-1}$ (which are revealed by the lemma) to exhibit the coerciveness properties of various energies.

Lemma 21.4 (Basic properties of the tensorfields from Defs. 21.1 and 21.3)**.** *The tensorfield $\overline{\Pi}_{\beta}^{\ \alpha}$ is a projection onto $\mathcal{P}_u$ in the following sense: $\overline{\Pi}\underline{L} = 0$, while if P is a $\mathcal{P}_u$-tangent vectorfield, then $\overline{\Pi}P = P$.*

Moreover, the tensorfields $\overline{e}$ and $\overline{E}$ are symmetric and positive semi-definite, and $\overline{e}$ restricts to a Riemannian metric on $\mathcal{P}_u$.

In addition, for all pairs (Y, Z) of Σ_t-tangent vectorfields, we have $\mathbf{h}(Y, Z) = \mathbf{g}(Y, Z)$, and if V is a Σ_t-tangent vectorfield, then $\mathbf{h}(V, \mathbf{B}) = 0$. Moreover, we have $\mathbf{h}(\mathbf{B}, \mathbf{B}) = 1$, and $\mathbf{h}$ is a Riemannian metric on spacetime. Furthermore, following identity holds:

$$(\mathbf{h}^{-1})^{\alpha\gamma}\mathbf{h}_{\gamma\beta} = \delta_{\beta}^{\alpha}, \tag{21.6}$$

where δ_{β}^{α} is the Kronecker delta. That is, $\mathbf{h}^{-1}$ is in fact the inverse metric of $\mathbf{h}$.

In addition, the following identities hold, where g and g^{-1} are respectively the first fundamental form of Σ_t and inverse first fundamental form of Σ_t from Def. 3.4:

$$\mathbf{h}_{\alpha\beta} = g_{\alpha\beta} + \mathbf{B}_{\alpha}\mathbf{B}_{\beta}, \tag{21.7a}$$

$$(\mathbf{h}^{-1})^{\alpha\beta} = (g^{-1})^{\alpha\beta} + \mathbf{B}^{\alpha}\mathbf{B}^{\beta}. \tag{21.7b}$$

Finally, the following identities hold:

$$\overline{e}_{\alpha\beta} = \mathbf{h}_{\alpha\beta} - \frac{1}{2}L_{\alpha}L_{\beta} = \not{g}_{\alpha\beta} + \frac{1}{2}\underline{L}_{\alpha}\underline{L}_{\beta}, \tag{21.8}$$

$$\overline{E}^{\alpha\beta} = (\mathbf{h}^{-1})^{\alpha\beta} - \frac{1}{2}\underline{L}^{\alpha}\underline{L}^{\beta} = (\not{g}^{-1})^{\alpha\beta} + \frac{1}{2}L^{\alpha}L^{\beta}, \tag{21.9}$$

$$\overline{E}^{\alpha\gamma}\overline{e}_{\gamma\beta} = \delta_{\beta}^{\alpha} + \frac{1}{2}\underline{L}^{\alpha}L_{\beta}, \tag{21.10}$$

$$\overline{e}_{\alpha\gamma}\overline{E}^{\gamma\beta} = \delta_{\alpha}^{\beta} + \frac{1}{2}\underline{L}_{\alpha}L^{\beta}, \tag{21.11}$$

$$\overline{E}^{\alpha\beta} = (\mathbf{h}^{-1})^{\alpha\gamma}(\mathbf{h}^{-1})^{\beta\delta}\overline{e}_{\gamma\delta}, \tag{21.12}$$

where $\not{g}$ and $\not{g}^{-1}$ are respectively the first fundamental form and inverse first fundamental form of the ***acoustic tori*** *$\ell_{t,u} = \Sigma_t \cap \mathcal{P}_u$ from Def. 3.4.*

Proof. The facts that $\overline{\Pi}\underline{L} = 0$, while if P is $\mathcal{P}_u$-tangent, then $\overline{\Pi}P = P$ follow from (7.3)–(7.4) and the fact that L is $\mathbf{g}$-orthogonal to $\mathcal{P}_u$.

(21.6) follows from a straightforward computation based on definitions (21.3a)–(21.3b) and the identity (3.23).

(21.7a) follows from definition (21.3a) and the identity (3.32a). Similarly, (21.7b) follows from definition (21.3b) and the identity (3.32b).

The remaining properties of $\mathbf{h}$ stated in the lemma follow in a straightforward fashion from (3.23) and (3.25).

To prove the first equality in (21.8), we substitute the definition (21.3a) of $\mathbf{h}_{\alpha\sigma}$ and the definition (21.1) $\overline{\Pi}_{\beta}^{\ \sigma}$ into of RHS (21.4) and carry out straightforward algebraic computations using Lemmas 3.9 and 7.3 (in particular, we use (7.2)). The second equality in (21.8) follows from similar arguments, where we in particular use (7.2) and (7.5). (21.8) also yields the symmetry of $\overline{e}$. The identities in (21.9) and (21.10)–(21.11) as well as the symmetry of $\overline{E}$ follow from similar arguments based on definitions (21.5) and (21.1).

The fact that $\overline{e}$ restricts to a Riemannian metric on $\mathcal{P}_u$ follows from (21.8), Lemmas 3.9 and 7.3, the fact that the tangent space of $\mathcal{P}_u$ is the direct sum of the tangent space of $\ell_{t,u}$ and the span of L, and the fact that $\not{g}_{\alpha\beta}$ is positive definite on $\ell_{t,u}$-tangent vectorfields and satisfies $\not{g}(L, \cdot) = 0$. The positive semi-definiteness of $\overline{e}$ and $\overline{E}$ follows from the second identities in (21.8)–(21.9) and the positive semi-definiteness of $\not{g}_{\alpha\beta}$ and $(\not{g}^{-1})^{\alpha\beta}$.

The identity (21.12) follows from definitions (21.4)–(21.5), the identity (21.6), and the symmetry of $\overline{e}$ and $\overline{E}$.

□

21.2. **Additional derivatives operators, pointwise norms, and comparison estimates.** In this section, we define some additional derivative operators and pointwise norms, and we establish some simple comparison estimates. Later, we will use them in our analysis of the terms in the elliptic-hyperbolic integral identity provided by Prop. 21.14.

21.2.1. *The Cartesian gradient of* ξ*, and* $|ZV|_g$*.*

Definition 21.5 (The Cartesian gradient of ξ, and $|ZV|_g$)**.**
Given a type $\binom{m}{n}$ spacetime tensorfield ξ, we define its *Cartesian gradient* $\boldsymbol{\partial}\xi$ to be the type $\binom{m}{n+1}$ spacetime tensorfield with the following Cartesian components:

$$(\boldsymbol{\partial}\xi)^{\alpha_1\cdots\alpha_m}_{\beta_1\beta_2\cdots\beta_{n+1}} \stackrel{\text{def}}{=} \partial_{\beta_1}\xi^{\alpha_1\cdots\alpha_m}_{\beta_2\cdots\beta_{n+1}}. \tag{21.13}$$

Let V be a Σ_t-tangent vectorfield and let Z be a spacetime vectorfield. Relative to the Cartesian coordinates, we define (see Remark 3.19) $|ZV|_g \geq 0$ as follows:

$$|ZV|_g^2 \stackrel{\text{def}}{=} g_{ab}(ZV^a)ZV^b, \tag{21.14}$$

where as usual $ZV^a = Z^\alpha\partial_\alpha V^a$.

Note that $|ZV|_g$ is the $|\cdot|_g$ norm of the Σ_t-tangent vectorfield with the Cartesian spatial components ZV^i, $i = 1, 2, 3$.

21.2.2. *The* $\mathbf{h}$*-norm of tensorfields.*

Definition 21.6 (The $\mathbf{h}$-norm of tensorfields)**.** Recall that in Lemma 21.4, we showed that $\mathbf{h}$ is a Riemannian metric on spacetime. If ξ is a type $\binom{m}{n}$ spacetime tensorfield, then we define $|\xi|_{\mathbf{h}} \geq 0$ by:

$$|\xi|_{\mathbf{h}}^2 \stackrel{\text{def}}{=} \mathbf{h}_{\alpha_1\widetilde{\alpha}_1}\cdots\mathbf{h}_{\alpha_m\widetilde{\alpha}_m}(\mathbf{h}^{-1})^{\beta_1\widetilde{\beta}_1}\cdots(\mathbf{h}^{-1})^{\beta_n\widetilde{\beta}_n}\xi^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\xi^{\widetilde{\alpha}_1\cdots\widetilde{\alpha}_m}_{\widetilde{\beta}_1\cdots\widetilde{\beta}_n}. \tag{21.15}$$

21.2.3. *Pointwise comparison results and the* $\mathbf{h}$*-size of* $\overline{\Pi}$*,* $\overline{e}$*, and* $\overline{E}$*.*

Lemma 21.7 (Pointwise comparison results and the $\mathbf{h}$-size of $\overline{\Pi}$, $\overline{e}$, and $\overline{E}$)**.** *For any type* $\binom{m}{n}$ *spacetime tensorfield* $\xi^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}$*, the following comparison estimates hold relative to the Cartesian coordinates on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$*:*

$$|\xi|_{\mathbf{h}} \approx \sum_{\substack{0\leq\alpha_1,\cdots,\alpha_m\leq 3\\ 0\leq\beta_1,\cdots,\beta_n\leq 3}}\left|\xi^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}\right|, \tag{21.16a}$$

$$|\boldsymbol{\partial}\xi|_{\mathbf{h}} \approx \sum_{\substack{0\leq\alpha_1,\cdots,\alpha_m\leq 3\\ 0\leq\beta_1,\cdots,\beta_n\leq 3\\ 0\leq\gamma\leq 3}}\left|\partial_\gamma\xi^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}\right|. \tag{21.16b}$$

Moreover, if V *is a* Σ_t*-tangent vectorfield and* Z *is a spacetime vectorfield, then the following comparison estimates hold relative to the Cartesian coordinates on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$*:*

$$|V|_g \approx \sum_{a=1,2,3}|V^a|, \tag{21.17a}$$

$$|ZV|_g \approx \sum_{a=1,2,3}|ZV^a|, \tag{21.17b}$$

$$|\boldsymbol{\partial}V|_{\mathbf{h}} \approx \sum_{\substack{\alpha=0,1,2,3\\ a=1,2,3}}|\partial_\alpha V^a| \approx \sum_{\alpha=0,1,2,3}|\partial_\alpha V|_g \approx |\mathbf{B}V|_g + \sum_{a=1,2,3}|\partial_a V|_g. \tag{21.17c}$$

Finally, the following identities hold:

$$|\overline{\Pi}|_{\mathbf{h}} = |\overline{e}|_{\mathbf{h}} = |\overline{E}|_{\mathbf{h}} = \sqrt{3}. \tag{21.18}$$

Proof. We prove (21.16a) when ξ is a type $\binom{1}{0}$ tensorfield; the case of general type $\binom{m}{n}$ tensorfields can be handled through similar arguments. To proceed, we first use (3.25), (3.33a), and (21.7a) to derive the following identity relative to the Cartesian coordinates:

$$|\xi|_{\mathbf{h}}^2 = \mathbf{h}_{\alpha\beta}\xi^\alpha\xi^\beta = c^{-2}\sum_{a=1,2,3}(\xi^a)^2 + \left\{1 + c^{-2}\sum_{a=1,2,3}(v^a)^2\right\}(\xi^0)^2 - 2c^{-2}\sum_{a=1,2,3}v^a\xi^a\xi^0. \tag{21.19}$$

From the bootstrap assumptions and (9.3e), we deduce that $|v^a| \lesssim 1$ and $c \approx 1$. From these estimates and (21.19), we conclude that $|\xi|_{\mathbf{h}} \lesssim \sum_{\alpha=0,1,2,3} |\xi^\alpha|$. To prove the reverse inequality, we note that the Cauchy–Schwarz inequality and Young's inequality imply that the cross-term $-2c^{-2}v^a\xi^a\xi^0$ on RHS (21.19) is bounded in magnitude by $\leq c^{-2}\sum_{a=1,2,3}(\xi^a)^2 + c^{-2}\sum_{a=1,2,3}(v^a)^2(\xi^0)^2$. From this bound, (21.19), and the estimate $c \approx 1$, it follows that:

$$|\xi|_{\mathbf{h}}^2 \geq (\xi^0)^2. \tag{21.20}$$

Moreover, the Cauchy–Schwarz inequality and Young's inequality imply that the cross-term $-2c^{-2}v^a\xi^a\xi^0$ is also bounded in magnitude by:

$$\leq \left\{ \frac{c^{-2}\sum_{d=1,2,3}(v^d)^2}{1 + c^{-2}\sum_{b=1,2,3}(v^b)^2} \right\} \times c^{-2} \sum_{a=1,2,3} (\xi^a)^2 + \left\{ 1 + c^{-2} \sum_{a=1,2,3} (v^a)^2 \right\} (\xi^0)^2.$$

From this bound, (21.19), and the aforementioned estimates $|v^a| \lesssim 1$ and $c \approx 1$, it follows that:

$$|\xi|_{\mathbf{h}}^2 \geq \left\{ \frac{1}{1 + c^{-2}\sum_{b=1,2,3}(v^b)^2} \right\} \times c^{-2} \sum_{a=1,2,3} (\xi^a)^2 \gtrsim \sum_{a=1,2,3} (\xi^a)^2. \tag{21.21}$$

Combining (21.20) and (21.21), we see that $\sum_{\alpha=0,1,2,3} |\xi^\alpha| \lesssim |\xi|_{\mathbf{h}}$, which completes the proof of (21.16a). The comparison estimates (21.17a), (21.17b), and (21.17c) follow from similar arguments, and we omit the straightforward details.

(21.16b) follows as a special case of (21.16a) with $\boldsymbol{\partial}\xi$ in the role of ξ.

We now prove (21.18). First, using (7.4), definitions (21.4)–(21.5), and (21.11), we compute that $|\overline{\Pi}|_{\mathbf{h}}^2 = \overline{E}^{\alpha\beta}\overline{e}_{\beta\alpha} = \delta_\alpha^\alpha + \frac{1}{2}\underline{L}_\alpha L^\alpha = 4 - 1 = 3$, as is desired. Next, we use (7.4), (21.1), (21.4), and Lemma 21.4 to compute that:

$$\begin{aligned} |\overline{e}|_{\mathbf{h}}^2 &= (\mathbf{h}^{-1})^{\alpha\beta}(\mathbf{h}^{-1})^{\gamma\delta}\overline{e}_{\alpha\gamma}\overline{e}_{\beta\delta} = \overline{\Pi}_\gamma^{\ \beta}\overline{\Pi}_\beta^{\ \gamma} = \left\{ \delta_\gamma^\beta + \frac{1}{2}L^\beta \underline{L}_\gamma \right\} \left\{ \delta_\beta^\gamma + \frac{1}{2}L^\gamma \underline{L}_\beta \right\} \\ &= 4 + L^\alpha \underline{L}_\alpha + \frac{1}{4}(L^\alpha \underline{L}_\alpha)^2 = 3. \end{aligned} \tag{21.22}$$

A similar calculation based on (21.5) yields that $|\overline{E}|_{\mathbf{h}}^2 = 3$. We have therefore proved (21.18).

□

21.3. **The coercive elliptic-hyperbolic quadratic form and its coerciveness.**

21.3.1. *The coercive elliptic-hyperbolic quadratic form.* In the next definition, we introduce the solution-adapted quadratic form $\mathscr{Q}$ that we will use to control the top-order derivatives of the specific vorticity and entropy gradient.

Definition 21.8 (The coercive elliptic-hyperbolic quadratic form)**.** Let $\not{\Pi}_b^{\ a}$ denote the Σ_t-components of the $\ell_{t,u}$ projection tensorfield defined in (3.5b), and let and $\overline{e}$ and $\overline{E}$ be the tensorfields defined in (21.4)–(21.5) respectively. Let V be a Σ_t-tangent vectorfield. Then relative to the Cartesian coordinates, we define $\mathscr{Q}[\boldsymbol{\partial}V, \boldsymbol{\partial}V]$ to be the following quadratic form associated to V:

$$\begin{aligned} \mathscr{Q}[\boldsymbol{\partial}V, \boldsymbol{\partial}V] &\stackrel{\text{def}}{=} \left(\overline{E}^{\alpha\beta} + 4\mathbf{B}^\alpha\mathbf{B}^\beta \right) \left(\overline{e}_{\gamma\delta} + 4\mathbf{B}_\gamma\mathbf{B}_\delta \right) (\partial_\alpha V^\gamma)\partial_\beta V^\delta \\ &\quad - \frac{1}{16}\left\{ -3(\mathbf{B}V^\alpha)\underline{L}_\alpha + (LV^\alpha)\underline{L}_\alpha + \not{\Pi}_b^{\ a}\partial_a V^b \right\}^2. \end{aligned} \tag{21.23}$$

21.3.2. *The coerciveness of the elliptic-hyperbolic quadratic form.* In the next lemma, we exhibit the coerciveness of $\mathscr{Q}[\boldsymbol{\partial}V, \boldsymbol{\partial}V]$.

Lemma 21.9 (Coerciveness of $\mathscr{Q}[\boldsymbol{\partial}V, \boldsymbol{\partial}V]$)**.** *On ${}^{(\text{Int})}\mathcal{M}_{[\tau_0, \tau_{\text{Boot}}), [-U_1, U_2]}$, the quadratic form $\mathscr{Q}$ from Def. 21.8 is quantitatively positive definite on the space of Cartesian gradients of Σ_t-tangent vectorfields V in the following sense, where $|\boldsymbol{\partial}V|_{\mathbf{h}}^2 \stackrel{def}{=} (\mathbf{h}^{-1})^{\alpha\beta}\mathbf{h}_{\gamma\delta}(\partial_\alpha V^\gamma)\partial_\beta V^\delta$:*

$$\mathscr{Q}[\boldsymbol{\partial}V, \boldsymbol{\partial}V] \approx |\boldsymbol{\partial}V|_{\mathbf{h}}^2 \approx \sum_{\alpha=0}^{3} |\partial_\alpha V|_g^2. \tag{21.24}$$

Proof. Throughout the proof, we will use the observations made in Remark 3.19. To start, we note that Young's inequality implies that the terms $-\frac{1}{16}\left\{-3(\mathbf{B}V^\alpha)\underline{L}_\alpha + (LV^\alpha)\underline{L}_\alpha + \not\Pi_b{}^a\partial_a V^b\right\}^2$ on RHS (21.23) are bounded in magnitude by:

$$\begin{aligned}
&\le \frac{4}{16}[3(\mathbf{B}V^\alpha)\underline{L}_\alpha]^2 + \frac{4}{3}\times\frac{1}{16}\left\{(LV^\alpha)\underline{L}_\alpha + \not\Pi_b{}^a\partial_a V^b\right\}^2 \\
&\le \frac{9}{4}[(\mathbf{B}V^\alpha)\underline{L}_\alpha]^2 + \frac{1}{6}[(LV^\alpha)\underline{L}_\alpha]^2 + \frac{1}{6}(\not\Pi_b{}^a\partial_a V^b)^2.
\end{aligned} \tag{21.25}$$

Moreover, the Cauchy–Schwarz inequality and the fact that $|\not\Pi|^2_{\not g} = \mathrm{tr}_{\not g}\not g = 2$ together imply that:

$$\frac{1}{6}(\not\Pi_b{}^a\partial_a V^b)^2 \le \frac{1}{3}(\not g^{-1})^{ab}\not g_{cd}(\partial_a V^c)\partial_b V^d \overset{\text{def}}{=} \frac{1}{3}|\partial V|^2_{\not g}. \tag{21.26}$$

Using Lemma 21.4 (in particular (21.8)–(21.9)), (21.23), and (21.25)–(21.26), we compute that:

$$\mathscr{Q}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] \approx \left(\overline{E}^{\alpha\beta} + 4\mathbf{B}^\alpha\mathbf{B}^\beta\right)\left(\overline{e}_{\gamma\delta} + 4\mathbf{B}_\gamma\mathbf{B}_\delta\right)(\partial_\alpha V^\gamma)\partial_\beta V^\delta. \tag{21.27}$$

From (21.27), the fact that (in Cartesian coordinates) $\mathbf{B}_\gamma\partial_\alpha V^\gamma = -\partial_\alpha V^0 = 0$, the identities $L = \mathbf{B} - X$ and $\underline{L} = \mathbf{B} + X$, the identities $g_{ab} = \not g_{ab} + X_a X_b$ and $(g^{-1})^{ab} = (\not g^{-1})^{ab} + X^a X^b$ proved in (3.34a)–(3.34b), (3.35a), and the identities (21.7a)–(21.7b) and (21.8)–(21.9), it follows that:

$$\begin{aligned}
\mathscr{Q}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] &\approx |\mathbf{B}V|^2_g + |LV|^2_g + (\not g^{-1})^{ab}g_{cd}(\partial_a V^c)\partial_b V^d \approx |\mathbf{B}V|^2_g + |XV|^2_g + (\not g^{-1})^{ab}g_{cd}(\partial_a V^c)\partial_b V^d \\
&\approx |\mathbf{B}V|^2_g + (g^{-1})^{ab}g_{cd}(\partial_a V^c)\partial_b V^d \approx |\boldsymbol{\partial}V|^2_{\mathbf{h}}.
\end{aligned} \tag{21.28}$$

(21.28) implies the first "≈" in (21.24). The second "≈" in (21.24) then follows from Lemma 21.7. □

21.4. **The characteristic currents.** The $\mathcal{P}_u$-tangent vectorfields $\mathscr{J}[V,\boldsymbol{\partial}V]$ in the next definition play a key role in our analysis. In our proof of Prop. 21.14, use them for bookkeeping when integrating by parts. We sometimes refer to the $\mathscr{J}[V,\boldsymbol{\partial}V]$ as "characteristic currents" since they are tangent to $\mathcal{P}_u$, or "elliptic-hyperbolic currents" since they are the basic ingredient for the elliptic-hyperbolic integral identities. In our prior work [4], we used related – but distinct – ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}$-tangent currents to derive elliptic-hyperbolic integral identities. Compared to the currents in [4], the ones featured in the next definition are better adapted to the structure of the singularity in the sense that they do not generate any critical-strength error terms in our top-order L^2 estimates. Moreover, when we integrate by parts over the spacetime region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}$, the $\mathcal{P}_u$-tangency of the $\mathscr{J}[V,\boldsymbol{\partial}V]$ allows us to avoid boundary integrals along $\mathcal{P}_u$. This is important because some of the acoustic geometry error terms (such as the top-order derivatives of μ) do not have sufficiently regularity to be controlled in L^2 along $\mathcal{P}_u$.

21.4.1. *The definition of the $\mathcal{P}_u$-tangent characteristic current.*

Definition 21.10 (The $\mathcal{P}_u$-tangent characteristic current)**.** Let V be a Σ_t-tangent vectorfield. We define the characteristic current to be the vectorfield $\mathscr{J} = \mathscr{J}[V,\boldsymbol{\partial}V]$ with the following components, $(\alpha = 0,1,2,3)$:

$$\mathscr{J}^\alpha[V,\boldsymbol{\partial}V] \overset{\text{def}}{=} V^\gamma\overline{\Pi}_\gamma{}^\lambda\overline{\Pi}_\kappa{}^\alpha\partial_\lambda V^\kappa - V^\gamma\overline{\Pi}_\gamma{}^\alpha\overline{\Pi}_\lambda{}^\kappa\partial_\kappa V^\lambda. \tag{21.29}$$

Remark 21.11 ($\mathscr{J}^\alpha[V,\boldsymbol{\partial}V]$ is $\mathcal{P}_u$-tangent)**.** Since for any vectorfield Z, the vectorfield $\overline{\Pi}_\beta{}^\alpha Z^\beta$ is $\mathcal{P}_u$-tangent, it follows from (21.29) that indeed, $\mathscr{J}^\alpha[V,\boldsymbol{\partial}V]$ is $\mathcal{P}_u$-tangent.

21.4.2. *The covariant divergence identity satisfied by the elliptic-hyperbolic current.* In the next lemma, we provide the main covariant divergence identity satisfied by the current $\mathscr{J}^\alpha[V,\boldsymbol{\partial}V]$ from Def. 21.10. The identity forms the starting point for the divergence-theorem-based proof of Prop. 21.14.

Lemma 21.12 (Covariant divergence identity for the elliptic-hyperbolic current)**.** *Let V be a Σ_t-tangent vectorfield, let $\mathscr{Q}[\boldsymbol{\partial}V,\boldsymbol{\partial}V]$ be the quadratic form defined by* (21.23)*, and let $\mathscr{W}$ be a "weight function." Then the following identity holds relative to the Cartesian coordinates, where $\mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}$:*

$$\begin{aligned}
\mathscr{W}\mathscr{Q}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] = \mathbf{D}_\alpha(\mathscr{W}\mathscr{J}^\alpha[V,\boldsymbol{\partial}V]) &+ \mathscr{W}\mathrm{J}_{(\text{Antisymmetric})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] + \mathscr{W}\mathrm{J}_{(\text{Div})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] \\
&+ \mathrm{J}_{(\partial\mathscr{W})}[V,\boldsymbol{\partial}V] + \mathscr{W}\mathrm{J}_{(\text{Absorb-1})}[V,\boldsymbol{\partial}V] + \mathscr{W}\mathrm{J}_{(\text{Absorb-2})}[V,\boldsymbol{\partial}V] \\
&+ \mathscr{W}\mathrm{J}_{(\text{Material})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] + \mathscr{W}\mathrm{J}_{(\text{Null Geometry})}[V,\boldsymbol{\partial}V],
\end{aligned} \tag{21.30}$$

where with $(\mathrm{d}V_\flat)_{\alpha\beta} \overset{def}{=} \partial_\alpha V_\beta - \partial_\beta V_\alpha$, *we have:*

$$\mathrm{J}_{(\text{Antisymmetric})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] \overset{def}{=} \frac{1}{2}\overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}(\mathrm{d}V_\flat)_{\alpha\gamma}(\mathrm{d}V_\flat)_{\beta\delta}, \tag{21.31a}$$

$$\mathrm{J}_{(\text{Div})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] \overset{def}{=} \frac{9}{16}(\partial_a V^a)^2, \tag{21.31b}$$

$$\mathrm{J}_{(\boldsymbol{\partial}\mathscr{W})}[V,\boldsymbol{\partial}V] \overset{def}{=} -\mathscr{J}^\alpha[V,\boldsymbol{\partial}V]\partial_\alpha\mathscr{W}, \tag{21.31c}$$

$$\begin{aligned}\mathrm{J}_{(\text{Absorb-1})}[V,\boldsymbol{\partial}V] &\overset{def}{=} -\overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}V^\kappa(\partial_\alpha\mathbf{g}_{\delta\kappa})(\mathrm{d}V_\flat)_{\beta\gamma} \\ &\quad + \frac{3}{8}(\partial_a V^a)\left\{-3(\mathbf{B}V^\alpha)\underline{L}_\alpha + (LV^\alpha)\underline{L}_\alpha + /\!\!\!\Pi_b{}^a\partial_a V^b\right\},\end{aligned} \tag{21.31d}$$

$$\mathrm{J}_{(\text{Absorb-2})}[V,\boldsymbol{\partial}V] \overset{def}{=} -\boldsymbol{\Gamma}_{\alpha\ \beta}^{\ \alpha}\mathscr{J}^\beta[V,\boldsymbol{\partial}V] - \overline{E}^{\alpha\beta}\overline{\Pi}_\gamma{}^\delta(\partial_\beta\mathbf{g}_{\delta\kappa})V^\kappa\partial_\alpha V^\gamma + \overline{E}^{\alpha\beta}\overline{\Pi}_\gamma{}^\delta V^\kappa(\partial_\delta\mathbf{g}_{\beta\kappa})\partial_\alpha V^\gamma, \tag{21.31e}$$

$$\mathrm{J}_{(\text{Material})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] \overset{def}{=} 4\overline{e}_{\alpha\beta}(\mathbf{B}V^\alpha)\mathbf{B}V^\beta, \tag{21.31f}$$

$$\mathrm{J}_{(\text{Null Geometry})}[V,\boldsymbol{\partial}V] \overset{def}{=} -V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_\gamma{}^\lambda\overline{\Pi}_\kappa{}^\alpha\right)\right\}\partial_\lambda V^\kappa + V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_\gamma{}^\alpha\overline{\Pi}_\lambda{}^\kappa\right)\right\}\partial_\kappa V^\lambda, \tag{21.31g}$$

and on RHS (21.31e), $\boldsymbol{\Gamma}_{\alpha\ \beta}^{\ \gamma} \overset{def}{=} \frac{1}{2}(\mathbf{g}^{-1})^{\gamma\delta}\left(\partial_\alpha\mathbf{g}_{\delta\beta} + \partial_\beta\mathbf{g}_{\alpha\delta} - \partial_\delta\mathbf{g}_{\alpha\beta}\right)$ *are the Cartesian Christoffel symbols of* $\mathbf{g}$.

Proof. Throughout the proof, we silently use the simple fact that $\mathbf{B}_\alpha V^\alpha = -V^0 = 0$ and thus $V_\alpha \overset{\text{def}}{=} \mathbf{g}_{\alpha\beta}V^\beta = \mathbf{h}_{\alpha\beta}V^\beta$ and $(\mathbf{g}^{-1})^{\alpha\beta}V_\beta = (\mathbf{h}^{-1})^{\alpha\beta}V_\beta$. We also silently use the simple identity $\partial_\alpha(\mathbf{h}^{-1})^{\beta\gamma} = -(\mathbf{h}^{-1})^{\beta\beta'}(\mathbf{h}^{-1})^{\gamma\gamma'}\partial_\alpha\mathbf{h}_{\beta'\gamma'}$ and the fact that in Cartesian coordinates, $\partial_\alpha\mathbf{g}_{\beta\gamma} = \partial_\alpha\mathbf{h}_{\beta\gamma}$ (this follows easily from (3.25) and (21.3a)). Moreover, we frequently relabel indices from line to line whenever convenient. We will also silently use the observations made in Remark 3.19 and the symmetry of $\overline{e}$ and $\overline{E}$ shown in Lemma 21.4.

We start by using (21.29) to compute that relative to the Cartesian coordinates, we have:

$$\begin{aligned}\mathbf{D}_\alpha\mathscr{J}^\alpha[V,\boldsymbol{\partial}V] &= \partial_\alpha\mathscr{J}^\alpha[V,\boldsymbol{\partial}V] + \boldsymbol{\Gamma}_{\alpha\ \beta}^{\ \alpha}\mathscr{J}^\beta[V,\boldsymbol{\partial}V] \\ &= \overline{\Pi}_\kappa{}^\alpha\overline{\Pi}_\gamma{}^\lambda(\partial_\alpha V^\gamma)\partial_\lambda V^\kappa - \left(\overline{\Pi}_\beta{}^\alpha\partial_\alpha V^\beta\right)^2 \\ &\quad + \boldsymbol{\Gamma}_{\alpha\ \beta}^{\ \alpha}\mathscr{J}^\beta[V,\boldsymbol{\partial}V] + V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_\gamma{}^\lambda\overline{\Pi}_\kappa{}^\alpha\right)\right\}\partial_\lambda V^\kappa - V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_\gamma{}^\alpha\overline{\Pi}_\lambda{}^\kappa\right)\right\}\partial_\kappa V^\lambda,\end{aligned} \tag{21.32}$$

where we stress that due to cancellations, the second derivatives of V are absent from RHS (21.32). Next, we compute the following identity:

$$\begin{aligned}\partial_\lambda V^\kappa &= \partial_\lambda[(\mathbf{h}^{-1})^{\kappa\sigma}V_\sigma] = (\mathbf{h}^{-1})^{\kappa\sigma}\partial_\lambda V_\sigma + [\partial_\lambda(\mathbf{h}^{-1})^{\kappa\sigma}]V_\sigma \\ &= (\mathbf{h}^{-1})^{\kappa\sigma}\partial_\sigma V_\lambda + (\mathbf{h}^{-1})^{\kappa\sigma}(\partial_\lambda V_\sigma - \partial_\sigma V_\lambda) + [\partial_\lambda(\mathbf{h}^{-1})^{\kappa\sigma}]V_\sigma \\ &= (\mathbf{h}^{-1})^{\kappa\sigma}\partial_\sigma(\mathbf{h}_{\lambda\lambda'}V^{\lambda'}) + (\mathbf{h}^{-1})^{\kappa\sigma}(\partial_\lambda V_\sigma - \partial_\sigma V_\lambda) - (\mathbf{h}^{-1})^{\kappa\kappa'}V^\sigma(\partial_\lambda\mathbf{h}_{\kappa'\sigma}) \\ &= (\mathbf{h}^{-1})^{\kappa\sigma}\mathbf{h}_{\lambda\lambda'}(\partial_\sigma V^{\lambda'}) + (\mathbf{h}^{-1})^{\kappa\sigma}(\partial_\lambda V_\sigma - \partial_\sigma V_\lambda) + (\mathbf{h}^{-1})^{\kappa\sigma}(\partial_\sigma\mathbf{h}_{\lambda\lambda'})V^{\lambda'} - (\mathbf{h}^{-1})^{\kappa\kappa'}V^\sigma(\partial_\lambda\mathbf{h}_{\kappa'\sigma}).\end{aligned} \tag{21.33}$$

Next, taking into account definition (21.5), we compute that the contribution of the second product $(\mathbf{h}^{-1})^{\kappa\sigma}(\partial_\lambda V_\sigma - \partial_\sigma V_\lambda)$ on RHS (21.33) to the first product on RHS (21.32) is as follows:

$$\begin{aligned}&\overline{\Pi}_\kappa{}^\alpha\overline{\Pi}_\gamma{}^\lambda(\partial_\alpha V^\gamma)(\mathbf{h}^{-1})^{\kappa\sigma}(\partial_\lambda V_\sigma - \partial_\sigma V_\lambda) \\ &= \overline{E}^{\alpha\sigma}\overline{\Pi}_\gamma{}^\lambda\left\{\partial_\alpha[(\mathbf{h}^{-1})^{\gamma\gamma'}V_{\gamma'}]\right\}(\mathrm{d}V_\flat)_{\lambda\sigma} \\ &= \overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}(\partial_\alpha V_\gamma)(\mathrm{d}V_\flat)_{\delta\beta} - \overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}V^\kappa(\partial_\alpha\mathbf{h}_{\delta\kappa})(\mathrm{d}V_\flat)_{\gamma\beta} \\ &= -\frac{1}{2}\overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}(\mathrm{d}V_\flat)_{\alpha\gamma}(\mathrm{d}V_\flat)_{\beta\delta} + \overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}V^\kappa(\partial_\alpha\mathbf{h}_{\delta\kappa})(\mathrm{d}V_\flat)_{\beta\gamma}.\end{aligned} \tag{21.34}$$

Next, using (21.4), (21.5), (21.33) and (21.34), we rewrite the first product on RHS (21.32) as follows:

$$\begin{aligned}\overline{\Pi}_{\kappa}^{\ \alpha}\overline{\Pi}_{\gamma}^{\ \lambda}(\partial_\alpha V^\gamma)\partial_\lambda V^\kappa &= \overline{E}^{\alpha\beta}\overline{e}_{\gamma\delta}(\partial_\alpha V^\gamma)\partial_\beta V^\delta - \frac{1}{2}\overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}(\mathrm{d}V_\flat)_{\alpha\gamma}(\mathrm{d}V_\flat)_{\beta\delta} \\ &\quad + \overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}V^\kappa(\partial_\alpha \mathbf{h}_{\delta\kappa})(\mathrm{d}V_\flat)_{\beta\gamma} \\ &\quad + \overline{E}^{\alpha\beta}\overline{\Pi}_{\gamma}^{\ \delta}(\partial_\beta \mathbf{h}_{\delta\kappa})V^\kappa\partial_\alpha V^\gamma - \overline{E}^{\alpha\beta}\overline{\Pi}_{\gamma}^{\ \delta}V^\kappa(\partial_\delta \mathbf{h}_{\beta\kappa})\partial_\alpha V^\gamma.\end{aligned} \tag{21.35}$$

Next, using (see Lemma 7.3) the identities $\mathbf{B} = \frac{1}{2}(L + \underline{L})$, $L = \mathbf{B} - X$, $\underline{L} = \mathbf{B} + X$, $\mathbf{B}_\alpha V^\alpha = 0$, and $\mathbf{B}_\alpha \partial_\beta V^\alpha = 0$, as well as the identity $\partial_a V^a = X_a X V^a + {\not\Pi}_b^{\ a}\partial_a V^b$ (see (3.5b)), we compute that:

$$L_\alpha \underline{L} V^\alpha = -\underline{L}_\alpha \underline{L} V^\alpha = \underline{L}_\alpha L V^\alpha - 2\underline{L}_\alpha \mathbf{B} V^\alpha, \tag{21.36}$$

$$L_\alpha \underline{L} V^\alpha = L_\alpha X V^\alpha + L_\alpha \mathbf{B} V^\alpha = -X_a X V^a + L_\alpha \mathbf{B} V^\alpha = {\not\Pi}_b^{\ a}\partial_a V^b - \partial_a V^a - \underline{L}_\alpha \mathbf{B} V^\alpha. \tag{21.37}$$

Next, using (21.1), (21.36), and (21.37), we compute that:

$$\overline{\Pi}_{\beta}^{\ \alpha}\partial_\alpha V^\beta = \partial_a V^a + \frac{1}{4}L_\alpha \underline{L} V^\alpha + \frac{1}{4}L_\alpha \underline{L} V^\alpha \tag{21.38}$$

$$= \frac{3}{4}\partial_a V^a - \frac{3}{4}(\mathbf{B} V^\alpha)\underline{L}_\alpha + \frac{1}{4}(L V^\alpha)\underline{L}_\alpha + \frac{1}{4}{\not\Pi}_b^{\ a}\partial_a V^b. \tag{21.39}$$

Using (21.38)–(21.39), we rewrite the second product on RHS (21.32) as follows:

$$\left(\overline{\Pi}_{\beta}^{\ \alpha}\partial_\alpha V^\beta\right)^2 = \frac{1}{16}\left\{3\partial_a V^a - 3(\mathbf{B} V^\alpha)\underline{L}_\alpha + (L V^\alpha)\underline{L}_\alpha + {\not\Pi}_b^{\ a}\partial_a V^b\right\}^2. \tag{21.40}$$

Combining (21.32), (21.35), and (21.40), we deduce the following identity:

$$\begin{aligned}\overline{E}^{\alpha\beta}\overline{e}_{\gamma\delta}(\partial_\alpha V^\gamma)\partial_\beta V^\delta &= \mathbf{D}_\alpha \mathscr{J}^\alpha[V, \boldsymbol{\partial} V] \\ &\quad + \frac{1}{2}\overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}(\mathrm{d}V_\flat)_{\alpha\gamma}(\mathrm{d}V_\flat)_{\beta\delta} + \frac{1}{16}\left\{3\partial_a V^a - 3(\mathbf{B} V^\alpha)\underline{L}_\alpha + (L V^\alpha)\underline{L}_\alpha + {\not\Pi}_b^{\ a}\partial_a V^b\right\}^2 \\ &\quad - \overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}V^\kappa(\partial_\alpha \mathbf{h}_{\delta\kappa})(\mathrm{d}V_\flat)_{\beta\gamma} \\ &\quad - V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_{\gamma}^{\ \lambda}\overline{\Pi}_{\kappa}^{\ \alpha}\right)\right\}\partial_\lambda V^\kappa + V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_{\gamma}^{\ \alpha}\overline{\Pi}_{\lambda}^{\ \kappa}\right)\right\}\partial_\kappa V^\lambda \\ &\quad - \mathbf{\Gamma}_{\alpha\ \beta}^{\ \alpha}\mathscr{J}^\beta[V, \boldsymbol{\partial} V] - \overline{E}^{\alpha\beta}\overline{\Pi}_{\gamma}^{\ \delta}(\partial_\beta \mathbf{h}_{\delta\kappa})V^\kappa\partial_\alpha V^\gamma + \overline{E}^{\alpha\beta}\overline{\Pi}_{\gamma}^{\ \delta}V^\kappa(\partial_\delta \mathbf{h}_{\beta\kappa})\partial_\alpha V^\gamma.\end{aligned} \tag{21.41}$$

Next, using (21.23), we observe that:

$$\begin{aligned}\mathscr{Q}[\boldsymbol{\partial} V, \boldsymbol{\partial} V] &= \overline{E}^{\alpha\beta}\overline{e}_{\gamma\delta}(\partial_\alpha V^\gamma)\partial_\beta V^\delta + 4\overline{e}_{\alpha\beta}(\mathbf{B} V^\alpha)\mathbf{B} V^\beta \\ &\quad + 4\left(\overline{E}^{\alpha\beta} + 4\mathbf{B}^\alpha\mathbf{B}^\beta\right)\mathbf{B}_\gamma\mathbf{B}_\delta(\partial_\alpha V^\gamma)\partial_\beta V^\delta \\ &\quad - \frac{1}{16}\left\{-3(\mathbf{B} V^\alpha)\underline{L}_\alpha + (L V^\alpha)\underline{L}_\alpha + {\not\Pi}_b^{\ a}\partial_a V^b\right\}^2.\end{aligned} \tag{21.42}$$

Hence, noting that the terms on the second line of RHS (21.42) vanish, we can add

$$4\overline{e}_{\alpha\beta}(\mathbf{B} V^\alpha)\mathbf{B} V^\beta - \frac{1}{16}\left\{-3(\mathbf{B} V^\alpha)\underline{L}_\alpha + (L V^\alpha)\underline{L}_\alpha + {\not\Pi}_b^{\ a}\partial_a V^b\right\}^2$$

to each side of (21.41) to obtain the following identity:

$$\begin{aligned}\mathscr{Q}[\boldsymbol{\partial} V, \boldsymbol{\partial} V] &= \mathbf{D}_\alpha \mathscr{J}^\alpha[V, \boldsymbol{\partial} V] \\ &\quad + \frac{1}{2}\overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}(\mathrm{d}V_\flat)_{\alpha\gamma}(\mathrm{d}V_\flat)_{\beta\delta} + \frac{9}{16}(\partial_a V^a)^2 \\ &\quad - \overline{E}^{\alpha\beta}\overline{E}^{\gamma\delta}V^\kappa(\partial_\alpha \mathbf{h}_{\delta\kappa})(\mathrm{d}V_\flat)_{\beta\gamma} + \frac{3}{8}(\partial_a V^a)\left\{-3(\mathbf{B} V^\alpha)\underline{L}_\alpha + (L V^\alpha)\underline{L}_\alpha + {\not\Pi}_b^{\ a}\partial_a V^b\right\} \\ &\quad + 4\overline{e}_{\alpha\beta}(\mathbf{B} V^\alpha)\mathbf{B} V^\beta \\ &\quad - V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_{\gamma}^{\ \lambda}\overline{\Pi}_{\kappa}^{\ \alpha}\right)\right\}\partial_\lambda V^\kappa + V^\gamma\left\{\partial_\alpha\left(\overline{\Pi}_{\gamma}^{\ \alpha}\overline{\Pi}_{\lambda}^{\ \kappa}\right)\right\}\partial_\kappa V^\lambda \\ &\quad - \mathbf{\Gamma}_{\alpha\ \beta}^{\ \alpha}\mathscr{J}^\beta[V, \boldsymbol{\partial} V] - \overline{E}^{\alpha\beta}\overline{\Pi}_{\gamma}^{\ \delta}(\partial_\beta \mathbf{h}_{\delta\kappa})V^\kappa\partial_\alpha V^\gamma + \overline{E}^{\alpha\beta}\overline{\Pi}_{\gamma}^{\ \delta}V^\kappa(\partial_\delta \mathbf{h}_{\beta\kappa})\partial_\alpha V^\gamma.\end{aligned} \tag{21.43}$$

We have therefore proved (21.30) in the special case $\mathscr{W} \stackrel{\text{def}}{=} 1$. To obtain (21.30) for a general weight $\mathscr{W}$, we simply multiply this special case identity by $\mathscr{W}$ and use the commutation identity $\mathscr{W}\mathbf{D}_\alpha \mathscr{J}^\alpha[V,\boldsymbol{\partial} V] = \mathbf{D}_\alpha(\mathscr{W}\mathscr{J}^\alpha[V,\boldsymbol{\partial} V]) + \mathfrak{J}_{(\partial\mathscr{W})}[V,\boldsymbol{\partial} V]$, where $\mathfrak{J}_{(\partial\mathscr{W})}[V,\boldsymbol{\partial} V]$ is defined by (21.31c).

□

21.4.3. *Key identity for the elliptic-hyperbolic boundary terms.* To derive our main elliptic-hyperbolic integral identities, which we state as Prop. 21.14, we will start by integrating (21.30) over the spacetime region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}$ and applying the divergence theorem. This procedure leads to boundary integrals, including the integral of ${}^{(\mathfrak{n})}\hat{N}_\alpha \mathscr{J}^\alpha[V,\boldsymbol{\partial} V]$ over ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}$, where ${}^{(\mathfrak{n})}\hat{N}$ is the future-directed $\mathbf{g}$-unit normal to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}$. To avoid uncontrollable error terms in the boundary integral, we will integrate by parts over ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}$ with the help of the identity for ${}^{(\mathfrak{n})}\hat{N}_\alpha \mathscr{J}^\alpha[V,\boldsymbol{\partial} V]$ provided by the next lemma. Of crucial importance for our top-order L^2 estimates is the sign of the first two terms ${}^{(\mathfrak{n})}\breve{R}\mathfrak{P}[V,V] + \frac{1}{2}\mathfrak{P}[V,V]\mathrm{tr}_{\widetilde{\not g}}{}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}$ on RHS (21.44); in the proof of the integral identity (21.63), we will integrate by parts and exploit the sign of these terms as well as the positive definiteness of the quadratic form $\mathfrak{P}[V,V]$ shown in (21.48).

Lemma 21.13 (Key identity for the elliptic-hyperbolic boundary terms)**.** *Let V be a Σ_t-tangent vectorfield, let $\not V$ be its $\mathbf{g}$-orthogonal projection onto the acoustic tori $\ell_{t,u}$, let $(\mathrm{d}V_\flat)_{\alpha\beta} \stackrel{\text{def}}{=} \partial_\alpha V_\beta - \partial_\beta V_\alpha$, and let $\mathscr{J}^\alpha[V,\boldsymbol{\partial} V]$ be the corresponding characteristic current defined in* (21.29)*. Then relative to the Cartesian coordinates, the following identity holds, where $\not\Pi$ is the $\ell_{t,u}$-projection from Def. 3.3, $\not\nabla$ is as in Def. 3.11, $|\cdot|_{\not g}$ and $|\cdot|_g$ are as in Def. 3.17, $\phi = \phi(u)$ is the cut-off function from Def. 4.1, ϕ' is its derivative, ${}^{(\mathfrak{n})}U$, ${}^{(\mathfrak{n})}\hat{N}$ and ${}^{(\mathfrak{n})}\breve{R}$ are as in Def. 6.4, $|{}^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}$ is as in* (6.20a)*, ${}^{(\mathfrak{n})}r$ is as in* (6.20b)*, $(\widetilde{\not g}^{-1})^{AB}$ is as in Lemma 6.5, $\widetilde{\mathrm{d}\not{\iota}\mathrm{v}}$ is the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-divergence operator from Def. 6.13, and $\mathrm{tr}_{\widetilde{\not g}}{}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}$ is the $\widetilde{\not g}$-trace (see Def. 6.10) of the deformation tensor ${}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}$ of ${}^{(\mathfrak{n})}\breve{R}$:*

$$
\begin{aligned}
\frac{|{}^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}}{\mu}{}^{(\mathfrak{n})}\hat{N}_\alpha \mathscr{J}^\alpha[V,\boldsymbol{\partial} V] &= {}^{(\mathfrak{n})}\breve{R}\mathfrak{P}[V,V] + \frac{1}{2}\mathfrak{P}[V,V]\mathrm{tr}_{\widetilde{\not g}}{}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi} \\
&\quad - \widetilde{\mathrm{d}\not{\iota}\mathrm{v}}\left\{\frac{\mu}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}\left[-\frac{1}{2}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|_g^2\right]{}^{(\mathfrak{n})}U\right\} \\
&\quad + \mathfrak{E}_{(\text{Principal})}[V,\boldsymbol{\partial} V] + \mathfrak{E}_{(\text{Lower-order})}[V,V],
\end{aligned}
\tag{21.44}
$$

where:

$$
\mathfrak{P}[V,V] \stackrel{\text{def}}{=} \frac{1}{4}\left\{\frac{1}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}\left[(X_a V^a)^2 + (1 + 2{}^{(\mathfrak{n})}r)|V|_{\not g}^2 - 2\frac{1}{L{}^{(\mathfrak{n})}\tau}X_a V^a \not V^\alpha \not\nabla_\alpha {}^{(\mathfrak{n})}\tau\right]\right\},
\tag{21.45}
$$

$$
\begin{aligned}
\mathfrak{E}_{(\text{Principal})}[V,\boldsymbol{\partial} V] &\stackrel{\text{def}}{=} \frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \overline{\Pi}_\beta^{\ \alpha} V^\gamma (\mathbf{g}^{-1})^{\beta\delta}(\mathrm{d}V_\flat)_{\gamma\delta} - \frac{1}{\mu}V^\beta L_\beta L^\gamma {}^{(\mathfrak{n})}\breve{R}^\delta (\mathrm{d}V_\flat)_{\gamma\delta} \\
&\quad - \frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \overline{\Pi}_\beta^{\ \alpha} V^\beta \partial_a V^a + \frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\beta L_\beta \mathbf{B} V^\alpha - \frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha L_\beta \mathbf{B} V^\beta \\
&\quad - \frac{1}{2(\mu - \phi\frac{\mathfrak{n}}{L\mu})}{}^{(\mathfrak{n})}\breve{R}_\alpha (\mathbf{B}V^\alpha) V^a X_a - \frac{1}{2(\mu - \phi\frac{\mathfrak{n}}{L\mu})}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha (\mathbf{B}V^a) X_a + \frac{1}{2(\mu - \phi\frac{\mathfrak{n}}{L\mu})}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha V_a \mathbf{B} V^a,
\end{aligned}
\tag{21.46}
$$

and:

$$
\begin{aligned}
\mathfrak{E}_{(\text{Lower-order})}[V,V] &\overset{def}{=} -\frac{1}{2}\mathfrak{P}[V,V]\mathrm{tr}_{\slashed{g}}\,{}^{({}^{(\mathfrak{n})}\check{R})}\boldsymbol{\pi} \\
&+ \frac{1}{2}\frac{({}^{(\mathfrak{n})}\check{R}\upmu)}{\upmu^2}|\slashed{V}|^2_{\slashed{g}} + \frac{1}{2}\frac{({}^{(\mathfrak{n})}\check{R}\upmu)\phi\frac{\mathfrak{n}}{L\upmu}}{\upmu^2(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}|\slashed{V}|^2_{\slashed{g}} + \frac{({}^{(\mathfrak{n})}\check{R}\upmu)}{\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})^2}\left\{\frac{1}{2}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a X_a - \frac{1}{4}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha|V|^2_g\right\} \\
&+ \frac{(L\upmu)}{\upmu^2}\left\{-\frac{1}{2}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha|V|^2_g\right\} + \frac{1}{2}\frac{(L\upmu)\phi\frac{\mathfrak{n}}{L\upmu}}{\upmu^2}|\slashed{V}|^2_{\slashed{g}} \\
&- \left\{\frac{({}^{(\mathfrak{n})}U\upmu)\phi\frac{\mathfrak{n}}{L\upmu}}{(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})^2} + \frac{\upmu\phi\mathfrak{n}\frac{{}^{(\mathfrak{n})}UL\upmu}{(L\upmu)^2}}{(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})^2}\right\}\left\{-\frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\upmu}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha|V|^2_g\right\} + \frac{1}{2}\frac{({}^{(\mathfrak{n})}U\upmu)\phi\frac{\mathfrak{n}}{L\upmu}}{\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}|\slashed{V}|^2_{\slashed{g}} \\
&+ \left\{\frac{1}{\upmu-\phi\frac{\mathfrak{n}}{L\upmu}}\left[-\frac{1}{2}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha|V|^2_g\right]\right\}\widetilde{\mathrm{d}\slashed{\imath}\mathrm{v}}\,{}^{(\mathfrak{n})}U \\
&- \frac{\phi\mathfrak{n}\frac{({}^{(\mathfrak{n})}\check{R}L\upmu)}{(L\upmu)^2}}{\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})^2}\left\{-\frac{1}{2}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha|V|^2_g\right\} \\
&+ \frac{1}{2}\phi\frac{\mathfrak{n}(\check{X}L\upmu)}{\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})(L\upmu)^2}|\slashed{V}|^2_{\slashed{g}} + \frac{1}{2}\phi\frac{\mathfrak{n}(LL\upmu)}{(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})(L\upmu)^2}|\slashed{V}|^2_{\slashed{g}} \\
&+ \frac{1}{4\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}{}^{(\mathfrak{n})}\check{R}_\alpha(\check{X}\underline{L}^\alpha)|V|^2_g + \frac{1}{\upmu}({}^{(\mathfrak{n})}\check{R}L_\alpha)L_\beta V^\alpha V^\beta - \frac{1}{2\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a\check{X}X_a \\
&+ \frac{1}{\upmu-\phi\frac{\mathfrak{n}}{L\upmu}}\left\{-\frac{1}{2}(\check{X}X_\alpha)V^\alpha V^a X_a + \frac{1}{4}(\check{X}X_\alpha)\underline{L}^\alpha|V|^2_g\right\} \\
&+ \frac{1}{\upmu-\phi\frac{\mathfrak{n}}{L\upmu}}\left(\phi\frac{\mathfrak{n}}{\upmu L\upmu} + {}^{(\mathfrak{n})}r\right)\left\{-\frac{1}{2}(\check{X}L_\alpha)V^\alpha V^a X_a + \frac{1}{4}(\check{X}L_\alpha)\underline{L}^\alpha|V|^2_g\right\} - \frac{1}{2}\frac{(\check{X}{}^{(\mathfrak{n})}r)}{\upmu-\phi\frac{\mathfrak{n}}{L\upmu}}|\slashed{V}|^2_{\slashed{g}} \\
&+ \frac{1}{(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}\frac{(\check{X}L{}^{(\mathfrak{n})}\uptau)}{(L{}^{(\mathfrak{n})}\uptau)^2}\left\{-\frac{1}{2}(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)V^\alpha V^a X_a + \frac{1}{4}(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)\underline{L}^\alpha|V|^2_g\right\} \\
&+ \frac{1}{(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}\frac{1}{L{}^{(\mathfrak{n})}\uptau}\left\{\frac{1}{2}[\check{X}(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)]V^\alpha V^a X_a - \frac{1}{4}[\check{X}(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)]\underline{L}^\alpha|V|^2_g\right\} \\
&- \frac{1}{2}\frac{1}{\upmu}(L{}^{(\mathfrak{n})}\check{R}_\alpha)V^\alpha L_\beta V^\beta - \frac{1}{4}\frac{1}{\upmu}\left\{L({}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha)\right\}|V|^2_g - \frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha(LL_\beta)V^\beta + \frac{1}{4(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}{}^{(\mathfrak{n})}\check{R}_\alpha(L\underline{L}^\alpha)|V|^2_g \\
&- \frac{1}{2(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a LX_a + \frac{\upmu}{\upmu-\phi\frac{\mathfrak{n}}{L\upmu}}\left\{-\frac{1}{2}(LX_\alpha)V^\alpha V^a X_a + \frac{1}{4}(LX_\alpha)\underline{L}^\alpha|V|^2_g\right\} \\
&+ \frac{\upmu}{\upmu-\phi\frac{\mathfrak{n}}{L\upmu}}\left(\phi\frac{\mathfrak{n}}{\upmu L\upmu} + {}^{(\mathfrak{n})}r\right)\left\{-\frac{1}{2}(LL_\alpha)V^\alpha V^a X_a + \frac{1}{4}(LL_\alpha)\underline{L}^\alpha|V|^2_g\right\} \\
&+ \frac{\upmu}{(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}\frac{(LL{}^{(\mathfrak{n})}\uptau)}{(L{}^{(\mathfrak{n})}\uptau)^2}\left\{-\frac{1}{2}(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)V^\alpha V^a X_a + \frac{1}{4}(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)\underline{L}^\alpha|V|^2_g\right\} \\
&+ \frac{\upmu}{(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}\frac{1}{L{}^{(\mathfrak{n})}\uptau}\left\{\frac{1}{2}\left[L(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)\right]V^\alpha V^a X_a - \frac{1}{4}\left[L(\slashed{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\uptau)\right]\underline{L}^\alpha|V|^2_g\right\} - \frac{1}{2}\frac{\upmu(L{}^{(\mathfrak{n})}r)}{\upmu-\phi\frac{\mathfrak{n}}{L\upmu}}|\slashed{V}|^2_{\slashed{g}} \\
&- \frac{1}{\upmu}({}^{(\mathfrak{n})}\check{R}\mathbf{g}_{\alpha\beta})L^\alpha L_\gamma V^\gamma V^\beta - \frac{1}{2}\frac{1}{\upmu}({}^{(\mathfrak{n})}\check{R}\mathbf{g}_{\alpha\beta})V^\alpha V^\beta + \frac{1}{4\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha(\check{X}g_{ab})V^a V^b \\
&+ \frac{1}{\upmu}{}^{(\mathfrak{n})}\check{R}^\alpha(L\mathbf{g}_{\alpha\beta})L_\gamma V^\gamma V^\beta - \frac{1}{4}\frac{1}{\upmu}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha(L\mathbf{g}_{\beta\gamma})V^\beta V^\gamma + \frac{1}{4(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha(Lg_{ab})V^a V^b \\
&+ \frac{1}{\upmu}{}^{(\mathfrak{n})}\check{R}_\alpha\overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_\gamma\mathbf{g}_{\delta\kappa})V^\delta V^\kappa - \frac{1}{\upmu}{}^{(\mathfrak{n})}\check{R}_\alpha\overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_\delta\mathbf{g}_{\gamma\kappa})V^\delta V^\kappa \\
&+ \frac{\phi'\frac{\mathfrak{n}}{L\upmu}}{\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})^2}\left\{-\frac{1}{2}{}^{(\mathfrak{n})}\check{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}{}^{(\mathfrak{n})}\check{R}_\alpha\underline{L}^\alpha|V|^2_g\right\} - \frac{1}{2}\frac{\phi'\frac{\mathfrak{n}}{L\upmu}}{\upmu(\upmu-\phi\frac{\mathfrak{n}}{L\upmu})}|\slashed{V}|^2_{\slashed{g}}.
\end{aligned}
\tag{21.47}
$$

Moreover, on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$, *the term* $\mathfrak{P}[V,V]$ *defined in* (21.45) *is quantitatively positive definite in the following sense:*

$$\mathfrak{P}[V,V] \approx \frac{1}{\mu - \phi\frac{\mathfrak{n}}{L\mu}}|V|_g^2. \tag{21.48}$$

Proof. First, using (21.1) and (21.29), we compute that:

$$\mathscr{J}^\alpha[V,\boldsymbol{\partial}V] = V^\beta \overline{\Pi}_\gamma^{\ \alpha}\partial_\beta V^\gamma + \frac{1}{2}V^\beta L_\beta \underline{L}V^\alpha - \frac{1}{2}V^\alpha L_\beta \underline{L}V^\beta - V^\beta \overline{\Pi}_\beta^{\ \alpha}\partial_a V^a. \tag{21.49}$$

Next, we note the following identity, where we recall that $(\mathrm{d}V_\flat)_{\alpha\beta} \stackrel{\mathrm{def}}{=} \partial_\alpha V_\beta - \partial_\beta V_\alpha$:

$$\partial_\beta V^\gamma = \mathbf{g}_{\beta\kappa}(\mathbf{g}^{-1})^{\gamma\lambda}\partial_\lambda V^\kappa + (\mathbf{g}^{-1})^{\gamma\kappa}(\mathrm{d}V_\flat)_{\beta\kappa} + (\mathbf{g}^{-1})^{\gamma\kappa}(\partial_\kappa \mathbf{g}_{\beta\lambda})V^\lambda - (\mathbf{g}^{-1})^{\gamma\kappa}(\partial_\beta \mathbf{g}_{\kappa\lambda})V^\lambda. \tag{21.50}$$

Using (21.50), we rewrite the first product on RHS (21.49) as follows:

$$\begin{aligned} V^\beta \overline{\Pi}_\gamma^{\ \alpha}\partial_\beta V^\gamma = {} & \overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}V_\delta \partial_\gamma V^\delta + \overline{\Pi}_\beta^{\ \alpha}V^\gamma(\mathbf{g}^{-1})^{\beta\delta}(\mathrm{d}V_\flat)_{\gamma\delta} \\ & + \overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_\gamma \mathbf{g}_{\delta\kappa})V^\delta V^\kappa - \overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_\delta \mathbf{g}_{\gamma\kappa})V^\delta V^\kappa. \end{aligned} \tag{21.51}$$

Next, using the identity $\underline{L} = 2\mathbf{B} - L$ (see Lemma 7.3), we rewrite the second and third products on RHS (21.49) as follows:

$$\frac{1}{2}V^\beta L_\beta \underline{L}V^\alpha - \frac{1}{2}V^\alpha L_\beta \underline{L}V^\beta = -\frac{1}{2}V^\beta L_\beta L V^\alpha + \frac{1}{2}V^\alpha L_\beta L V^\beta + V^\beta L_\beta \mathbf{B}V^\alpha - V^\alpha L_\beta \mathbf{B}V^\beta. \tag{21.52}$$

Differentiating by parts in the second term on RHS (21.52), we rewrite (21.52) as follows:

$$\begin{aligned} \frac{1}{2}V^\beta L_\beta \underline{L}V^\alpha - \frac{1}{2}V^\alpha L_\beta \underline{L}V^\beta = {} & -V^\beta L_\beta L V^\alpha + \frac{1}{2}L(V^\alpha L_\beta V^\beta) \\ & - \frac{1}{2}V^\alpha (LL_\beta)V^\beta + V^\beta L_\beta \mathbf{B}V^\alpha - V^\alpha L_\beta \mathbf{B}V^\beta. \end{aligned} \tag{21.53}$$

Next, with the help of (21.50), we rewrite the first product on RHS (21.53) as follows:

$$\begin{aligned} V^\beta L_\beta L V^\alpha = {} & V^\beta L_\beta (\mathbf{g}^{-1})^{\alpha\gamma}(\partial_\gamma V^\delta)L_\delta + V^\beta L_\beta L^\gamma (\mathbf{g}^{-1})^{\alpha\delta}(\mathrm{d}V_\flat)_{\gamma\delta} \\ & + (\mathbf{g}^{-1})^{\alpha\beta}(\partial_\beta \mathbf{g}_{\gamma\delta})L^\gamma L_\kappa V^\kappa V^\delta - (\mathbf{g}^{-1})^{\alpha\beta}(L\mathbf{g}_{\beta\gamma})L_\delta V^\delta V^\gamma. \end{aligned} \tag{21.54}$$

Combining (21.49)–(21.54), we find that:

$$\begin{aligned} \mathscr{J}^\alpha[V,\boldsymbol{\partial}V] = {} & \frac{1}{2}L(V^\alpha L_\beta V^\beta) + \overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}V_\delta \partial_\gamma V^\delta - V^\beta L_\beta (\mathbf{g}^{-1})^{\alpha\gamma}(\partial_\gamma V^\delta)L_\delta \\ & + \overline{\Pi}_\beta^{\ \alpha}V^\gamma (\mathbf{g}^{-1})^{\beta\delta}(\mathrm{d}V_\flat)_{\gamma\delta} - V^\beta L_\beta L^\gamma (\mathbf{g}^{-1})^{\alpha\delta}(\mathrm{d}V_\flat)_{\gamma\delta} \\ & + \overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_\gamma \mathbf{g}_{\delta\kappa})V^\delta V^\kappa - \overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_\delta \mathbf{g}_{\gamma\kappa})V^\delta V^\kappa \\ & - (\mathbf{g}^{-1})^{\alpha\beta}(\partial_\beta \mathbf{g}_{\gamma\delta})L^\gamma L_\kappa V^\kappa V^\delta + (\mathbf{g}^{-1})^{\alpha\beta}(L\mathbf{g}_{\beta\gamma})L_\delta V^\delta V^\gamma - \frac{1}{2}V^\alpha (LL_\beta)V^\beta \\ & + V^\beta L_\beta \mathbf{B}V^\alpha - V^\alpha L_\beta \mathbf{B}V^\beta - \overline{\Pi}_\beta^{\ \alpha}V^\beta \partial_a V^a. \end{aligned} \tag{21.55}$$

Next, taking into account definition (21.1) and differentiating by parts, we express the second and third products on RHS (21.55) as follows:

$$\begin{aligned} & \overline{\Pi}_\beta^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}V_\delta \partial_\gamma V^\delta - V^\beta L_\beta (\mathbf{g}^{-1})^{\alpha\gamma}(\partial_\gamma V^\delta)L_\delta \\ & = \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta(|V|_g^2) + \frac{1}{4}\underline{L}^\alpha L(|V|_g^2) - \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta\left\{(V^\gamma L_\gamma)^2\right\} \\ & \quad - \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}(\partial_\beta \mathbf{g}_{\gamma\delta})V^\gamma V^\delta - \frac{1}{4}\underline{L}^\alpha (L\mathbf{g}_{\gamma\delta})V^\gamma V^\delta + (\mathbf{g}^{-1})^{\alpha\beta}(\partial_\beta L_\gamma)V^\gamma V^\delta L_\delta. \end{aligned} \tag{21.56}$$

Using (21.56) to substitute for the second and third products on RHS (21.55), we deduce that:

$$\begin{aligned}
\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V] &= \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}\partial_{\beta}(|V|_g^2) - \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}\partial_{\beta}\left\{(V^{\gamma}L_{\gamma})^2\right\} + \frac{1}{2}L(V^{\alpha}L_{\beta}V^{\beta}) + \frac{1}{4}\underline{L}^{\alpha}L(|V|_g^2) \\
&\quad + \overline{\Pi}_{\beta}^{\ \alpha}V^{\gamma}(\mathbf{g}^{-1})^{\beta\delta}(\mathrm{d}V_{\flat})_{\gamma\delta} - V^{\beta}L_{\beta}L^{\gamma}(\mathbf{g}^{-1})^{\alpha\delta}(\mathrm{d}V_{\flat})_{\gamma\delta} \\
&\quad + \overline{\Pi}_{\beta}^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_{\gamma}\mathbf{g}_{\delta\kappa})V^{\delta}V^{\kappa} - \overline{\Pi}_{\beta}^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_{\delta}\mathbf{g}_{\gamma\kappa})V^{\delta}V^{\kappa} \\
&\quad - (\mathbf{g}^{-1})^{\alpha\beta}(\partial_{\beta}\mathbf{g}_{\gamma\delta})L^{\gamma}L_{\kappa}V^{\kappa}V^{\delta} + (\mathbf{g}^{-1})^{\alpha\beta}(L\mathbf{g}_{\beta\gamma})L_{\delta}V^{\delta}V^{\gamma} - \frac{1}{2}V^{\alpha}(LL_{\beta})V^{\beta} \\
&\quad + V^{\beta}L_{\beta}\mathbf{B}V^{\alpha} - V^{\alpha}L_{\beta}\mathbf{B}V^{\beta} - \overline{\Pi}_{\beta}^{\ \alpha}V^{\beta}\partial_a V^a \\
&\quad - \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}(\partial_{\beta}\mathbf{g}_{\gamma\delta})V^{\gamma}V^{\delta} - \frac{1}{4}\underline{L}^{\alpha}(L\mathbf{g}_{\gamma\delta})V^{\gamma}V^{\delta} + (\mathbf{g}^{-1})^{\alpha\beta}(\partial_{\beta}L_{\gamma})V^{\gamma}V^{\delta}L_{\delta}.
\end{aligned} \tag{21.57}$$

Next, we note that since $\mathscr{J}[V,\boldsymbol{\partial}V]$ is tangent to $\mathcal{P}_u$, we have $L_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V] = 0$. From this fact and equations (6.7) and (6.20d), it follows that $\frac{|^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}}{\upmu}{}^{(\mathfrak{n})}\hat{N}_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V] = \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V]$. Using this identity, (21.57), the identity $V^{\gamma}L_{\gamma} = -V^aX_a$ implied by the fact that $V^{\gamma}\mathbf{B}_{\gamma} = 0$ and the identity $\mathbf{B} = L + X$ (see (3.24)), and using that $|V|_g^2 - (V^aX_a)^2 = |V|_{g\!\!\!/}^2$ (see (3.34a)) we deduce that:

$$\begin{aligned}
\frac{|^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}}{\upmu}{}^{(\mathfrak{n})}\hat{N}_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V] &= \frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}\left\{|V\!\!\!/|_{g\!\!\!/}^2\right\} + \frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}L(V^{\alpha}L_{\beta}V^{\beta}) + \frac{1}{4}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\underline{L}^{\alpha}L(|V|_g^2) \\
&\quad + \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\overline{\Pi}_{\beta}^{\ \alpha}V^{\gamma}(\mathbf{g}^{-1})^{\beta\delta}(\mathrm{d}V_{\flat})_{\gamma\delta} - \frac{1}{\upmu}V^{\beta}L_{\beta}L^{\gamma(\mathfrak{n})}\breve{R}^{\delta}(\mathrm{d}V_{\flat})_{\gamma\delta} \\
&\quad + \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\overline{\Pi}_{\beta}^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_{\gamma}\mathbf{g}_{\delta\kappa})V^{\delta}V^{\kappa} - \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\overline{\Pi}_{\beta}^{\ \alpha}(\mathbf{g}^{-1})^{\beta\gamma}(\partial_{\delta}\mathbf{g}_{\gamma\kappa})V^{\delta}V^{\kappa} \\
&\quad - \frac{1}{\upmu}({}^{(\mathfrak{n})}\breve{R}\mathbf{g}_{\alpha\beta})L^{\alpha}L_{\gamma}V^{\gamma}V^{\beta} + \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}^{\alpha}(L\mathbf{g}_{\alpha\beta})L_{\gamma}V^{\gamma}V^{\beta} - \frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}V^{\alpha}(LL_{\beta})V^{\beta} \\
&\quad + \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}V^{\beta}L_{\beta}\mathbf{B}V^{\alpha} - \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}V^{\alpha}L_{\beta}\mathbf{B}V^{\beta} - \frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\overline{\Pi}_{\beta}^{\ \alpha}V^{\beta}\partial_a V^a \\
&\quad - \frac{1}{2}\frac{1}{\upmu}({}^{(\mathfrak{n})}\breve{R}\mathbf{g}_{\alpha\beta})V^{\alpha}V^{\beta} - \frac{1}{4}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\underline{L}^{\alpha}(L\mathbf{g}_{\beta\gamma})V^{\beta}V^{\gamma} + \frac{1}{\upmu}({}^{(\mathfrak{n})}\breve{R}L_{\alpha})L_{\beta}V^{\alpha}V^{\beta}.
\end{aligned} \tag{21.58}$$

Next, using the identity $L_{\beta}V^{\beta} = -V^aX_a$ noted above and differentiating by parts, we rewrite the terms on the first line of RHS (21.58) as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}\left\{|V\!\!\!/|_{g\!\!\!/}^2\right\} + \frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}L(V^{\alpha}L_{\beta}V^{\beta}) + \frac{1}{4}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\underline{L}^{\alpha}L(|V|_g^2) \\
&= \frac{1}{2}{}^{(\mathfrak{n})}\breve{R}\left\{\frac{1}{\upmu}|V\!\!\!/|_{g\!\!\!/}^2\right\} + L\left\{-\frac{1}{2}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}V^{\alpha}V^aX_a + \frac{1}{4}\frac{1}{\upmu}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\underline{L}^{\alpha}|V|_g^2\right\} \\
&\quad + \frac{1}{2}\frac{({}^{(\mathfrak{n})}\breve{R}\upmu)}{\upmu^2}|V\!\!\!/|_{g\!\!\!/}^2 + \frac{(L\upmu)}{\upmu^2}\left\{-\frac{1}{2}{}^{(\mathfrak{n})}\breve{R}_{\alpha}V^{\alpha}V^aX_a + \frac{1}{4}{}^{(\mathfrak{n})}\breve{R}_{\alpha}\underline{L}^{\alpha}|V|_g^2\right\} \\
&\quad - \frac{1}{2}\frac{1}{\upmu}(L{}^{(\mathfrak{n})}\breve{R}_{\alpha})V^{\alpha}L_{\beta}V^{\beta} - \frac{1}{4}\frac{1}{\upmu}\left\{L({}^{(\mathfrak{n})}\breve{R}_{\alpha}\underline{L}^{\alpha})\right\}|V|_g^2.
\end{aligned} \tag{21.59}$$

Next, we use the identities (7.12a), (7.14b), (7.10), and (7.11), and differentiation by parts to rewrite the terms on the first line of RHS (21.59) as follows, where $\widetilde{\mathrm{d}\not{\mathrm{i}}\mathrm{v}}$ is the divergence operator on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent vectorfields from Def. 6.13:

$$
\begin{aligned}
&\frac{1}{2}{}^{(\mathfrak{n})}\breve{R}\left\{\frac{1}{\mu}|\not{V}|^2_{\not{g}}\right\} + L\left\{-\frac{1}{2}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|^2_g\right\} \\
&= \frac{1}{4}{}^{(\mathfrak{n})}\breve{R}\left\{\frac{1}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\left[(X_aV^a)^2 + (1+2{}^{(\mathfrak{n})}r)|V|^2_{\not{g}} - 2\frac{1}{L{}^{(\mathfrak{n})}\tau}X_aV^a\not{V}^\alpha\not\nabla_\alpha{}^{(\mathfrak{n})}\tau\right]\right\} \\
&\quad + \left\{-\frac{{}^{(\mathfrak{n})}\breve{R}\mu}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})^2} + \frac{\mathfrak{n}\frac{\phi'}{L\mu}}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})^2} - \frac{\phi\mathfrak{n}\frac{{}^{(\mathfrak{n})}\breve{R}L\mu}{(L\mu)^2}}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})^2}\right\}\left\{-\frac{1}{2}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|^2_g\right\} \\
&\quad - \widetilde{\mathrm{d}\not{\mathrm{i}}\mathrm{v}}\left\{\frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\left[-\frac{1}{2}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|^2_g\right]{}^{(\mathfrak{n})}U\right\} \\
&\quad + \left\{\frac{1}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\left[-\frac{1}{2}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|^2_g\right]\right\}\widetilde{\mathrm{d}\not{\mathrm{i}}\mathrm{v}}\,{}^{(\mathfrak{n})}U \\
&\quad - \left\{\frac{({}^{(\mathfrak{n})}U\mu)\phi\frac{\mathfrak{n}}{L\mu}}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})^2} + \frac{\mu\phi\mathfrak{n}\frac{{}^{(\mathfrak{n})}UL\mu}{(L\mu)^2}}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})^2}\right\}\left\{-\frac{1}{2}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|^2_g\right\} \\
&\quad + \frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B}\left\{-\frac{1}{2}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|^2_g\right\}.
\end{aligned}
\tag{21.60}
$$

Before proceeding, we note the following identity, which follows from substituting RHS (7.9) (after lowering indices on both sides via $\mathbf{g}$) for the factor of $\frac{{}^{(\mathfrak{n})}\breve{R}_\alpha}{\mu}$ on LHS (21.61), from using the Leibniz and chain rules, from using (7.12a) to re-express the vectorfield differential operator $\frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B}$ as differentiation with respect to $L + \frac{1}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}{}^{(\mathfrak{n})}\breve{R} + \frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}{}^{(\mathfrak{n})}U$ when the operator falls on the factor of $\frac{1}{\mu}$ on RHS (7.9), from using (3.24) to re-express the vectorfield differential operator $\frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B}$ as differentiation with respect to $\frac{1}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})}\breve{X} + \frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}L$ when the operator falls on any other factor besides $\frac{1}{\mu}$ on RHS (7.9), from using that $\not\nabla_\alpha{}^{(\mathfrak{n})}\tau = \not{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\tau$, from using that $\breve{X}\phi = \phi'$, and from using that $P\phi = 0$ for any $\mathcal{P}_u$-tangent vectorfield P:

$$
\begin{aligned}
\frac{\mu}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})}\mathbf{B}\left\{\frac{{}^{(\mathfrak{n})}\breve{R}_\alpha}{\mu}\right\} &= -\frac{({}^{(\mathfrak{n})}\breve{R}\mu)\phi\frac{\mathfrak{n}}{L\mu}}{\mu^2(\mu-\phi\frac{\mathfrak{n}}{L\mu})}L_\alpha - \frac{(L\mu)\phi\frac{\mathfrak{n}}{L\mu}}{\mu^2}L_\alpha - \frac{({}^{(\mathfrak{n})}U\mu)\phi\frac{\mathfrak{n}}{L\mu}}{\mu(\mu-\phi\frac{\mathfrak{n}}{L\mu})}L_\alpha + \frac{\phi'\frac{\mathfrak{n}}{L\mu}}{\mu(\mu-\phi\frac{\mathfrak{n}}{L\mu})}L_\alpha \\
&\quad - \phi\frac{\mathfrak{n}(\breve{X}L\mu)}{\mu(\mu-\phi\frac{\mathfrak{n}}{L\mu})(L\mu)^2}L_\alpha - \phi\frac{\mathfrak{n}(LL\mu)}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})(L\mu)^2}L_\alpha \\
&\quad + \frac{1}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\breve{X}X_\alpha + \frac{\mu}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})}LX_\alpha + \frac{1}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\left(\phi\frac{\mathfrak{n}}{\mu L\mu} + {}^{(\mathfrak{n})}r\right)\breve{X}L_\alpha \\
&\quad + \frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\left(\phi\frac{\mathfrak{n}}{\mu L\mu} + {}^{(\mathfrak{n})}r\right)LL_\alpha + \frac{(\breve{X}{}^{(\mathfrak{n})}r)}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}L_\alpha + \frac{\mu(L{}^{(\mathfrak{n})}r)}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}L_\alpha \\
&\quad + \frac{1}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})}\frac{(\breve{X}L{}^{(\mathfrak{n})}\tau)}{(L{}^{(\mathfrak{n})}\tau)^2}(\not{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\tau) + \frac{\mu}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})}\frac{(LL{}^{(\mathfrak{n})}\tau)}{(L{}^{(\mathfrak{n})}\tau)^2}(\not{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\tau) \\
&\quad - \frac{1}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})}\frac{1}{L{}^{(\mathfrak{n})}\tau}\breve{X}(\not{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\tau) - \frac{\mu}{(\mu-\phi\frac{\mathfrak{n}}{L\mu})}\frac{1}{L{}^{(\mathfrak{n})}\tau}L(\not{\Pi}_\alpha^{\ \beta}\partial_\beta{}^{(\mathfrak{n})}\tau).
\end{aligned}
\tag{21.61}
$$

Next, we rewrite the terms on the last line of RHS (21.60) using the following strategy. First, we use the Leibniz and chain rules to expand the differentiation with respect to $\frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B}$ in Cartesian coordinates. Second, when this derivative operator falls on a Cartesian component of V, we make no further adjustments. Third, when the derivative operator $\frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B}$ falls on either of the two factors of $\frac{1}{\mu}{}^{(\mathfrak{n})}\breve{R}_\alpha$ in braces on LHS (21.62), we use (21.61) for substitution. Finally, when the derivative operator $\frac{\mu}{\mu-\phi\frac{\mathfrak{n}}{L\mu}}\mathbf{B}$ falls on any other quantity in braces on LHS (21.62), we use (3.24) to re-express the operator as differentiation with respect to $\frac{1}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \breve{X} + \frac{\upmu}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} L$. These four steps allow us to deduce the following identity:

$$
\begin{aligned}
&\frac{\upmu}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \mathbf{B} \left\{ -\frac{1}{2}\frac{1}{\upmu} {}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a X_a + \frac{1}{4}\frac{1}{\upmu} {}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha |V|_g^2 \right\} \\
&= -\frac{1}{2(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha (\mathbf{B} V^\alpha) V^a X_a - \frac{1}{2(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha (\mathbf{B} V^a) X_a + \frac{1}{2(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha V_a \mathbf{B} V^a \\
&\quad - \frac{1}{2\upmu(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a \breve{X} X_a - \frac{1}{2(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha V^\alpha V^a L X_a \\
&\quad + \frac{1}{4\upmu(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha (\breve{X} \underline{L}^\alpha) |V|_g^2 + \frac{1}{4(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha (L \underline{L}^\alpha) |V|_g^2 \\
&\quad + \frac{1}{4\upmu(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha (\breve{X} g_{ab}) V^a V^b + \frac{1}{4(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} {}^{(\mathfrak{n})}\breve{R}_\alpha \underline{L}^\alpha (L g_{ab}) V^a V^b \\
&\quad + \frac{({}^{(\mathfrak{n})}\breve{R}\upmu)\phi \frac{\mathfrak{n}}{L\upmu}}{\upmu^2(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} \left\{ \frac{1}{2} L_\alpha V^\alpha V^a X_a - \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} + \frac{(L\upmu)\phi \frac{\mathfrak{n}}{L\upmu}}{\upmu^2} \left\{ \frac{1}{2} L_\alpha V^\alpha V^a X_a - \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{({}^{(\mathfrak{n})}U\upmu)\phi \frac{\mathfrak{n}}{L\upmu}}{\upmu(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} \left\{ \frac{1}{2} L_\alpha V^\alpha V^a X_a - \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} + \frac{\phi' \frac{\mathfrak{n}}{L\upmu}}{\upmu(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} \left\{ -\frac{1}{2} L_\alpha V^\alpha V^a X_a + \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \phi \frac{\mathfrak{n}(\breve{X} L\upmu)}{\upmu(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})(L\upmu)^2} \left\{ \frac{1}{2} L_\alpha V^\alpha V^a X_a - \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \phi \frac{\mathfrak{n}(LL\upmu)}{(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})(L\upmu)^2} \left\{ \frac{1}{2} L_\alpha V^\alpha V^a X_a - \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{1}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \left\{ -\frac{1}{2} (\breve{X} X_\alpha) V^\alpha V^a X_a + \frac{1}{4} (\breve{X} X_\alpha) \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{\upmu}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \left\{ -\frac{1}{2} (L X_\alpha) V^\alpha V^a X_a + \frac{1}{4} (L X_\alpha) \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{1}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \left(\phi \frac{\mathfrak{n}}{\upmu L\upmu} + {}^{(\mathfrak{n})}r \right) \left\{ -\frac{1}{2} (\breve{X} L_\alpha) V^\alpha V^a X_a + \frac{1}{4} (\breve{X} L_\alpha) \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{\upmu}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \left(\phi \frac{\mathfrak{n}}{\upmu L\upmu} + {}^{(\mathfrak{n})}r \right) \left\{ -\frac{1}{2} (L L_\alpha) V^\alpha V^a X_a + \frac{1}{4} (L L_\alpha) \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{(\breve{X}{}^{(\mathfrak{n})}r)}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \left\{ -\frac{1}{2} L_\alpha V^\alpha V^a X_a + \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{\upmu(L{}^{(\mathfrak{n})}r)}{\upmu - \phi \frac{\mathfrak{n}}{L\upmu}} \left\{ -\frac{1}{2} L_\alpha V^\alpha V^a X_a + \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{1}{(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} \frac{(\breve{X} L{}^{(\mathfrak{n})}\uptau)}{(L{}^{(\mathfrak{n})}\uptau)^2} \left\{ -\frac{1}{2} (\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau) V^\alpha V^a X_a + \frac{1}{4} (\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau) \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{\upmu}{(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} \frac{(L L{}^{(\mathfrak{n})}\uptau)}{(L{}^{(\mathfrak{n})}\uptau)^2} \left\{ -\frac{1}{2} (\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau) V^\alpha V^a X_a + \frac{1}{4} (\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau) \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{1}{(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} \frac{1}{L{}^{(\mathfrak{n})}\uptau} \left\{ \frac{1}{2} [\breve{X}(\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau)] V^\alpha V^a X_a - \frac{1}{4} [\breve{X}(\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau)] \underline{L}^\alpha |V|_g^2 \right\} \\
&\quad + \frac{\upmu}{(\upmu - \phi \frac{\mathfrak{n}}{L\upmu})} \frac{1}{L{}^{(\mathfrak{n})}\uptau} \left\{ \frac{1}{2} [L(\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau)] V^\alpha V^a X_a - \frac{1}{4} [L(\Pi\!\!\!/_\alpha^{\ \beta} \partial_\beta {}^{(\mathfrak{n})}\uptau)] \underline{L}^\alpha |V|_g^2 \right\}.
\end{aligned}
\tag{21.62}
$$

Combining (21.58)–(21.62), using the identity $\frac{1}{2} L_\alpha V^\alpha V^a X_a - \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 = \frac{1}{2} |V\!\!\!/|_{g\!\!\!/}^2$ (which follows from (7.4) and the identities $V^\gamma L_\gamma = -V^a X_a$ and $|V|_g^2 - (V^a X_a)^2 = |V\!\!\!/|_{g\!\!\!/}^2$ mentioned above) to substitute for the factors $\left\{ \frac{1}{2} L_\alpha V^\alpha V^a X_a - \frac{1}{4} L_\alpha \underline{L}^\alpha |V|_g^2 \right\}$

on RHS (21.62), and carrying out straightforward algebraic calculations, we deduce the desired identity (21.44). We clarify that we explicitly placed the perfect ${}^{(\mathfrak{n})}\breve{R}$-derivative and $\widetilde{\mathrm{div}}$-derivative terms as the first and third terms on RHS (21.44), that we placed the remaining terms involving a first derivative of V on RHS (21.46), that we placed all terms that are quadratic in V (without depending on the first derivatives of V) on RHS (21.47), and that we added the term $\frac{1}{2}\mathfrak{P}[V,V]\mathrm{tr}_{\widetilde{g\!\!/}}{}^{({}^{(\mathfrak{n})}\breve{R})}\boldsymbol{\pi}$ to RHS (21.44) (as the second term) and subtracted it on RHS (21.47) (as the first term).

To prove (21.48), we first use (3.31b), Lemma 5.5, and Prop. 9.1 to deduce that $|\nabla\!\!\!/\,{}^{(\mathfrak{n})}\tau|_{g\!\!/}^2 = (g\!\!/^{-1})^{AB}(\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau)\frac{\partial}{\partial x^B}{}^{(\mathfrak{n})}\tau = \mathrm{f}(\gamma)\cdot(\frac{\partial}{\partial x^2}{}^{(\mathfrak{n})}\tau, \frac{\partial}{\partial x^3}{}^{(\mathfrak{n})}\tau)\cdot(\frac{\partial}{\partial x^2}{}^{(\mathfrak{n})}\tau, \frac{\partial}{\partial x^3}{}^{(\mathfrak{n})}\tau)$, where the last expression is schematic. From this expression, (15.11b), the bootstrap assumptions, and Cor. 17.2, we deduce that $|\nabla\!\!\!/\,{}^{(\mathfrak{n})}\tau|_{g\!\!/} \lesssim \varepsilon$. From this estimate, (18.9b), the $g\!\!/$ Cauchy–Schwarz inequality, and Young's inequality, we deduce the following bound for the last product on RHS (21.45): $\left|\frac{1}{L{}^{(\mathfrak{n})}\tau}X_a V^a V\!\!\!/^{\alpha}\nabla\!\!\!/_{\alpha}{}^{(\mathfrak{n})}\tau\right| \lesssim \varepsilon(X_a V^a)^2 + \varepsilon|V\!\!\!/|_{g\!\!/}^2$. From this estimate, the estimate (18.27) for $|{}^{(\mathfrak{n})}r|$, definition (21.45), and the identity $|V|_g^2 = (V^a X_a)^2 + |V|_{g\!\!/}^2$ noted just above (21.58), we conclude (21.48).

□

21.5. **The main elliptic-hyperbolic integral identity.** We now state and prove the main elliptic-hyperbolic integral identity.

Proposition 21.14 (The main elliptic-hyperbolic integral identity)**.** *Let V be a Σ_t-tangent vectorfield on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$, and let $\mathscr{Q}[\boldsymbol{\partial}V,\boldsymbol{\partial}V]$ be the quadratic form from Def. 21.8. Then the following integral identity holds for any $\tau_0 \leq \tau_1 \leq \tau_2 < \tau_{\mathrm{Boot}}$ and $-U_1 \leq u_1 \leq u_2 \leq U_2$:*

$$
\begin{aligned}
&\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}} \frac{1}{L{}^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}V,\boldsymbol{\partial}V]\,\mathrm{d}\varpi + \int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u_2}} \mathfrak{P}[V,V]\,\mathrm{d}\varpi_{\widetilde{g\!\!/}} \\
&= \int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u_1}} \mathfrak{P}[V,V]\,\mathrm{d}\varpi_{\widetilde{g\!\!/}} + \int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_1,u_2}} \mathfrak{P}[V,V]\,\mathrm{d}\varpi_{\widetilde{g\!\!/}} - \int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_1,u_1}} \mathfrak{P}[V,V]\,\mathrm{d}\varpi_{\widetilde{g\!\!/}} \\
&\quad + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_1}^{[u_1,u_2]}} \left\{\mathfrak{E}_{(\mathrm{Principal})}[V,\boldsymbol{\partial}V] + \mathfrak{E}_{(\mathrm{Lower\text{-}order})}[V,V]\right\}\mathrm{d}\underline{\varpi} \\
&\quad - \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}} \left\{\mathfrak{E}_{(\mathrm{Principal})}[V,\boldsymbol{\partial}V] + \mathfrak{E}_{(\mathrm{Lower\text{-}order})}[V,V]\right\}\mathrm{d}\underline{\varpi} + \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}} \mathfrak{M}[V,\boldsymbol{\partial}V]\,\mathrm{d}\varpi,
\end{aligned}
\tag{21.63}
$$

where $\mathfrak{P}[V,V]$ is defined by (21.45) *and is positive definite in the sense that the pointwise estimate* (21.48) *holds, the error terms $\mathfrak{E}_{(\mathrm{Principal})}[V,\boldsymbol{\partial}V]$ and $\mathfrak{E}_{(\mathrm{Lower\text{-}order})}[V,V]$ are defined in* (21.46)–(21.47)*,*

$$
\begin{aligned}
\mathfrak{M}[V,\boldsymbol{\partial}V] \overset{\mathit{def}}{=} \frac{1}{L{}^{(\mathfrak{n})}\tau}\Big\{&\mathfrak{J}_{(\mathrm{Antisymmetric})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] + \mathfrak{J}_{(\mathrm{Div})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] + \mu\mathfrak{J}_{(\partial\frac{1}{\mu})}[V,\boldsymbol{\partial}V] \\
&+ \mathfrak{J}_{(\mathrm{Absorb\text{-}1})}[V,\boldsymbol{\partial}V] + \mathfrak{J}_{(\mathrm{Absorb\text{-}2})}[V,\boldsymbol{\partial}V] + \mathfrak{J}_{(\mathrm{Material})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V] + \mathfrak{J}_{(\mathrm{Null\ Geometry})}[V,\boldsymbol{\partial}V]\Big\},
\end{aligned}
\tag{21.64}
$$

and the error terms $\mathfrak{J}_{(\mathrm{Antisymmetric})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V],\cdots,\mathfrak{J}_{(\mathrm{Null\ Geometry})}[V,\boldsymbol{\partial}V]$ on RHS (21.64) *are defined in* (21.31a)–(21.31g).

Proof. We consider the divergence identity (21.30) with $\mathscr{W} \overset{\mathrm{def}}{=} \frac{1}{\mu}$. We integrate the identity over ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}$ with respect to the canonical volume form (8.14b) of $\mathbf{g}$ in adapted rough coordinates, apply the divergence theorem, and take into account Def. 8.3 and the identity (8.14a), thereby obtaining the following identity, where ${}^{(\mathfrak{n})}\hat{N}$ is the future-directed $\mathbf{g}$-unit normal to the $\mathbf{g}$-spacelike hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[u_1,u_2]}$ (see Prop. 6.7):

$$
\begin{aligned}
&\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}} \frac{1}{L{}^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}V,\boldsymbol{\partial}V]\,\mathrm{d}\varpi \\
&= -\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}} \frac{|{}^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}}{\mu}{}^{(\mathfrak{n})}\hat{N}_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V]\,\mathrm{d}\underline{\varpi} + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_1}^{[u_1,u_2]}} \frac{|{}^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}}{\mu}{}^{(\mathfrak{n})}\hat{N}_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V]\,\mathrm{d}\underline{\varpi} \\
&\quad + \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}} \mathfrak{M}[V,\boldsymbol{\partial}V]\,\mathrm{d}\varpi.
\end{aligned}
\tag{21.65}
$$

We emphasize that there are no null hypersurface boundary integrals in (21.65) because the current $\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V]$ defined in (21.29) is tangent to the null hypersurface $\mathcal{P}_u$ (and thus $L_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V]=0$), and the signs of the first two integrals on RHS (21.65) are tied to the Lorentzian nature of $\mathbf{g}$ and the fact that the $\mathbf{g}$-timelike vectorfield ${}^{(\mathfrak{n})}\hat{N}^{\alpha}$ is future-pointing (and thus outward-pointing to ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}$ along ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}$ and inward-pointing to ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}$ along ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_1}^{[u_1,u_2]}$). Next, we use the identities (21.44) and (20.2) to re-express the integral $\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}}\cdots$ on RHS (21.65) as follows, where we note that integral of the perfect divergence term $-\widetilde{\mathrm{d}\!\!/\mathrm{iv}}\cdots$ on RHS (21.44) vanishes when integrated over any rough torus ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$:

$$
\begin{aligned}
-\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}}\frac{|{}^{(\mathfrak{n})}\breve{R}|_{\widetilde{g}}}{\mu}{}^{(\mathfrak{n})}\hat{N}_{\alpha}\mathscr{J}^{\alpha}[V,\boldsymbol{\partial}V]\,\mathrm{d}\underline{\varpi} &= -\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u_2}}\mathfrak{P}[V,V]\,\mathrm{d}\varpi_{\widetilde{g\!\!/}}+\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u_1}}\mathfrak{P}[V,V]\,\mathrm{d}\varpi_{\widetilde{g\!\!/}} \\
&\quad -\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_2}^{[u_1,u_2]}}\left\{\mathfrak{E}_{(\mathrm{Principal})}[V,\boldsymbol{\partial}V]+\mathfrak{E}_{(\mathrm{Lower\text{-}order})}[V,V]\right\}\mathrm{d}\underline{\varpi}.
\end{aligned}
\tag{21.66}
$$

Moreover, we note that (21.66) also holds with τ_1 in place of τ_2. Using these two identities to substitute for the first two integrals on RHS (21.65), we arrive at the identity (21.63).

□

22. Pointwise estimates for the error terms in the commuted wave equations

We continue to work under the assumptions of Sect. 13.2. In this section, we derive pointwise estimates for the error terms that arise when we commute the wave equations (2.22a)–(2.22d) up to N_{top} times, where we recall that N_{top} is a fixed integer satisfying (10.6). More precisely, for $\Psi\in\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$ and $1\le N\le N_{\mathrm{top}}$, we derive pointwise estimates for the inhomogeneous term ${}^{(\mathcal{P}^N;\Psi)}\mathfrak{G}$ in the μ-weighted geometric wave equation $\mu\square_{\mathbf{g}}\mathcal{P}^N\Psi={}^{(\mathcal{P}^N;\Psi)}\mathfrak{G}$ satisfied by $\mathcal{P}^N\Psi$. These pointwise estimates are a preliminary ingredient for the L^2 estimates that we derive later on. Many of the terms appearing in ${}^{(\mathcal{P}^N;\Psi)}\mathfrak{G}$ are harmless from the point of view of regularity and the strength of their singularity; the bulk of our effort goes towards the most difficult terms, which involve the top-order derivatives of the eikonal function and which we handle by using the modified quantities from Def. 19.2.

22.1. Identification of the most difficult error terms in the commuted wave equations.

Most of the terms in the commuted wave equations are harmless from the point of view of regularity and the strength of their singularity. The next definition captures these "harmless" error terms.

22.1.1. *Harmless wave equation error terms.*

Definition 22.1 (Harmless wave equation error terms)**.** Let $1\le N\le N_{\mathrm{top}}$. We define $\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N]}$ to be any term that satisfies the following pointwise estimate on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$:

$$
\left|\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N]}\right|\lesssim\left|\mathscr{Z}_*^{[1,N+1];1}\vec{\Psi}\right|+\left|\mathscr{Z}_*^{[1,N];1}\gamma\right|+\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|.
\tag{22.1}
$$

The following simple lemma shows that $\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N_{\mathrm{top}}-12]}$ terms are small in the norm $\|\cdot\|_{L^{\infty}({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$.

Lemma 22.2 (L^{∞} estimates for $\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N_{\mathrm{top}}-12]}$)**.** *Let* $\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N_{\mathrm{top}}-12]}$ *be as in Def. 22.1. Then the following estimate holds for* $(\tau,u)\in[\tau_0,\tau_{\mathrm{Boot}})\times[-U_1,U_2]$*:*

$$
\left\|\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N_{\mathrm{top}}-12]}\right\|_{L^{\infty}({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}\lesssim\varepsilon.
\tag{22.2}
$$

Proof. The estimate (22.2) follows from definition (22.1) and Prop. 17.1. □

22.1.2. *The most difficult error terms.*

In the following proposition, we identify the most difficult error terms in the commuted wave equations satisfied by the wave-variables $\vec{\Psi}$.

Proposition 22.3 (Identification of the most difficult error terms in the commuted wave equations)**.** *Let* $\vec{\Psi}\stackrel{\mathrm{def}}{=}(\Psi_0,\Psi_1,\Psi_2,\Psi_3,\Psi_4)\stackrel{\mathrm{def}}{=}(\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s)$ *be solutions to the covariant wave equations* (2.22)*, and let* $N\le N_{\mathrm{top}}-1$*. We denote the product of*

$\upmu$ and the RHS of the covariant wave equation satisfied by Ψ_ι by $\mathfrak{G}_\iota$, i.e., $\upmu\square_{\mathbf{g}}\Psi_\iota = \mathfrak{G}_\iota$. Then the following wave equations hold for $\iota = 0,1,2,3,4$ and $A = 2,3$:

$$\upmu\square_{\mathbf{g}}(\mathcal{Y}^{N-1}L\Psi_\iota) = {d\!\!\!/}^{\#}\Psi_\iota \cdot \upmu {d\!\!\!/}\mathcal{Y}^{N-1}\mathrm{tr}_{g\!\!\!/}\chi + \mathcal{Y}^{N-1}L\mathfrak{G}_\iota + \mathrm{Harmless}^{[1,N]}_{(\mathrm{Wave})}, \tag{22.3a}$$

$$\begin{aligned}\upmu\square_{\mathbf{g}}(\mathcal{Y}^{N-1}Y_{(A)}\Psi_\iota) &= (\breve{X}\Psi_\iota)\mathcal{Y}^{N-1}Y_{(A)}\mathrm{tr}_{g\!\!\!/}\chi + (c^{-2}X^A){d\!\!\!/}^{\#}\Psi_\iota \cdot \upmu{d\!\!\!/}\mathcal{Y}^{N-1}\mathrm{tr}_{g\!\!\!/}\chi \\ &\quad + \mathcal{Y}^{N-1}Y_{(A)}\mathfrak{G}_\iota + \mathrm{Harmless}^{[1,N]}_{(\mathrm{Wave})}.\end{aligned} \tag{22.3b}$$

Moreover, if $1 \leq N \leq N_{\mathrm{top}}$ and $\mathcal{P}^N$ denotes any order N string of $\mathcal{P}_u$-tangent commutator vectorfields other than the ones appearing on LHSs (22.3a)–(22.3b)*, (i.e., if $\mathcal{P}^N$ features at least two copies of L or only a single L that does not hit Ψ_ι first), then $\mathcal{P}^N\Psi_\iota$ obeys the following wave equation:*

$$\upmu\square_{\mathbf{g}}(\mathcal{P}^N\Psi_\iota) = \mathcal{P}^N\mathfrak{G}_\iota + \mathrm{Harmless}^{[1,N]}_{(\mathrm{Wave})}. \tag{22.3c}$$

Proof. In $2D$, a detailed proof was provided in [50, Proposition 13.2], except there are no vorticity-involving or entropy-involving terms in our definition of $\mathrm{Harmless}^{[1,N]}_{(\mathrm{Wave})}$ because we have soaked such terms into our definition of $\mathfrak{G}_\iota$. Only minor changes are needed to account for the third space dimension, so we omit the details here. □

22.2. **Pointwise estimates for the difficult product $(\breve{X}\Psi_\iota)\mathcal{Y}^N\mathbf{tr}_{g\!\!\!/}\chi$.** In this section, we derive pointwise estimates tied to the most difficult terms appearing in the commuted wave equations, specifically the products $(\breve{X}\Psi_\iota)\mathcal{Y}^N\mathrm{tr}_{g\!\!\!/}\chi$ on RHS (22.3b) (note that $\mathcal{Y}^{N-1}Y_{(A)}$ can be expressed as $\mathcal{Y}^N$). Our analysis relies on the fully modified quantities ${}^{(\mathcal{Y}^N)}\mathscr{X}$ from Def. 19.2.

22.2.1. *Pointwise estimates for the inhomogeneous terms in the transport equations satisfied by the modified quantities.* We start with the following lemma, which provides pointwise estimates for the inhomogeneous terms in the transport equations satisfied by the fully modified quantities. The lemma also provides, for use in Sect. 22.3, pointwise estimates for the inhomogeneous terms in the transport equations satisfied by the partially modified quantities.

Lemma 22.4 (Pointwise estimates for inhomogeneous terms tied to the modified quantities)**.** *Let $N = N_{\mathrm{top}}$.*

***Estimates tied to the fully modified quantities**. Let $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$ where $\mathfrak{Y}^{(N)}$ is the set of order N $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Let $\mathfrak{X}$ be the term defined in* (19.6b)*, and let $\mathfrak{A}$ be the term appearing on RHS* (19.1) *and enjoying the schematic structure* (19.2)*. Then the following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$:*

$$\left|\mathcal{Y}^N\mathfrak{X} + \vec{G}_{LL}\diamond\breve{X}\mathcal{Y}^N\vec{\Psi}\right| \lesssim \upmu\left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \left|\mathcal{Z}_*^{[1,N];1}\vec{\Psi}\right| + \left|\mathcal{P}^{[1,N]}\gamma\right| + \left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|, \tag{22.4a}$$

$$\left|\mathcal{P}^N\mathfrak{X}\right| \lesssim \left|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}\right| + \left|\mathcal{P}^{[1,N]}\gamma\right| + \left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|, \tag{22.4b}$$

$$\begin{aligned}|\mathcal{Y}^N\mathfrak{A}| &\lesssim \upmu|\mathcal{Y}^N(\mathcal{C},\mathcal{D})| + |\mathcal{Y}^{\leq N-1}(\mathcal{C},\mathcal{D})| + |\mathcal{Y}^{\leq N}(\Omega,S)| \\ &\quad + |\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}| + |\mathcal{P}^{[1,N]}\gamma| + |\mathcal{P}_*^{[1,N]}\underline{\gamma}|.\end{aligned} \tag{22.4c}$$

***Estimates tied to the partially modified quantities**. If $\mathcal{Y}^{N-1} \in \mathfrak{Y}^{(N-1)}$ and $\widetilde{\mathfrak{X}}$, ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}$, ${}^{(\mathcal{Y}^{N-1})}\mathfrak{B}$ are as defined in* (19.8)*,* (19.7b) *(with $\mathcal{Y}^{N-1}$ in the role of $\mathcal{P}^N$), and* (19.12) *(with $\mathcal{Y}^{N-1}$ in the role of $\mathcal{P}^{N-1}$) respectively, then the following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$:*

$$\left|{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}\right| \lesssim \left|\mathcal{P}^{[1,N]}\vec{\Psi}\right|, \tag{22.5a}$$

$$\left|L^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}\right|, \left|Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}\right| \lesssim \left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right|, \tag{22.5b}$$

$$\left|{}^{(\mathcal{Y}^{N-1})}\mathfrak{B}\right| \lesssim \left|\mathcal{P}^{[1,N]}\gamma\right|. \tag{22.5c}$$

Proof. All estimates stated in the lemma are straightforward consequences of Lemma 3.13, Prop. 9.1, Lemma 9.7, the commutator estimate (13.6a), and the estimates of Prop. 17.1. □

22.2.2. *Preliminary pointwise estimates for* ${}^{(\mathcal{Y}^N)}\mathscr{X}$. In the next lemma, we use the transport equation (19.10) to derive a preliminary pointwise estimate for the fully modified quantity ${}^{(\mathcal{P}^N)}\mathscr{X}$ in the case $\mathcal{P}^N = \mathcal{Y}^N$ with $N = N_{\mathrm{top}}$, which in practice is the only case in which we need to use the fully modified quantities. The estimates in the lemma are crucial ingredient in our proof of Prop. 22.8, in which we derive the main pointwise estimate for the difficult product $(\check{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not g}\chi$. The proof of the lemma is similar to the proof of [73, Lemma 11.9], but due to our reliance on the adapted rough coordinates, which are a new feature of the present paper, we provide complete details here.

Remark 22.5 (Boxed constants affect high order energy blowup-rates)**.** In Lemma 22.6 and its proof, and also throughout the rest of the paper, the important boxed constants such as $\boxed{2}$ affect the blowup-rate of our top-order energies with respect to powers of $|\tau|^{-1}$; see Prop. 24.1. We therefore carefully track these boxed constants.

Lemma 22.6 (Pointwise estimates for ${}^{(\mathcal{Y}^{N_{\mathrm{top}}})}\mathscr{X}$)**.** *Let $N = N_{\mathrm{top}}$, and let $\mathfrak{Y}^{(N)}$ and $\not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}$ be the sets of order N $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Let $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$, and let ${}^{(\mathcal{Y}^N)}\mathscr{X}$ be the corresponding fully modified quantity defined in* (19.6a) *(with $\mathcal{Y}^N$ in the role of $\mathcal{P}^N$). Let ${}^{(\mathfrak{n})}\widetilde{L}$ be the rough null vectorfield defined in* (6.3)*, and let ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ be the τ_0-normalized flow map of ${}^{(\mathfrak{n})}\widetilde{L}$ with respect to the adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$ appearing in Lemma 16.1. Moreover, let ${}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}$ be the spacetime neighborhood constructed in Prop. 18.1 (specifically, in* (18.12)*), on which we have derived especially sharp control of μ. In addition, if $\mathfrak{K}$ is a spacetime subset, let $\mathbf{1}_{\mathfrak{K}}$ denote the characteristic function $\mathfrak{K}$. Then relative to the adapted rough coordinates (τ, u, x^2, x^3) (see Remark 5.3), the following pointwise estimate holds on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$ (i.e., for $(\tau,u,x^2,x^3) \in [\tau_0,\tau_{\mathrm{Boot}}) \times [-U_1,U_2] \times \mathbb{T}^2$), where $\mathcal{Y}^N$ denotes the same operator in each term in* (22.6) *(except for the one term in which $\max_{\not{\mathcal{L}}_{\mathcal{Y}^N} \in \not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}}$ is taken):*

$$
\begin{aligned}
&\left|{}^{(\mathcal{Y}^N)}\mathscr{X}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
&\leq C\left|{}^{(\mathcal{Y}^N)}\mathscr{X}\right|(\tau_0,u,x^2,x^3) \\
&\quad + \boxed{2}\int_{\tau'=\tau_0}^{\tau} \left\{ \left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right| \cdot \left|\mathcal{Y}^N \mathfrak{X}\right| \cdot \mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]} \cap {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}} \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\, d\tau' \\
&\quad + C\varepsilon \int_{\tau'=\tau_0}^{\tau} \left\{ \max_{\not{\mathcal{L}}_{\mathcal{Y}}^N \in \not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}} \left|\mu \not{\mathcal{L}}_{\mathcal{Y}}^N \chi\right| \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\, d\tau' \\
&\quad + C\int_{\tau'=\tau_0}^{\tau} \left\{ \mu|\mathcal{Y}^N(\mathcal{C},\mathcal{D})| + |\mathcal{Y}^{\leq N-1}(\mathcal{C},\mathcal{D})| \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\, d\tau' \\
&\quad + C\int_{\tau'=\tau_0}^{\tau} |\mathcal{Y}^{\leq N}(\Omega,S)| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\, d\tau' \\
&\quad + C\int_{\tau'=\tau_0}^{\tau} \left\{ |\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}| + |\mathcal{P}^{[1,N]}\gamma| + |\mathcal{P}_*^{[1,N]}\underline{\gamma}| \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\, d\tau'.
\end{aligned}
\tag{22.6}
$$

Proof. Our analysis relies on the transport equation (19.10). To control solutions to this equation, we will use the integrating factor $\mathscr{I}$, which we define as follows relative to the adapted rough coordinates:

$$
\begin{aligned}
\mathscr{I}(\tau,u,x^2,x^3) &\stackrel{\mathrm{def}}{=} \frac{\mu^2 \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_0,u,x^2,x^3)}{\mu^2 \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)} \\
&= \exp\left\{ -2\int_{\tau_0}^{\tau} \left(\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right) \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\, d\tau' \right\},
\end{aligned}
\tag{22.7}
$$

where to obtain the second equality in (22.7), we have used (16.1) and the fundamental theorem of calculus. We also recall that by (16.1), we have ${}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau_0,u,x^2,x^3) = (\tau_0,u,x^2,x^3)$ and thus $\mathscr{I}(\tau_0,u,x^2,x^3) = 1$.

We now fix $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$. Since ${}^{(\mathfrak{n})}\widetilde{L} = \frac{1}{L^{(\mathfrak{n})}\tau} L$ (see (6.3)), we can multiply both sides of (19.10) (with $\mathcal{Y}^N$ in the role of $\mathcal{P}^N$) by $\frac{1}{L^{(\mathfrak{n})}\tau}$, evaluate at ${}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)$, then multiply both sides by $\mathscr{I}(\tau',u,x^2,x^3)$, integrate in rough time, use (16.1) and the fundamental theorem of calculus, use that $\mathscr{I}(\tau_0,u,x^2,x^3) = 1$, and finally divide the resulting identity by

$\mathscr{I}(\tau,u,x^2,x^3)$ to deduce the following equation, valid for $(\tau,u,x^2,x^3)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]\times\mathbb{T}^2$:

$$\begin{aligned}{}^{(\mathcal{Y}^N)}\mathscr{X}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) &= \frac{1}{\mathscr{I}(\tau,u,x^2,x^3)}{}^{(\mathcal{Y}^N)}\mathscr{X}(\tau_0,u,x^2,x^3)\\ &\quad+\int_{\tau_0}^{\tau}\frac{\mathscr{I}(\tau',u,x^2,x^3)}{\mathscr{I}(\tau,u,x^2,x^3)}\times\left\{\frac{1}{L^{(\mathfrak{n})}\tau}\times(\text{RHS (19.10)})\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.\end{aligned} \tag{22.8}$$

Next, we note the following estimate, which follows from (18.10a)–(18.10b):

$$\sup_{\substack{\tau\in[\tau_0,\tau_{\text{Boot}}]\\(\tau',u,x^2,x^3)\in[\tau_0,\tau]\times[-U_1,U_2]\times\mathbb{T}^2}}\frac{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)}{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)}\leq C. \tag{22.9}$$

From (22.9) and definition (22.7), we see that:

$$\sup_{(\tau,u,x^2,x^3)\in[\tau_0,\tau_{\text{Boot}}]\times[-U_1,U_2]\times\mathbb{T}^2}\frac{1}{\mathscr{I}(\tau,u,x^2,x^3)}\leq C. \tag{22.10}$$

We will now derive pointwise estimates for the terms on RHS (22.8). First, we use (22.10) to deduce that the first term on RHS (22.8) is bounded in magnitude by the first term on RHS (22.6) as desired.

We now bound the term on RHS (22.8) generated by the first term on RHS (19.10) (with $\mathcal{Y}^N$ in the role of $\mathcal{P}^N$) as follows, where throughout the rest of the proof, we use that $\frac{\mathscr{I}(\tau',u,x^2,x^3)}{\mathscr{I}(\tau,u,x^2,x^3)}=\frac{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)}{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)}$:

$$\begin{aligned}&\left|-2\int_{\tau_0}^{\tau}\frac{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)}{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)}\left\{\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathcal{Y}^N\mathfrak{X}\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\right|\\ &\leq 2\int_{\tau_0}^{\tau}\frac{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)}{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)}\left\{\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right|\cdot\left|\mathcal{Y}^N\mathfrak{X}\right|\cdot\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}}\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\ &\quad+2\int_{\tau_0}^{\tau}\frac{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)}{\mu^2\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)}\left\{\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right|\cdot\left|\mathcal{Y}^N\mathfrak{X}\right|\cdot\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\setminus{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}}\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.\end{aligned} \tag{22.11}$$

To handle the first integral on RHS (22.11), we use (18.10a) to bound it by:

$$\leq\boxed{2}\int_{\tau_0}^{\tau}\left\{\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right|\cdot\left|\mathcal{Y}^N\mathfrak{X}\right|\cdot\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}}\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau', \tag{22.12}$$

which is $\leq$ the second term on RHS (22.6) as desired. To handle the second integral on RHS (22.11), we use the crude bounds $|{}^{(\mathfrak{n})}\widetilde{L}\mu|\lesssim 1$ and $|\mu|\lesssim 1$ (which follow from the bootstrap assumptions), as well as (18.16) and (22.4b), to bound it by the last integral on RHS (22.6) as desired.

Next, to handle the term on RHS (22.8) generated by the second term on RHS (19.10), we first use (22.9) to bound it in magnitude as follows:

$$\leq C\int_{\tau'=\tau_0}^{\tau}\left|\frac{1}{L^{(\mathfrak{n})}\tau}\times\mu[L,\mathcal{Y}^N]\mathrm{tr}_{\not{g}}\chi\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'. \tag{22.13}$$

Next, using the commutator estimate (13.6a), Lemma 13.3, Prop. 13.7, Prop. 17.1, Cor. 17.2, and (18.9b), we deduce the following bound:

$$\begin{aligned}|\text{RHS (22.13)}| &\leq C\varepsilon\int_{\tau'=\tau_0}^{\tau}\left\{\max_{\not{\mathcal{L}}_{\mathcal{Y}}^N\in\not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}}\left|\mu\not{\mathcal{L}}_{\mathcal{Y}}^N\chi\right|\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\ &\quad+C\int_{\tau'=\tau_0}^{\tau}\left\{|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}|+|\mathcal{P}^{[1,N]}\gamma|+|\mathcal{P}_*^{[1,N]}\underline{\gamma}|\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.\end{aligned} \tag{22.14}$$

The first term on RHS (22.14) is bounded by the $C\varepsilon$-multiplied term on RHS (22.6), while the last term on RHS (22.14) is bounded by the last term on RHS (22.6).

To handle the terms on RHS (22.8) generated by the three commutator terms on the second line of RHS (19.10), we can use the same arguments given above to bound them as follows:

$$\leq C \int_{\tau'=\tau_0}^{\tau} \left\{ |\mathcal{Z}_*^{[1,N+1];1} \vec{\Psi}| + |\mathcal{P}^{[1,N]}\gamma| + |\mathcal{P}_*^{[1,N]}\underline{\gamma}| \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\, d\tau', \tag{22.15}$$

which in turn is bounded by the last term on RHS (22.6) as desired.

To handle the terms on RHS (22.8) arising from the term $\mathcal{Y}^N(\mu|\chi|_{\not{g}}^2)$ on RHS (19.10), we expand this term using the Leibniz rule for the operators $\not{\mathcal{L}}_{Y_{(A)}}$. Then using the same arguments we used in proving (22.14) (except no commutator estimates are needed), we find that:

$$\left|\mathcal{Y}^N(\mu|\chi|_{\not{g}}^2)\right| \lesssim \varepsilon \left|\mu \not{\mathcal{L}}_{\mathcal{Y}}^N \chi\right|_{\not{g}} + \left|\mathcal{P}^{[1,N]}\gamma\right| + \left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|. \tag{22.16}$$

Using (18.9b) and (22.16), we see that the time integral of the terms in (22.8) generated by the term $\mathcal{Y}^N(\mu|\chi|_{\not{g}}^2)$ can be bounded as follows:

$$\begin{aligned} &\leq C\varepsilon \int_{\tau'=\tau_0}^{\tau} \left\{ |\mu \not{\mathcal{L}}_{\mathcal{Y}}^N \chi| \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\, d\tau' \\ &\quad + C \int_{\tau'=\tau_0}^{\tau} \left\{ |\mathcal{P}^{[1,N]}\gamma| + |\mathcal{P}_*^{[1,N]}\underline{\gamma}| \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\, d\tau', \end{aligned} \tag{22.17}$$

which in turn is bounded by RHS (22.6) as desired.

Finally, to handle the terms on RHS (22.8) arising from the term $\mathcal{Y}^N \mathfrak{A}$ on RHS (19.10), we use (18.9b) and the pointwise estimate (22.4c) to bound the magnitude of the time integral of the product of $\frac{1}{L^{(\mathfrak{n})}\tau}$ and $\mathcal{Y}^N \mathfrak{A}$ by the sum of the last three integrals on RHS (22.6) as desired. We have therefore proved the lemma. □

22.2.3. *The main pointwise estimates for* $(\check{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not{g}}\chi$. We are now ready to prove Prop. 22.8, which provides the main pointwise estimate for the product $(\check{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not{g}}\chi$ on RHS (22.3b) (with $\iota = 0$) in the case $N = N_{\mathrm{top}}$. The proof relies on the pointwise estimates for ${}^{(\mathcal{Y}^{N_{\mathrm{top}}})}\mathscr{X}$ provided by Lemma 22.6.

Remark 22.7 (The role of the notation C_*)**.** The constants C_* on RHS (22.18) have the same properties as the constants C appearing throughout the paper. We have used the notation C_* for some of the constants in (22.18) because this will aid our analysis of the coupling of the different wave energies, especially in in Sect. 29.7.1, when we prove our Grönwall-type estimates. Similar remarks apply for constants C_* appearing throughout the rest of the paper.

Proposition 22.8 (The key pointwise estimate for $(\check{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not{g}}\chi$)**.** *Let $N = N_{\mathrm{top}}$, let $\mathfrak{Y}^{(N)}$ and $\not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}$ be the sets of order N $\ell_{t,u}$-tangential commutator operators from Def. 8.10, and let $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$. Recall that the arrays $\vec{\Psi}$ and $\vec{\Psi}_{(\mathrm{Partial})}$ are defined in* (2.11a)–(2.11b) *respectively. Then relative to the adapted rough coordinates (τ, u, x^2, x^3) (see Remark 5.3), the following pointwise estimate holds on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$ (i.e., for $(\tau, u, x^2, x^3) \in [\tau_0, \tau_{\mathrm{Boot}}) \times [-U_1, U_2] \times \mathbb{T}^2$), where*

$\mathcal{Y}^N$ *is the same in all appearances in* (22.18), *(except for the one term in which* $\max_{\not{\mathcal{L}}_{\mathcal{Y}^N} \in \not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}}$ *is taken):*

$$
\begin{aligned}
&\left|\frac{1}{L^{(\mathfrak{n})}\tau}(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not{g}}\chi\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
&\leq \boxed{2}\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\cdot|\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
&\quad + \frac{C_*}{|\tau|}|\breve{X}\mathcal{Y}^N\vec{\Psi}_{(\mathrm{Partial})}| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
&\quad + \boxed{4}\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
&\qquad \times \int_{\tau'=\tau_0}^{\tau}\left\{\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right|\left|\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right|\right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
&\quad + \frac{C_*}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}|\breve{X}\mathcal{Y}^N\vec{\Psi}_{(\mathrm{Partial})}| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
&\quad + \frac{C\varepsilon}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\mu\max_{\not{\mathcal{L}}_{\mathcal{Y}}^N\in\not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}}\left|\not{\mathcal{L}}_{\mathcal{Y}}^N\chi\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
&\quad + \frac{C}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\left\{\mu|\mathcal{Y}^N(\mathcal{C},\mathcal{D})|+|\mathcal{Y}^{\leq N-1}(\mathcal{C},\mathcal{D})|\right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
&\quad + \frac{C}{|\tau|}\int_{\tau'=\tau_0}^{\tau}|\mathcal{Y}^{\leq N}(\Omega,S)| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
&\quad + \mathrm{Error} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3),
\end{aligned}
\tag{22.18}
$$

and the error term $\mathrm{Error} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}$ *satisfies the following bound:*

$$
\begin{aligned}
|\mathrm{Error}| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) &\lesssim \frac{1}{|\tau|}\left|{}^{(\mathcal{Y}^N)}\mathscr{X}\right|(\tau_0,u,x^2,x^3) \\
&\quad + \frac{\varepsilon}{|\tau|}\left|\breve{X}\mathcal{Y}^N\vec{\Psi}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) + \left|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
&\quad + \frac{1}{|\tau|}\left|\mathcal{Z}_*^{[1,N];1}\vec{\Psi}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) + \frac{1}{|\tau|}\left|\begin{pmatrix}\mathcal{P}^{[1,N]}\gamma \\ \mathcal{P}_*^{[1,N]}\underline{\gamma}\end{pmatrix}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
&\quad + \frac{\varepsilon}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\left|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
&\quad + \frac{1}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\left|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
&\quad + \frac{1}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\left\{\left|\mathcal{Z}_*^{[1,N];1}\vec{\Psi}\right| + \left|\begin{pmatrix}\mathcal{P}^{[1,N]}\gamma \\ \mathcal{P}_*^{[1,N]}\underline{\gamma}\end{pmatrix}\right|\right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.
\end{aligned}
\tag{22.19}
$$

Moreover, the following less precise pointwise estimate holds on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$:

$$
\begin{aligned}
\left|\mu\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) &\lesssim \left|{}^{(\mathcal{Y}^N)}\mathscr{X}\right|(\tau_0,u,x^2,x^3)+\left\{\mu\left|\mathcal{P}^{N+1}\vec{\Psi}\right|+\left|\breve{X}\mathcal{P}^N\vec{\Psi}\right|\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\\
&\quad+\left\{\left|\mathcal{Z}_*^{[1,N];1}\vec{\Psi}\right|+\left|\begin{pmatrix}\mathcal{P}^{[1,N]}\gamma\\ \mathcal{P}_*^{[1,N]}\underline{\gamma}\end{pmatrix}\right|\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\\
&\quad+\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\left|\breve{X}\mathcal{P}^N\vec{\Psi}\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\
&\quad+\int_{\tau'=\tau_0}^{\tau}\left|\mathcal{Z}_*^{N+1;1}\vec{\Psi}\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\
&\quad+\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\left\{\left|\mathcal{Z}_*^{[1,N];1}\vec{\Psi}\right|+\left|\begin{pmatrix}\mathcal{P}^{[1,N]}\gamma\\ \mathcal{P}_*^{[1,N]}\underline{\gamma}\end{pmatrix}\right|\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\
&\quad+\varepsilon\int_{\tau'=\tau_0}^{\tau}\mu\max_{\not{\mathcal{L}}_{\mathcal{Y}}^N\in\not{\mathcal{L}}_{\mathfrak{Y}}^{(N)}}\left|\not{\mathcal{L}}_{\mathcal{Y}}^N\chi\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\
&\quad+\int_{\tau'=\tau_0}^{\tau}\left\{\mu|\mathcal{Y}^N(\mathcal{C},\mathcal{D})|+|\mathcal{Y}^{\leq N-1}(\mathcal{C},\mathcal{D})|\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\
&\quad+\int_{\tau'=\tau_0}^{\tau}|\mathcal{Y}^{\leq N}(\Omega,S)|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.
\end{aligned}
\tag{22.20}
$$

Proof. Throughout this proof, Error denotes any term such that $|\mathrm{Error}|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$ satisfies (22.19). To prove (22.18), we begin by using the definition (19.6a) of ${}^{(\mathcal{Y}^N)}\mathscr{X}$ and the estimates (17.10), (18.1), (18.9b), and (22.4a) to deduce:

$$
\frac{1}{L^{(\mathfrak{n})}\tau}(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^N\mathrm{tr}_{\not g}\chi=\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right){}^{(\mathcal{Y}^N)}\mathscr{X}+\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right)\vec{G}_{LL}\diamond\breve{X}\mathcal{Y}^N\vec{\Psi}+\mathrm{Error},
\tag{22.21}
$$

where in (22.21), we view all terms as being evaluated at ${}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$. Next, using the transport equation (3.44), (6.3), the fact that $\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\overset{\text{def}}{=}G_{LL}^0\breve{X}\mathcal{R}_{(+)}+G_{LL}^1\breve{X}\mathcal{R}_{(-)}+G_{LL}^2\breve{X}v^2+G_{LL}^3\breve{X}v^3+G_{LL}^4\breve{X}s$, and the identity $1=\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}+\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\backslash{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}$, we deduce the following identity for the second product on RHS (22.21):

$$
\begin{aligned}
&\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right)\vec{G}_{LL}\diamond\breve{X}\mathcal{Y}^N\vec{\Psi}\\
&=2\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\\
&\quad+2\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\backslash{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\\
&\quad-\frac{1}{\mu L^{(\mathfrak{n})}\tau}G_{LL}^1(\breve{X}\mathcal{R}_{(-)})\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}-\sum_{A=2}^{3}\frac{1}{\mu L^{(\mathfrak{n})}\tau}G_{LL}^A(\breve{X}v^A)\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}-\frac{1}{\mu L^{(\mathfrak{n})}\tau}G_{LL}^4(\breve{X}s)\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\\
&\quad+\left(\vec{G}_{LL}\diamond{}^{(\mathfrak{n})}\widetilde{L}\vec{\Psi}\right)\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}+2\left(\vec{G}_{LX}\diamond{}^{(\mathfrak{n})}\widetilde{L}\vec{\Psi}\right)\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\\
&\quad+\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right)G_{LL}^1\breve{X}\mathcal{Y}^N\mathcal{R}_{(-)}+\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right)G_{LL}^A\breve{X}\mathcal{Y}^Nv^A+\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right)G_{LL}^4\breve{X}\mathcal{Y}^Ns.
\end{aligned}
\tag{22.22}
$$

Clearly, the first product $2\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}$ on RHS (22.22) is bounded in magnitude by the first product on RHS (22.18) as desired. Similarly, using Prop. 9.1, (18.1), (18.9b), and the estimates of Prop. 17.1, we see that the products on the last line of RHS (22.22) are bounded by the C_*-multiplied product on the second line of RHS (22.18). Also using the estimate (18.16), we see that the product $2\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\backslash{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}$ on RHS (22.22) is bounded by the second term $\left|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$ on the second line of RHS (22.19). Finally, using Prop. 9.1, (18.1), (18.9b), and the estimates of Prop. 17.1, we see that the remaining products on RHS (22.22) are bounded in magnitude by $\lesssim \frac{\varepsilon}{|\tau|}\left|\breve{X}\mathcal{Y}^N\vec{\Psi}\right|$ (and thus are of type Error), where we have used that all these remaining products gain an overall smallness factor of ε from the factors $\breve{X}\mathcal{R}_{(-)}$, $\breve{X}v^A$, $\breve{X}s$, and ${}^{(\mathfrak{n})}\widetilde{L}\vec{\Psi}$.

We now bound the first product $\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right)^{(\mathcal{Y}^N)}\mathscr{X}$ on RHS (22.21). We start by multiplying the inequality (22.6) by $\left(\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right)\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$. To bound the product corresponding to the term $\boxed{2}\cdots$ on RHS (22.6), we first use Prop. 9.1, (18.1), (18.9b), the estimates of Prop. 17.1, and (22.4a) to express the product as follows:

$$
\begin{aligned}
&\boxed{2}\left|\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\\
&\qquad\times\int_{\tau'=\tau_0}^{\tau}\left\{\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right|\cdot\left|\vec{G}_{LL}\diamond\breve{X}\mathcal{Y}^N\vec{\Psi}\right|\cdot\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'\\
&\quad+\mathrm{Error}.
\end{aligned}
\tag{22.23}
$$

We now decompose the second factor in the integrand in (22.23) as follows:

$$
\vec{G}_{LL}\diamond\breve{X}\mathcal{Y}^N\vec{\Psi}=G_{LL}^0\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}+G_{LL}^1\breve{X}\mathcal{Y}^N\mathcal{R}_{(-)}+\sum_{A=2}^{3}G_{LL}^A\breve{X}\mathcal{Y}^N v^A+G_{LL}^4\breve{X}\mathcal{Y}^N s,
\tag{22.24}
$$

where we view (22.24) as being evaluated at ${}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)$. We now use (22.24) to substitute for the integrand factor in (22.23). Using Prop. 9.1, (18.1), (18.9b), and the estimates of Prop. 17.1, we see that the time integral corresponding to all the products on RHS (22.24) except the first one $G_{LL}^0\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}$ in (22.24) are bounded in magnitude by the term $\frac{C_*}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}|\breve{X}\mathcal{Y}^N\vec{\Psi}_{(\mathrm{Partial})}|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'$ on RHS (22.18). We now handle the one remaining integral, which is:

$$
\begin{aligned}
&2\left|\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\\
&\qquad\times\int_{\tau'=\tau_0}^{\tau}\left\{\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right|\cdot\left|G_{LL}^0\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right|\cdot\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\cap\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}^{(\mathfrak{n})}}\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.
\end{aligned}
\tag{22.25}
$$

To proceed, we first use (16.13) and Cor. 17.2 to obtain the following relation, valid for $\tau'\in[\tau_{\mathrm{Boot}},\tau]$:

$$
\begin{aligned}
&\left(G_{LL}^0\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\right)\left(\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\right)\\
&=\left(G_{LL}^0\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\right)\left(\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\right)+\mathcal{O}(\varepsilon)\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3),
\end{aligned}
\tag{22.26}
$$

where we emphasize that the G_{LL}^0 factor on RHS (22.26) is evaluated at rough-time τ (as opposed to τ'). We now substitute (22.26) into the integral (22.25). Using Prop. 9.1, (18.1), (18.9b), and the estimates of Prop. 17.1, we see that the integral corresponding to the last product on RHS (22.26) is bounded by $\lesssim$ the ε-multiplied time integral on RHS (22.19) and thus is of type Error. Next, we consider the time integral corresponding to the first product on RHS (22.26). Since the factor $G_{LL}^0\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$ does not depend on the integration variable τ', we can pull it out of the integral to obtain the following integral:

$$
\begin{aligned}
&2\left|\frac{\breve{X}\mathcal{R}_{(+)}}{\mu L^{(\mathfrak{n})}\tau}G_{LL}^0\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\\
&\qquad\int_{\tau'=\tau_0}^{\tau}\left\{\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\right|\cdot\left|\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right|\cdot\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right\}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.
\end{aligned}
\tag{22.27}
$$

Using the transport equation (3.44) satisfied by $\upmu$, (6.3), Prop. 9.1, (18.1), (18.9b), and the estimates of Prop. 17.1, we rewrite the product in (22.27) that is outside of the integral as follows:

$$\begin{aligned}
\frac{\breve{X}\mathcal{R}_{(+)}}{\upmu L^{(\mathfrak{n})}\tau} G^0_{LL} &= 2\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu} - \frac{1}{\upmu L^{(\mathfrak{n})}\tau} G^1_{LL}\breve{X}\mathcal{R}_{(-)} - \sum_{A=2}^{3} \frac{1}{\upmu L^{(\mathfrak{n})}\tau} G^A_{LL}\breve{X}v^A - \frac{1}{\upmu L^{(\mathfrak{n})}\tau} G^4_{LL}\breve{X}s \\
&\quad + \vec{G}_{LL} \diamond {}^{(\mathfrak{n})}\widetilde{L}\vec{\Psi} + 2\vec{G}_{LX} \diamond {}^{(\mathfrak{n})}\widetilde{L}\vec{\Psi} \\
&= 2\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu} + \mathcal{O}(\varepsilon)\frac{1}{|\tau|}.
\end{aligned} \tag{22.28}$$

Substituting (22.28) for the product outside the integral in (22.27), and using the bound $\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3) \lesssim \frac{1}{|\tau'|}$ (which follows from (18.1), and the estimates of Prop. 17.1), we bound the resulting term as follows:

$$\begin{aligned}
&\leq 4\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3) \int_{\tau'=\tau_0}^{\tau} \left\{ \left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\right| \cdot \left|\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right| \cdot \mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]} \cap {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}} \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\,\mathrm{d}\tau' \\
&\quad + \mathrm{Error}.
\end{aligned} \tag{22.29}$$

Next, we consider the simple estimate $\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\right| \leq \left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \cap {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right| + \left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \backslash {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right|$, which we substitute into the factor outside the integral in (22.29). The term corresponding to $\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \cap {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right|$ is bounded by the term spanning the third and fourth lines of RHS (22.18). Moreover, using (18.16), and the estimates of Prop. 17.1, we bound the term corresponding to $\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\upmu}{\upmu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \backslash {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right|$ by:

$$\begin{aligned}
&\leq C\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|}\left|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\,\mathrm{d}\tau' \\
&\leq C\frac{1}{|\tau|}\int_{\tau'=\tau_0}^{\tau} \left|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau', u, x^2, x^3)\,\mathrm{d}\tau' \leq \mathrm{Error}
\end{aligned} \tag{22.30}$$

as desired. Finally, we use (18.1), (18.9b), and (17.10) to deduce $\left|\frac{\breve{X}\mathcal{R}_{(+)}}{\upmu L^{(\mathfrak{n})}\tau} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3)\right| \lesssim \frac{1}{|\tau|}$, and from this pointwise estimate, it is easy to see that the products of $\frac{\breve{X}\mathcal{R}_{(+)}}{\upmu L^{(\mathfrak{n})}\tau}{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3)$ and the remaining terms on RHS (22.6) are bounded in magnitude by $\leq$ RHS (22.18) as desired. We have therefore proved (22.18).

The proof of (22.20) is similar, but much less delicate because it does not rely on careful decompositions like (22.22) and (22.28); we omit the details. □

22.3. **Pointwise estimates for the partially modified quantities.** In this section, we derive pointwise estimates for the partially modified quantities from Def. 19.2.

Lemma 22.9 (Pointwise estimates for partially modified quantities and their L-derivative)**.** *Let $N = N_{\mathrm{top}}$, and let $\mathcal{Y}^{N-1} \in \mathfrak{Y}^{(N-1)}$, where $\mathfrak{Y}^{(N-1)}$ is the set of order $N-1$ $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Let ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ be the corresponding partially modified quantity defined in* (19.7a) *(with $N-1$ in the role of N and $\mathcal{Y}^{N-1}$ in the role of $\mathcal{P}^N$). Recall that the arrays $\vec{\Psi}$ and $\vec{\Psi}_{(\mathrm{Partial})}$ are defined in* (2.11a)–(2.11b) *respectively. Let ${}^{(\mathfrak{n})}\widetilde{L}$ be the rough null vectorfield defined in* (6.3)*, and let ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ be the τ_0-normalized flow map of ${}^{(\mathfrak{n})}\widetilde{L}$ with respect to the adapted rough coordinates $({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$ appearing in Lemma 16.1. Then there exist constants $C > 0$ and $C_* > 0$ such that relative to the adapted rough coordinates (see Remark 5.3), the following pointwise estimate holds on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$ (i.e., for $(\tau, u, x^2, x^3) \in [\tau_0, \tau_{\mathrm{Boot}}) \times [-U_1, U_2] \times \mathbb{T}^2$):*

$$\begin{aligned}
\left|{}^{(\mathfrak{n})}\widetilde{L}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3) &\leq \frac{1}{2}\left|\frac{1}{L^{(\mathfrak{n})}\tau}(G^0_{LL})\rlap{/}\Delta\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3) \\
&\quad + C_*\left|\rlap{/}\Delta\mathcal{Y}^{N-1}\vec{\Psi}_{(\mathrm{Partial})}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3) \\
&\quad + C\left|\mathcal{P}^{[1,N]}\gamma\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau, u, x^2, x^3).
\end{aligned} \tag{22.31a}$$

Moreover, the following pointwise estimate holds relative to the adapted rough coordinates on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$:

$$
\begin{aligned}
\left|^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \leq & \left|^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right|(\tau_0,u,x^2,x^3) \\
& + \frac{1}{2}\left|\frac{1}{L^{(\mathfrak{n})}\tau}G_{LL}^0\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \\
& \quad \times \int_{\tau'=\tau_0}^{\tau}\left|{\not\Delta}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
& + C_* \int_{\tau'=\tau_0}^{\tau}\left|{\not\Delta}\mathcal{Y}^{N-1}\vec{\Psi}_{(\text{Partial})}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\
& + C\int_{\tau'=\tau_0}^{\tau}\left\{\varepsilon\left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right| + \left|\mathcal{P}^{[1,N]}\gamma\right|\right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'.
\end{aligned}
\tag{22.31b}
$$

Proof. To prove (22.31a), we first expand the first term on RHS (19.11) as follows:

$$
\frac{1}{2}G_{LL}^0{\not\Delta}\mathcal{Y}^{N-1}\mathcal{R}_{(+)} + \frac{1}{2}G_{LL}^1{\not\Delta}\mathcal{Y}^{N-1}\mathcal{R}_{(-)} + \sum_{A=2,3}\frac{1}{2}G_{LL}^A{\not\Delta}\mathcal{Y}^{N-1}v^A + \frac{1}{2}G_{LL}^4{\not\Delta}\mathcal{Y}^{N-1}s. \tag{22.32}
$$

Multiplying (19.11) by $\frac{1}{L^{(\mathfrak{n})}\tau}$ (in view of (6.3)) and evaluating at ${}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$, we see that the first product $\frac{1}{2}G_{LL}^0{\not\Delta}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}$ in (22.32) yields precisely the first term on RHS (22.31a). Next, using the bound $\sum_{\iota=1}^4|G_{LL}^\iota| \lesssim 1$ (which follows from Prop. 9.1 and Prop. 17.1), as well as the estimate $\frac{1}{|L^{(\mathfrak{n})}\tau|} \approx 1$ (see (18.9b)), we see that the terms generated by the remaining products in (22.32) are bounded in magnitude by the C_*-multiplied term on RHS (22.31a). Also using (22.5c), we conclude (22.31a).

To prove (22.31b), we first use (16.12) with $f \stackrel{\text{def}}{=} {}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$, $\tau_1 \stackrel{\text{def}}{=} \tau_0$, and $\tau_2 \stackrel{\text{def}}{=} \tau$. We take the absolute value and then substitute the estimate (22.31a) into the integrand. The (rough) time integrals of all products on RHS (22.31a) except the first one are clearly bounded in magnitude by RHS (22.31b). To handle the remaining the integral of the remaining product $\frac{1}{2}\left|\frac{1}{L^{(\mathfrak{n})}\tau}(G_{LL}^0){\not\Delta}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}$, we use (16.13) and Cor. 17.2 to replace the integrand factor $|G_{LL}^0 \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)|$ with $|G_{LL}^0 \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)|$ (which we can pull out of the integral, as is indicated in the second term on RHS (22.31b)) factor from the τ'-integral, at the expense of error terms featuring a small ε factor. Using the comparison estimates (13.11a)–(13.11b), we bound these error terms by the terms on the last line of RHS (22.31b) as desired. □

23. Pointwise estimates for controlling the specific vorticity, entropy gradient, and modified fluid variables

We continue to work under the assumptions of Sect. 13.2. In this section, we derive a variety of pointwise estimates that we will use in Sects. 26–(27), when we derive L^2 estimates for Ω, S, $\mathcal{C}$, and $\mathcal{D}$ up to top-order. Among the estimates we derive are pointwise estimates for the error terms in the elliptic-hyperbolic integral identity (21.63).

23.1. Pointwise estimates for Ω, S, $\mathcal{C}$, $\mathcal{D}$, and their derivatives.

In this section, we derive pointwise estimates for Ω, S, $\mathcal{C}$, $\mathcal{D}$, and various derivatives of these quantities. We provide the main estimates in Prop. 23.3.

23.1.1. *A simple identity for* $\mathrm{d}V_\flat$.

In our proof of Prop. 23.3, we will use the following lemma, which provides an identity for $\mathrm{d}V_\flat$ when V is Σ_t-tangent.

Lemma 23.1. *Let V be a Σ_t-tangent vectorfield, and let $\mathrm{d}V_\flat$ be the two-form with the following components: $(\mathrm{d}V_\flat)_{\alpha\beta} \stackrel{\text{def}}{=} \partial_\alpha V_\beta - \partial_\beta V_\alpha$. Then relative to the Cartesian coordinates, the following identity holds, where $c = c(\rho,s)$ is the speed of sound:*

$$
\begin{aligned}
(\mathrm{d}V_\flat)_{\alpha\beta} = \partial_\alpha V_\beta - \partial_\beta V_\alpha = & \, 2(\partial_\beta \ln c)V_\alpha - 2(\partial_\alpha \ln c)V_\beta + \delta_\alpha^0 V_a \partial_\beta v^a - \delta_\beta^0 V_a \partial_\alpha v^a \\
& + \left\{\delta_\alpha^0 \mathbf{g}_{\beta\gamma} - \delta_\beta^0 \mathbf{g}_{\alpha\gamma}\right\}\mathbf{B}V^\gamma + c^{-2}\epsilon_{\alpha\beta\gamma\delta}\mathbf{B}^\gamma(\operatorname{curl} V)^\delta.
\end{aligned}
\tag{23.1}
$$

Proof. The same proof of [4, Lemma 5.6] holds. □

23.1.2. *Simple commutator estimates involving* Ω *and* S. We will use the following simple commutator estimates in our proof of Prop. 23.3.

Lemma 23.2 (Commuting $\upmu$-weighted Cartesian derivatives with the geometric vectorfields)**.** *Let* $1 \leq N \leq N_{\mathrm{top}}$. *Then the following commutator estimates hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$:

$$\begin{aligned} &\left|[\upmu\partial_i,\mathcal{P}^N](\Omega,S)\right|, \left|[\upmu\,\mathrm{curl},\mathcal{P}^N](\Omega,S)\right|, \left|[\upmu\,\mathrm{div},\mathcal{P}^N](\Omega,S)\right| \\ &\lesssim \left|\mathcal{P}^{\leq N}(\Omega,S)\right| + \left|\breve{X}\mathcal{P}^{\leq N-1}(\Omega,S)\right| + \varepsilon|\breve{X}\mathcal{P}^{[1,N-1]}\vec{\Psi}| + \varepsilon\upmu|\mathcal{P}^N\vec{\Psi}| + \varepsilon|\mathcal{P}_*^{[1,N]}\underline{\gamma}|. \end{aligned} \tag{23.2}$$

Proof. To derive the estimates for $\left|[\upmu\partial_i,\mathcal{P}^N](\Omega,S)\right|$, we first use Lemma 5.6 to express the $\upmu$-weighted Cartesian partial derivatives in terms of the geometric commutation vectorfields $\{L,\breve{X},Y_{(2)},Y_{(3)}\}$. Also using Prop. 9.1, the commutator estimates (13.6a)–(13.6b), and the estimates of Prop. 17.1, we conclude the desired estimate (23.2) for $\left|[\upmu\partial_i,\mathcal{P}^N](\Omega,S)\right|$. The desired estimates for $\left|[\upmu\,\mathrm{curl},\mathcal{P}^N](\Omega,S)\right|$ and $\left|[\upmu\,\mathrm{div},\mathcal{P}^N](\Omega,S)\right|$ follow immediately from the estimate for $\left|[\upmu\partial_i,\mathcal{P}^N](\Omega,S)\right|$. □

23.1.3. *The main pointwise estimates.*

Proposition 23.3 (Pointwise estimates for $\Omega,S,\mathcal{C},\mathcal{D}$, and their derivatives)**.** *The following pointwise estimates hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$:

Transport estimates*. For* $0 \leq N \leq N_{\mathrm{top}}$, *we have:*

$$|\upmu\mathbf{B}\mathcal{P}^N(\Omega,S)| \lesssim |\mathcal{P}^{\leq N}(\Omega,S)| + \varepsilon|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}| + \varepsilon\upmu|\mathcal{P}^{N+1}\vec{\Psi}| + \varepsilon|\mathcal{P}_*^{[1,N]}\underline{\gamma}|, \tag{23.3}$$

$$|\upmu\mathbf{B}\mathcal{P}^N\mathcal{C}| \lesssim |\mathcal{P}^{\leq N}\mathcal{C}| + |\mathcal{P}^{\leq N+1}(\Omega,S)| + \varepsilon|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}| + \varepsilon|\mathcal{P}^{N+1}\vec{\Psi}| + \varepsilon|\mathcal{P}_*^{[1,N]}\underline{\gamma}|, \tag{23.4a}$$

$$|\upmu\mathbf{B}\mathcal{P}^N\mathcal{D}| \lesssim |\mathcal{P}^{\leq N}\mathcal{D}| + |\mathcal{P}^{\leq N+1}(\Omega,S)| + \varepsilon|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}| + \varepsilon|\mathcal{P}^{N+1}\vec{\Psi}| + \varepsilon|\mathcal{P}_*^{[1,N]}\underline{\gamma}|. \tag{23.4b}$$

Algebraic estimates for transversal derivatives in terms of tangential derivatives*. For* $0 \leq N \leq N_{\mathrm{top}}$, *we have:*

$$|\breve{X}\mathcal{P}^N\Omega|, |\mathcal{P}^N\breve{X}\Omega| \lesssim \upmu|L\mathcal{P}^N\Omega| + |\mathcal{P}^{\leq N}(\Omega,S)| + \varepsilon|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}| + \varepsilon\upmu|\mathcal{P}^{N+1}\vec{\Psi}| + \varepsilon|\mathcal{P}_*^{[1,N]}\underline{\gamma}|, \tag{23.5a}$$

$$|\breve{X}\mathcal{P}^N S|, |\mathcal{P}^N\breve{X}S| \lesssim \upmu|L\mathcal{P}^N S| + |\mathcal{P}^{\leq N}(\Omega,S)| + \varepsilon|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}| + \varepsilon\upmu|\mathcal{P}^{N+1}\vec{\Psi}| + \varepsilon|\mathcal{P}_*^{[1,N]}\underline{\gamma}|. \tag{23.5b}$$

Algebraic estimates for $(\mathrm{div}\,\Omega,\mathrm{div}\,S)$ ***and*** $(\mathrm{curl}\,\Omega,\mathrm{curl}\,S)$ ***in terms of*** $(\mathcal{C},\mathcal{D})$. *For* $0 \leq N \leq N_{\mathrm{top}}$, *we have:*

$$\begin{aligned} \left|\mathrm{curl}\,\mathcal{P}^N\Omega\right| &\lesssim \left|\mathcal{P}^N\mathcal{C}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N-1}\mathcal{C}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N}(\Omega,S)\right| \\ &\quad + \frac{\varepsilon}{\upmu}\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{N+1}\vec{\Psi}\right| + \frac{\varepsilon}{\upmu}\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|, \end{aligned} \tag{23.6a}$$

$$\begin{aligned} \left|\mathrm{div}\,\mathcal{P}^N S\right| &\lesssim \left|\mathcal{P}^N\mathcal{D}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N-1}\mathcal{D}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N}(\Omega,S)\right| \\ &\quad + \frac{\varepsilon}{\upmu}\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{N+1}\vec{\Psi}\right| + \frac{\varepsilon}{\upmu}\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|, \end{aligned} \tag{23.6b}$$

$$\left|\mathrm{div}\,\mathcal{P}^N\Omega\right|, \left|\mathrm{curl}\,\mathcal{P}^N S\right| \lesssim \frac{1}{\upmu}|\mathcal{P}^{\leq N}(\Omega,S)| + \frac{\varepsilon}{\upmu}|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}| + \varepsilon|\mathcal{P}^{N+1}\vec{\Psi}| + \frac{\varepsilon}{\upmu}|\mathcal{P}_*^{[1,N]}\underline{\gamma}|. \tag{23.6c}$$

Estimates for the exterior derivative of $(\mathcal{P}^{N_{\mathrm{top}}}\Omega)_\flat$ ***and*** $(\mathcal{P}^{N_{\mathrm{top}}}S)_\flat$. *The following estimates hold, where* $(\mathrm{d}\mathcal{P}^{N_{\mathrm{top}}}\Omega)_\flat$ *is the two-form with the Cartesian components components* $\partial_\alpha(\mathcal{P}^{N_{\mathrm{top}}}\Omega)_\beta - \partial_\beta(\mathcal{P}^{N_{\mathrm{top}}}\Omega)_\alpha$ *(where* $(\mathcal{P}^{N_{\mathrm{top}}}\Omega)_\alpha = \mathbf{g}_{\alpha\gamma}\mathcal{P}^{N_{\mathrm{top}}}\Omega^\gamma$*),* *and similarly for* $\mathrm{d}(\mathcal{P}^{N_{\mathrm{top}}}S)_\flat$*, and* $\mathbf{h}$ *is the Riemannian metric from Def. 21.3 and Lemma 21.4:*

$$\begin{aligned}|\mathrm{d}(\mathcal{P}^{N_{\mathrm{top}}}\Omega)_\flat|_{\mathbf{h}} &\lesssim \left|\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N_{\mathrm{top}}-1}\mathcal{C}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega,S)\right| \\ &\quad + \frac{\varepsilon}{\upmu}\left|\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{N_{\mathrm{top}}+1}\vec{\Psi}\right| + \frac{\varepsilon}{\upmu}\left|\mathcal{P}_*^{[1,N_{\mathrm{top}}]}\underline{\gamma}\right|,\end{aligned} \tag{23.7a}$$

$$|\mathrm{d}(\mathcal{P}^{N_{\mathrm{top}}}S)_\flat|_{\mathbf{h}} \lesssim \frac{1}{\upmu}\left|\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega,S)\right| + \frac{\varepsilon}{\upmu}\left|\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{N_{\mathrm{top}}+1}\vec{\Psi}\right| + \frac{\varepsilon}{\upmu}\left|\mathcal{P}_*^{[1,N_{\mathrm{top}}]}\underline{\gamma}\right|. \tag{23.7b}$$

Proof.

Proof of (23.3): We first prove the estimate (23.3) for Ω^i. We start by multiplying the transport equation (2.23a) by $\upmu$ and commuting with $\mathcal{P}^N$ to deduce that $|\upmu\mathbf{B}\mathcal{P}^N\Omega^i| \lesssim |\mathcal{P}^N(\upmu\mathfrak{L}^i_{(\Omega)})| + |[\upmu\mathbf{B},\mathcal{P}^N]\Omega^i|$. Next, using Lemma 13.11, we see that $|\mathcal{P}^N(\upmu\mathfrak{L}^i_{(\Omega)})| \lesssim$ RHS (23.3). Next, using the relation $\upmu\mathbf{B} = \upmu L + \breve{X}$ (see (3.24)), we derive the commutator identity $[\upmu\mathbf{B},\mathcal{P}^N] = \upmu[L,\mathcal{P}^N] + [\upmu,\mathcal{P}^N]L + [\breve{X},\mathcal{P}^N]$. Using the Leibniz rule, the commutator estimates (13.6a)–(13.6b), and the estimates of Prop. 17.1, we find that $|[\upmu\mathbf{B},\mathcal{P}^N]\Omega^i| \lesssim$ RHS (23.3). Combining these estimates, we conclude the desired estimate (23.3) for Ω^i. Using similar arguments, based on the transport equation (2.23c), we also conclude (23.3) for S^i.

Proof of (23.5a) **and** (23.5b): To deduce (23.5a) for $\mathcal{P}^N\breve{X}\Omega$ we differentiate the identity (9.6a) with $\mathcal{P}^N$ and use the estimates of Prop. 17.1 and the commutator estimate (13.6a) (to commute $\mathcal{P}^N$ under the factor of L in the first term on RHS (9.6a)). From this estimate for $\mathcal{P}^N\breve{X}\Omega$, the commutator estimate (13.6b), and the estimates of Prop. 17.1, we also conclude the desired bound (23.5a) for $\breve{X}\mathcal{P}^N\Omega$. The estimates stated in (23.5b) follow from a nearly identical argument based on the identity (9.6b).

Proof of (23.4a) **and** (23.4b): To prove (23.4a), we first multiply the transport equation (2.24b) by $\upmu$ and commute with $\mathcal{P}^N$ to deduce that $|\upmu\mathbf{B}\mathcal{P}^N\mathcal{C}^i| \lesssim \left|\mathcal{P}^N\left\{\upmu\mathfrak{M}^i_{(\mathcal{C})} + \upmu\mathfrak{Q}^i_{(\mathcal{C})} + \upmu\mathfrak{L}^i_{(\mathcal{C})}\right\}\right| + |[\upmu\mathbf{B},\mathcal{P}^N]\mathcal{C}|$. Next, using Lemmas 13.10 and 13.11, we see that $\left|\mathcal{P}^N\left\{\upmu\mathfrak{M}^i_{(\mathcal{C})} + \upmu\mathfrak{Q}^i_{(\mathcal{C})} + \upmu\mathfrak{L}^i_{(\mathcal{C})}\right\}\right| \lesssim$ RHS (23.4a). To show that $|[\upmu\mathbf{B},\mathcal{P}^N]\mathcal{C}| \lesssim$ RHS (23.4a), we can use the same argument that we used in the proof of (23.3). We have therefore proved the estimate (23.4a). The estimate (23.4b) can be proved by applying nearly identical arguments based on the transport equation (2.25a).

Proof of (23.6a)–(23.6c): To prove (23.6a), we first multiply (2.10a) by $\upmu$ and use Lemmas 5.6 and 9.1 to write the resulting equation in the schematic form $\upmu(\mathrm{curl}\,\Omega)^i \overset{\text{def}}{=} F = \upmu\mathrm{f}(\vec{\Psi})\mathcal{C}^i + \mathrm{f}(\gamma)\cdot S^a\cdot\breve{X}\vec{\Psi} + \upmu\mathrm{f}(\gamma)\cdot S^a\cdot\mathcal{P}\vec{\Psi}$. Hence, if $0 \leq N \leq N_{\mathrm{top}}$, we can commute this equation with $\mathcal{P}^N$ and then divide by $\upmu$ to deduce $\mathrm{curl}\,\mathcal{P}^N\Omega^i = \frac{1}{\upmu}[\upmu\,\mathrm{curl},\mathcal{P}^N]\Omega^i + \frac{1}{\upmu}\mathcal{P}^N F$. The bootstrap assumptions and the estimates of Prop. 17.1 imply that $\frac{1}{\upmu}|\mathcal{P}^N F| \lesssim \left|\mathcal{P}^N\mathcal{C}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N-1}\mathcal{C}\right| + \frac{1}{\upmu}\left|\mathcal{P}^{\leq N}(\Omega,S)\right| + \frac{\varepsilon}{\upmu}\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{P}^{N+1}\vec{\Psi}\right| + \frac{\varepsilon}{\upmu}\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right| \lesssim$ RHS (23.6a) as desired. To show that $\frac{1}{\upmu}\left|[\upmu\,\mathrm{curl},\mathcal{P}^N]\Omega^i\right| \lesssim$ RHS (23.6a), we use the commutator estimate (23.2), the estimates (23.5a)–(23.5b), and the estimate $|\upmu| \lesssim 1$ (which follows from the bootstrap assumptions).

To prove (23.6b), we first multiply (2.10b) by $\upmu$ and use Lemmas 5.6 and 9.1 to deduce the schematic equation $\upmu\,\mathrm{div}\,S = \upmu f(\vec{\Psi})\mathcal{D} + \mathrm{f}(\gamma)\cdot S^a\cdot\breve{X}\vec{\Psi} + \upmu\mathrm{f}(\gamma)\cdot S^a\cdot\mathcal{P}\vec{\Psi}$. We now argue as in the proof of (23.6a), where we use (23.2), the estimates (23.5a)–(23.5b), and the estimate $|\upmu| \lesssim 1$ to bound the commutator term $\frac{1}{\upmu}\left|[\upmu\,\mathrm{div},\mathcal{P}^N]S^i\right|$, thereby concluding (23.6b).

We now prove (23.6c) for $\mathrm{div}\,\mathcal{P}^N\Omega$. Multiplying (2.24a) by $\upmu$, commuting with $\mathcal{P}^N$, and then dividing by $\upmu$, we find that $|\mathrm{div}\,\mathcal{P}^N\Omega| \lesssim \frac{1}{\upmu}|\mathcal{P}^N(\upmu\mathfrak{L}_{(\mathrm{div}\,\Omega)})| + \frac{1}{\upmu}|[\upmu\,\mathrm{div},\mathcal{P}^N]\Omega|$. The desired estimate now follows from Lemma 13.11, the commutator estimate (23.2), (23.5a)–(23.5b), and the estimate $|\upmu| \lesssim 1$. The proof of (23.6c) for $\mathrm{curl}\,\mathcal{P}^N S$ follows from a similar argument based on equation (2.25b), the commutator estimate (23.2), the estimates (23.5a)–(23.5b), and the estimate $|\upmu| \lesssim 1$.

Proof of (23.7a)–(23.7b): To prove (23.7a), we first use the identity (23.1) with $\mathcal{P}^{N_{\mathrm{top}}}\Omega$ in the role of V, Lemmas 5.6 and 9.1, the estimates of Prop. 17.1, and (21.16a) to deduce that $|(\mathrm{d}\mathcal{P}^{N_{\mathrm{top}}}\Omega)_\flat|_{\mathbf{h}} \lesssim \frac{1}{\upmu}|\mathcal{P}^{N_{\mathrm{top}}}\Omega| + |\mathbf{B}\mathcal{P}^{N_{\mathrm{top}}}\Omega| + |\mathrm{curl}\,\mathcal{P}^{N_{\mathrm{top}}}\Omega|$. From this estimate and the pointwise estimates (23.3) and (23.6a) with $N \overset{\text{def}}{=} N_{\mathrm{top}}$, we conclude the desired estimate (23.7a). The estimate (23.7b) follows from a similar argument that relies on the pointwise estimate (23.3) for $|\mathbf{B}\mathcal{P}^{N_{\mathrm{top}}}S|$ and the pointwise estimate (23.6c) for $\left|\mathrm{curl}\,\mathcal{P}^{N_{\mathrm{top}}}S\right|$.

□

23.2. **Pointwise estimates for the elliptic-hyperbolic integral identity error terms.** Recall that to prove our top-order L^2 estimates for the specific vorticity and entropy gradient, we will rely on the elliptic-hyperbolic integral identity (21.63) with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V. In the next proposition, we derive pointwise estimates for the error terms appearing in the identity.

Proposition 23.4 (Pointwise estimates for the elliptic-hyperbolic integral identity error terms)**.** *Let $\varsigma \in (0,1]$. Then the error terms appearing in the elliptic-hyperbolic integral identity* (21.63) *(with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V) satisfy the following pointwise estimates on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, where the implicit constants are* ***independent*** *of ς.*

Estimates for controlling spacetime error integrals.

$$\begin{aligned}\left|\mathrm{J}_{(\text{Antisymmetric})}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| &\lesssim \left|\mathcal{P}^{N_{\text{top}}}\mathcal{C}\right|^2+\frac{1}{\mu^2}\left|\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{C}\right|^2+\frac{1}{\mu^2}\left|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)\right|^2\\ &\quad+\frac{\varepsilon^2}{\mu^2}\left|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}\right|^2+\varepsilon^2\left|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}\right|^2+\frac{\varepsilon^2}{\mu^2}\left|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}\right|^2,\end{aligned} \tag{23.8a}$$

$$\begin{aligned}\left|\mathrm{J}_{(\text{Antisymmetric})}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| &\lesssim \frac{1}{\mu^2}\left|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)\right|^2\\ &\quad+\frac{\varepsilon^2}{\mu^2}\left|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}\right|^2+\varepsilon^2\left|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}\right|^2+\frac{\varepsilon^2}{\mu^2}\left|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}\right|^2,\end{aligned} \tag{23.8b}$$

$$\begin{aligned}\left|\mathrm{J}_{(\text{Div})}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| &\lesssim \frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2\\ &\quad+\frac{\varepsilon^2}{\mu^2}|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}|^2+\varepsilon^2|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}|^2+\frac{\varepsilon^2}{\mu^2}|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}|^2,\end{aligned} \tag{23.9a}$$

$$\begin{aligned}\left|\mathrm{J}_{(\text{Div})}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| &\lesssim \left|\mathcal{P}^{N_{\text{top}}}\mathcal{D}\right|^2+\frac{1}{\mu^2}\left|\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{D}\right|^2+\frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2\\ &\quad+\frac{\varepsilon^2}{\mu^2}|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}|^2+\varepsilon^2|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}|^2+\frac{\varepsilon^2}{\mu^2}|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}|^2,\end{aligned} \tag{23.9b}$$

$$\mu\left|\mathrm{J}_{(\boldsymbol{\partial}\frac{1}{\mu})}[\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| \lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]+\frac{1}{\varsigma}\frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}\Omega|^2, \tag{23.10a}$$

$$\mu\left|\mathrm{J}_{(\boldsymbol{\partial}\frac{1}{\mu})}[\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| \lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]+\frac{1}{\varsigma}\frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}S|^2, \tag{23.10b}$$

$$\begin{aligned}
\left|\mathrm{J}_{(\text{Absorb-1})}[\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| &\lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega] \\
&\quad+\left(1+\frac{1}{\varsigma}\right)\frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2 \\
&\quad+\left(1+\frac{1}{\varsigma}\right)\frac{\varepsilon^2}{\mu^2}|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}|^2+\left(1+\frac{1}{\varsigma}\right)\varepsilon^2|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}|^2 \\
&\quad+\left(1+\frac{1}{\varsigma}\right)\frac{\varepsilon^2}{\mu^2}|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}|^2,
\end{aligned} \tag{23.11a}$$

$$\begin{aligned}
\left|\mathrm{J}_{(\text{Absorb-1})}[\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| &\lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S] \\
&\quad+\left(1+\frac{1}{\varsigma}\right)\left|\mathcal{P}^{N_{\text{top}}}\mathcal{D}\right|^2+\left(1+\frac{1}{\varsigma}\right)\frac{1}{\mu^2}\left|\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{D}\right|^2 \\
&\quad+\left(1+\frac{1}{\varsigma}\right)\frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2 \\
&\quad+\left(1+\frac{1}{\varsigma}\right)\frac{\varepsilon^2}{\mu^2}|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}|^2+\left(1+\frac{1}{\varsigma}\right)\varepsilon^2|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}|^2 \\
&\quad+\left(1+\frac{1}{\varsigma}\right)\frac{\varepsilon^2}{\mu^2}|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}|^2,
\end{aligned} \tag{23.11b}$$

$$\left|\mathrm{J}_{(\text{Absorb-2})}[\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| \lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]+\frac{1}{\varsigma}\frac{1}{\mu^2}|\mathcal{P}^{N_{\text{top}}}\Omega|^2, \tag{23.12a}$$

$$\left|\mathrm{J}_{(\text{Absorb-2})}[\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| \lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]+\frac{1}{\varsigma}\frac{1}{\mu^2}|\mathcal{P}^{N_{\text{top}}}S|^2, \tag{23.12b}$$

$$\begin{aligned}
\left|\mathrm{J}_{(\text{Material})}[\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| &\lesssim \frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2 \\
&\quad+\frac{\varepsilon^2}{\mu^2}|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}|^2+\varepsilon^2|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}|^2+\frac{\varepsilon^2}{\mu^2}|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}|^2,
\end{aligned} \tag{23.13a}$$

$$\begin{aligned}
\left|\mathrm{J}_{(\text{Material})}[\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| &\lesssim \frac{1}{\mu^2}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2 \\
&\quad+\frac{\varepsilon^2}{\mu^2}|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}|^2+\varepsilon^2|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}|^2+\frac{\varepsilon^2}{\mu^2}|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}|^2,
\end{aligned} \tag{23.13b}$$

$$\left|\mathrm{J}_{(\text{Null Geometry})}[\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| \lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]+\frac{1}{\varsigma}\frac{1}{\mu^2}|\mathcal{P}^{N_{\text{top}}}\Omega|^2, \tag{23.14a}$$

$$\left|\mathrm{J}_{(\text{Null Geometry})}[\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| \lesssim \varsigma\frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]+\frac{1}{\varsigma}\frac{1}{\mu^2}|\mathcal{P}^{N_{\text{top}}}S|^2. \tag{23.14b}$$

Estimates for controlling spatial error integrals.

$$\begin{aligned}\left|\mathfrak{E}_{(\text{Principal})}[\mathcal{P}^{N_{\text{top}}}\Omega,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\right| &\lesssim \left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{N_{\text{top}}}\mathcal{C}\right|^2+\frac{1}{\mu^{3/2}}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{C}\right|^2 \\ &\quad+\frac{1}{\mu^{5/2}}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)\right|^2 \\ &\quad+\frac{\varepsilon^2}{\mu^{3/2}}\left|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}\right|^2+\varepsilon^2\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}\right|^2 \\ &\quad+\frac{\varepsilon^2}{\mu^{3/2}}\left|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}\right|^2,\end{aligned} \tag{23.15a}$$

$$\begin{aligned}\left|\mathfrak{E}_{(\text{Principal})}[\mathcal{P}^{N_{\text{top}}}S,\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\right| &\lesssim \left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{N_{\text{top}}}\mathcal{D}\right|^2+\frac{1}{\mu^{3/2}}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{D}\right|^2 \\ &\quad+\frac{1}{\mu^{5/2}}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)\right|^2 \\ &\quad+\frac{\varepsilon^2}{\mu^{3/2}}\left|\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}\right|^2+\varepsilon^2\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{N_{\text{top}}+1}\vec{\Psi}\right|^2 \\ &\quad+\frac{\varepsilon^2}{\mu^{3/2}}\left|\mathcal{P}_*^{[1,N_{\text{top}}]}\underline{\gamma}\right|^2,\end{aligned} \tag{23.15b}$$

$$\left|\mathfrak{E}_{(\text{Lower-order})}[\mathcal{P}^{N_{\text{top}}}\Omega,\mathcal{P}^{N_{\text{top}}}\Omega]\right| \lesssim \frac{1}{\mu^{5/2}}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{N_{\text{top}}}\Omega\right|^2, \tag{23.16a}$$

$$\left|\mathfrak{E}_{(\text{Lower-order})}[\mathcal{P}^{N_{\text{top}}}S,\mathcal{P}^{N_{\text{top}}}S]\right| \lesssim \frac{1}{\mu^{5/2}}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}\left|\mathcal{P}^{N_{\text{top}}}S\right|^2. \tag{23.16b}$$

Proof. We recall that $\mathbf{h}$ is the Riemannian metric from Def. 21.3 and Lemma 21.4.

Proof of (23.8a)–(23.8b) **and** (23.9a)–(23.9b)**:** First, using (21.18) and (21.31a), we deduce that $|\mathrm{J}_{(\text{Antisymmetric})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V]| \lesssim |\mathrm{d}V_\flat|_{\mathbf{h}}^2$. From this bound with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V and the pointwise estimates (23.7a)–(23.7b), we conclude the desired bounds (23.8a)–(23.8b).

(23.9a)–(23.9b) follow from similar arguments based on (21.31b) and the pointwise estimates (23.6b)–(23.6c).

Proof of (23.10a)–(23.10b)**:** Let V be any Σ_t-tangent vectorfield. We first note that since the elliptic hyperbolic current $\mathscr{J}^\alpha[V,\boldsymbol{\partial}V]$ defined by (21.29) is tangent to $\mathcal{P}_u$, we can use the identity (7.6) to obtain the following decomposition, where $\not{\Pi}$ is the $\ell_{t,u}$-projection tensorfield from Def. 3.3: $\mathscr{J}^\alpha[V,\boldsymbol{\partial}V] = -\frac{1}{2}\underline{L}_\beta\mathscr{J}^\beta[V,\boldsymbol{\partial}V]L^\alpha + \not{\Pi}_\beta^{\ \alpha}\mathscr{J}^\beta[V,\boldsymbol{\partial}V]$. From this decomposition and definition (21.31c), we deduce the following pointwise bound, where $\not{\Pi}\mathscr{J}[V,\boldsymbol{\partial}V]$ is the $\ell_{t,u}$-projection of $\mathscr{J}[V,\boldsymbol{\partial}V]$:

$$\mu|\mathrm{J}_{(\boldsymbol{\partial}\frac{1}{\mu})}[V,\boldsymbol{\partial}V]| \lesssim \frac{1}{\mu}|L\mu||\underline{L}_\beta\mathscr{J}^\beta[V,\boldsymbol{\partial}V]| + |\not{\nabla}\mu|_{\not{g}}|\not{\Pi}\mathscr{J}[V,\boldsymbol{\partial}V]|_{\not{g}}. \tag{23.17}$$

Next, using (23.17), definition (21.29), Prop. 9.1, the estimates of Prop. 17.1, and the pointwise comparison estimates provided by Lemma 21.7, we deduce the following pointwise bound:

$$\mu|\mathrm{J}_{(\boldsymbol{\partial}\frac{1}{\mu})}[V,\boldsymbol{\partial}V]| \lesssim \frac{1}{\mu}|V|_g|\boldsymbol{\partial}V|_{\mathbf{h}}. \tag{23.18}$$

Using (23.18) with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V, (21.24), the bound $\frac{1}{L^{(\mathfrak{n})}\tau} \approx 1$ (see (18.9b)), (21.17a), and Young's inequality (where we multiply and divide by powers of ς), we deduce the desired bounds (23.10a)–(23.10b).

Proof of (23.11a)–(23.11b)**:** Let V be any Σ_t-tangent vectorfield. We first use definition (21.31d), (21.9), Prop. 9.1, the estimates of Prop. 17.1, the bound $|\partial_\alpha\mathbf{g}_{\beta\gamma}| \lesssim |\boldsymbol{\partial}\vec{\Psi}| \lesssim \frac{1}{\mu}$ (which follows from Lemma 5.6 Prop. 9.1, and Prop. 17.1), and the pointwise comparison estimates provided by Lemma 21.7 to deduce that:

$$\left|\mathrm{J}_{(\text{Absorb-1})}[V,\boldsymbol{\partial}V]\right| \lesssim \frac{1}{\mu}|V|_g|\boldsymbol{\partial}V|_{\mathbf{h}} + |\partial_a V^a||\boldsymbol{\partial}V|_{\mathbf{h}}. \tag{23.19}$$

Using (23.19) with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V, (21.24), the bound $\frac{1}{L^{(\mathfrak{n})}\tau} \approx 1$ (see (18.9b)), the pointwise estimates (23.6b)–(23.6c), (21.17a), and Young's inequality (where we multiply and divide by powers of μ and ς as needed), we deduce the desired bounds (23.11a)–(23.11b).

Proof of (23.12a)–(23.12b)**:** Let V be any Σ_t-tangent vectorfield. We first use definitions (21.31e) and (21.29) and the same arguments used in the proof of (23.19) to deduce the pointwise bound $\left|\mathfrak{J}_{(\text{Absorb-2})}[V,\boldsymbol{\partial}V]\right| \lesssim \frac{1}{\mu}|V|_g|\boldsymbol{\partial}V|_{\mathbf{h}}$. Then using the same arguments given just below (23.18), we conclude (23.12a)–(23.12b).

Proof of (23.13a)–(23.13b)**:** Let V be any Σ_t-tangent vectorfield. We first use definition (21.31f), (21.18), and the pointwise comparison estimates provided by Lemma 21.7 to deduce the pointwise bound $\left|\mathfrak{J}_{(\text{Material})}[\boldsymbol{\partial}V,\boldsymbol{\partial}V]\right| \lesssim |\mathbf{B}V|_{\mathbf{h}}^2 \lesssim \sum_{a=1,2,3}|\mathbf{B}V^a|^2$. From this bound with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V and the pointwise estimates (23.3), we conclude (23.13a)–(23.13b).

Proof of (23.14a)–(23.14b)**:** Let V be any Σ_t-tangent vectorfield. We first use definitions (21.31g) and (21.1), Lemma 5.6, Prop. 9.1, and the estimates of Prop. 17.1 to deduce the pointwise bound $\left|\mathfrak{J}_{(\text{Null Geometry})}[V,\boldsymbol{\partial}V]\right| \lesssim \frac{1}{\mu}|V|_g|\boldsymbol{\partial}V|_{\mathbf{h}}$. Then using the same arguments given just below (23.18), we conclude (23.14a)–(23.14b).

Proof of (23.15a)–(23.15b)**:** Let V be any Σ_t-tangent vectorfield. We first use definitions (21.46) and (21.1), Prop. 9.1, the estimates of Prop. 17.1, (18.24), (18.8a), and the pointwise comparison estimates provided by Lemma 21.7 to deduce the following pointwise estimate, where ϕ is the cut-off function from Def. 4.1:

$$\left|\mathfrak{E}_{(\text{Principal})}[V,\boldsymbol{\partial}V]\right| \lesssim \frac{1}{\mu}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}|V|_g\left\{|\mathbf{B}V|_g+|\mathrm{d}V_\flat|_{\mathbf{h}}+|\mathrm{div}\,V|\right\}. \tag{23.20}$$

Using (23.20) with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V, the pointwise estimates (23.3) and (23.6a)–(23.6c) with N_{top} in the role of N, the pointwise estimates (23.7a)–(23.7b), the pointwise comparison estimates (21.17a)–(21.17b), the estimate $\left|\mu-\phi\frac{\mathfrak{n}}{L\mu}\right| \lesssim 1$ (which follows from Prop. 17.1 and (18.8a)) and Young's inequality (where we multiply and divide by powers of μ and ς as needed), we conclude that the desired estimates (23.15a)–(23.15b) hold for any $\varsigma \in (0,1]$.

Proof of (23.16a)–(23.16b)**:** Let V be any Σ_t-tangent vectorfield. We first use definitions (21.47) and (21.1), Lemma 5.5, Cor. 5.7, Prop. 9.1, Lemma 15.5, Prop. 17.1, Lemma 18.6, (18.7), (18.8a), (21.48), and the pointwise comparison estimates provided by Lemma 21.7 to deduce the following pointwise bound, where ϕ is the cut-off function from Def. 4.1: $\left|\mathfrak{E}_{(\text{Lower-order})}[V,V]\right| \lesssim \frac{1}{\mu^{5/2}}\left\{\mu-\phi\frac{\mathfrak{n}}{L\mu}\right\}|V|_g^2$. Using this bound with $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$ in the role of V and the comparison estimate (21.17a), we conclude the desired bounds (23.16a)–(23.16b).

□

24. Statement of the a priori L^2 estimates, data-estimates for the L^2-controlling quantities, and bootstrap assumptions for the wave variable energies

We continue to work under the assumptions of Sect. 13.2. In Sect. 24.1, we state all of the a priori energy estimates for the fluid variables and the acoustic geometry. In particular, we state the main estimates for the L^2-controlling quantities defined in Sect. 20.5. The proofs of these estimates take considerable effort and form the focus of the paper through Sect. 29. In Sect. 24.2, we show that along the data-hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$ and ${}^{(\mathfrak{n})}\mathcal{P}_{-U_1}^{[\tau_0,\tau_{\text{Boot}})}$, the L^2-controlling quantities are bounded by $\lesssim \mathring{\epsilon}^2$, i.e., the L^2-controlling quantities have small data. Finally, in Sect. 24.3, we state bootstrap assumptions for the $\mathbb{W}_N$, i.e., for the L^2-controlling quantities of the wave-variables. Our proof of Prop. 24.1, given in Sect. 29.7.1, will yield strict improvements of the bootstrap assumptions for the $\mathbb{W}_N$.

24.1. Statement of the a priori L^2 estimates.

24.1.1. *Statement of the a priori L^2 estimates for the wave-variables.* In the following proposition, we state our main a priori energy estimates for the wave-variables. Its proof is located in Sect. 29.7.1.

Proposition 24.1 (The main a priori estimates for $\mathbb{W}_{[1,N_{\text{top}}]}$)**.** *Let $\mathbb{W}_N(\tau,u)$ be the L^2-controlling quantity for the wave-variables $\vec{\Psi}$, as defined in* (20.43c)*. Under the data-assumptions of Sect. 11, the parameter size-assumptions of Sect. 10.2,* *and the bootstrap assumptions of Sect. 12, there exists a constant $C > 0$ such that the following estimates hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\mathbb{W}_{N_{\text{top}}-K}(\tau,u) \leq C\mathring{\epsilon}^2|\tau|^{-15.6+2K}, \qquad \text{if } 0 \leq K \leq 7, \tag{24.1a}$$

$$\mathbb{W}_N(\tau,u) \leq C\mathring{\epsilon}^2, \qquad \text{if } 1 \leq N \leq N_{\text{top}} - 8. \tag{24.1b}$$

24.1.2. *Statement of the a priori L^2 estimates for the transport-variables.* The following proposition provides an analog of Prop. 24.1 for the transport-variables. Its proof is located in Sects. 26.2 and 27.3.

Proposition 24.2 (The main a priori L^2 estimates for the transport-variables on hypersurfaces)**.** *Let $\mathbb{V}_N(\tau,u)$ and $\mathbb{S}_N(\tau,u)$ be the L^2-controlling quantities for Ω and S, as defined in* (20.45a)–(20.45b)*, and let $\mathbb{C}_N(\tau,u)$ and $\mathbb{D}_N(\tau,u)$ be the L^2-controlling quantities for the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$, as defined in* (20.47a)–(20.47b)*. Under the data-assumptions of Sect. 11, the parameter size-assumptions of Sect. 10.2, and the bootstrap assumptions of Sect. 12, there exists a constant $C > 0$ such that the following estimates hold for $(\tau,u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\mathbb{V}_{N_{\text{top}}-K}(\tau,u), \mathbb{S}_{N_{\text{top}}-K}(\tau,u) \leq C\mathring{\epsilon}^2|\tau|^{-14.6+2K}, \qquad \text{if } 0 \leq K \leq 7, \tag{24.2a}$$

$$\mathbb{V}_N(\tau,u), \mathbb{S}_N(\tau,u) \leq C\mathring{\epsilon}^2, \qquad \text{if } 0 \leq N \leq N_{\text{top}} - 8, \tag{24.2b}$$

$$\mathbb{C}_{N_{\text{top}}}(\tau,u), \mathbb{D}_{N_{\text{top}}}(\tau,u) \leq C\mathring{\epsilon}^2|\tau|^{-17.1}, \tag{24.3a}$$

$$\mathbb{C}_{N_{\text{top}}-1-K}(\tau,u), \mathbb{D}_{N_{\text{top}}-1-K}(\tau,u) \leq C\mathring{\epsilon}^2|\tau|^{-14.6+2K}, \qquad \text{if } 0 \leq K \leq 7, \tag{24.3b}$$

$$\mathbb{C}_N(\tau,u), \mathbb{D}_N(\tau,u) \leq C\mathring{\epsilon}^2, \qquad \text{if } 0 \leq N \leq N_{\text{top}} - 9. \tag{24.3c}$$

In addition to the L^2 estimates of Prop. 24.2, which are estimates for the transport-variables on constant-rough-time hypersurfaces and null hypersurfaces, we also derive L^2 estimates for the transport-variables on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. We need these estimates because rough tori integrals are featured in the main elliptic-hyperbolic integral identity (see Prop. 21.14) that we use to control various top-order spacetime L^2 norms of the Ω and S. The rough tori estimates of interest are provided by the next proposition. Its proof is located in Sects. 26.3 and 27.7.

Proposition 24.3 (The main a priori L^2 estimates for the transport-variables on the rough tori)**.** *Let $\mathbb{V}_N^{(\text{Rough Tori})}(\tau,u)$ and $\mathbb{S}_N^{(\text{Rough Tori})}(\tau,u)$ be the rough tori-L^2-controlling quantities for Ω and S defined in* (20.46a)–(20.46b)*, and let $\mathbb{C}_N^{(\text{Rough Tori})}$ and $\mathbb{D}_N^{(\text{Rough Tori})}(\tau,u)$ be the rough tori-L^2-controlling quantities for $\mathcal{C}$ and $\mathcal{D}$ defined in* (20.48a)–(20.48b)*. Under the data-assumptions of Sect. 11, the parameter size-assumptions of Sect. 10.2, and the bootstrap assumptions of Sect. 12, there exists a constant $C > 0$ such that the following estimates hold for $(\tau,u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\mathbb{V}_{N_{\text{top}}}^{(\text{Rough Tori})}(\tau,u), \mathbb{S}_{N_{\text{top}}}^{(\text{Rough Tori})} \leq C\mathring{\epsilon}^2|\tau|^{-17.1}, \tag{24.4a}$$

$$\mathbb{V}_{N_{\text{top}}-1-K}^{(\text{Rough Tori})}(\tau,u), \mathbb{S}_{N_{\text{top}}-1-K}^{(\text{Rough Tori})} \leq C\mathring{\epsilon}^2|\tau|^{-14.6+2K}, \qquad \text{if } 0 \leq K \leq 7, \tag{24.4b}$$

$$\mathbb{V}_N^{(\text{Rough Tori})}(\tau,u), \mathbb{S}_N^{(\text{Rough Tori})}(\tau,u) \leq C\mathring{\epsilon}^2, \qquad \text{if } 0 \leq N \leq N_{\text{top}} - 9, \tag{24.4c}$$

$$\mathbb{C}_{N_{\text{top}}-1}^{(\text{Rough Tori})}(\tau,u), \mathbb{D}_{N_{\text{top}}-1}^{(\text{Rough Tori})}(\tau,u) \leq C\mathring{\epsilon}^2|\tau|^{-17.1}, \tag{24.5a}$$

$$\mathbb{C}_{N_{\text{top}}-2-K}^{(\text{Rough Tori})}(\tau,u), \mathbb{D}_{N_{\text{top}}-2-K}^{(\text{Rough Tori})}(\tau,u) \leq C\mathring{\epsilon}^2|\tau|^{-14.6+2K}, \qquad \text{if } 0 \leq K \leq 7, \tag{24.5b}$$

$$\mathbb{C}_N^{(\text{Rough Tori})}(\tau,u), \mathbb{D}_N^{(\text{Rough Tori})}(\tau,u) \leq C\mathring{\epsilon}^2, \qquad \text{if } 0 \leq N \leq N_{\text{top}} - 10. \tag{24.5c}$$

24.1.3. *Statement of the a priori L^2 estimates for the acoustic geometry.* The following proposition provides our main a priori energy estimates for the acoustic geometry. Its proof is located in Sect. 29.8.

Proposition 24.4 (The main a priori estimates for the acoustic geometry along the rough foliations)**.** *Under the data-assumptions of Sect. 11, the parameter size-assumptions of Sect. 10.2, and the bootstrap assumptions of Sect. 12, there exists a* *constant* $C > 0$ *such that the following estimates hold for* $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$*:*

$$\begin{pmatrix} \left\| \mathcal{P}^{N_{\text{top}}} \text{tr}_{\not{g}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \\ \left\| \not{\mathcal{L}}_{\mathcal{P}}^{N_{\text{top}}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \end{pmatrix} \leq C \mathring{\epsilon} |\tau|^{-8.8}, \tag{24.6a}$$

$$\begin{pmatrix} \left\| \mathcal{P}^{N-1} \text{tr}_{\not{g}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \\ \left\| \not{\mathcal{L}}_{\mathcal{P}}^{N-1} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \\ \left\| \mathcal{P}_*^{N} \mu \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \\ \left\| \mathcal{P}^{N} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \end{pmatrix} \leq C \mathring{\epsilon} |\tau|^{-7.3+N_{\text{top}}-N}, \qquad \text{if } N_{\text{top}} - 7 \leq N \leq N_{\text{top}}, \tag{24.6b}$$

$$\begin{pmatrix} \left\| \mathcal{P}^{\leq N_{\text{top}}-9} \text{tr}_{\not{g}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \\ \left\| \not{\mathcal{L}}_{\mathcal{P}}^{N_{\text{top}}-9} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \\ \left\| \mathcal{P}_*^{[1,N_{\text{top}}-8]} \mu \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \\ \left\| \mathcal{P}^{[1,N_{\text{top}}-8]} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}\right)} \end{pmatrix} \leq C \mathring{\epsilon}. \tag{24.6c}$$

24.2. **Data-estimates for the L^2-controlling quantities.** In this section, we derive estimates for the data of the L^2-controlling quantities $\mathbb{W}_{[1,N_{\text{top}}]}$, $\mathbb{V}_{\leq N_{\text{top}}}$, etc. We will use these data-estimates in our proofs of Props. 24.1–24.4.

Lemma 24.5 (The L^2-controlling quantities are initially small)**.** *The following estimates hold for* $\tau \in [\tau_0, \tau_{\text{Boot}})$ *and* $u \in [-U_1, U_2]$*:*

$$\mathbb{W}_{[1,N_{\text{top}}]}(\tau, -U_1) \leq C \mathring{\epsilon}^2, \qquad \mathbb{W}_{[1,N_{\text{top}}]}(\tau_0, u) \leq C \mathring{\epsilon}^2, \tag{24.7}$$

$$\mathbb{V}_{\leq N_{\text{top}}}(\tau_0, u) \leq C \mathring{\epsilon}^2, \qquad \mathbb{V}_{\leq N_{\text{top}}}(\tau, -U_1) \leq C \mathring{\epsilon}^2, \tag{24.8a}$$

$$\mathbb{S}_{\leq N_{\text{top}}}(\tau_0, u) \leq C \mathring{\epsilon}^2, \qquad \mathbb{S}_{\leq N_{\text{top}}}(\tau, -U_1) \leq C \mathring{\epsilon}^2, \tag{24.8b}$$

$$\mathbb{C}_{\leq N_{\text{top}}}(\tau_0, u) \leq C \mathring{\epsilon}^2, \qquad \mathbb{C}_{\leq N_{\text{top}}}(\tau, -U_1) \leq C \mathring{\epsilon}^2, \tag{24.9a}$$

$$\mathbb{D}_{\leq N_{\text{top}}}(\tau_0, u) \leq C \mathring{\epsilon}^2, \qquad \mathbb{D}_{\leq N_{\text{top}}}(\tau, -U_1) \leq C \mathring{\epsilon}^2, \tag{24.9b}$$

$$\mathbb{V}^{(\text{Rough Tori})}_{\leq N_{\text{top}}}(\tau_0, u) \leq C \mathring{\epsilon}^2, \tag{24.10a}$$

$$\mathbb{S}^{(\text{Rough Tori})}_{\leq N_{\text{top}}}(\tau_0, u) \leq C \mathring{\epsilon}^2, \tag{24.10b}$$

$$\mathbb{C}^{(\text{Rough Tori})}_{\leq N_{\text{top}}-1}(\tau_0, u) \leq C \mathring{\epsilon}^2, \tag{24.11a}$$

$$\mathbb{D}^{(\text{Rough Tori})}_{\leq N_{\text{top}}-1}(\tau_0, u) \leq C \mathring{\epsilon}^2. \tag{24.11b}$$

Proof. The estimates stated in (24.7) are straightforward consequences of the data-assumptions (11.11a), (11.11b), and (11.12a), definitions (20.23a)–(20.23b), (20.25), and (20.43a)–(20.43c), the identities (20.51a), (20.51d), (3.31a), (5.8c)–(5.8d), (5.13c), (6.20a), (7.7), and (6.11)–(6.13), Prop. 9.1, the bootstrap assumptions (see in particular (**BA** $t-$ **SIZE**)), and the estimates (15.24) and (18.9a)–(18.9b).

The estimates (24.8a)–(24.8b) follow from applying similar reasoning based on definitions (20.24a)–(20.24b) and (20.45a)–(20.45b) and the data-assumptions (11.11c) and (11.12b).

The estimates (24.9a)–(24.9b) follow from applying similar reasoning based on definitions (20.24a)–(20.24b) and (20.47a)–(20.47b) and the data-assumptions (11.11d) and (11.12c).

The estimates (24.10a)–(24.10b) follow from definitions (20.46a)–(20.46b) and the data-assumptions (11.13b).
The estimates (24.11a)–(24.11b) follow from definitions (20.48a)–(20.48b) and the data-assumptions (11.13c). □

24.3. **Bootstrap assumptions for the $\mathbb{W}_N$.** In proving Props. 24.1–24.4, we find it convenient to make bootstrap assumptions for the L^2-controlling quantities for the wave-variables. Specifically, with $\mathbb{W}_N(\tau, u)$ (see definition (20.43c)) denoting the L^2-controlling quantity for the wave-variables $\vec{\Psi}$, we assume that the following bootstrap assumptions hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, where ε is the bootstrap parameter from Sect. 12.3.1:

$$\mathbb{W}_{N_{\text{top}}-K}(\tau, u) \leq \varepsilon|\tau|^{-15.6+2K}, \qquad \text{if } 0 \leq K \leq 7, \tag{24.12a}$$

$$\mathbb{W}_{[1,N_{\text{top}}-8]}(\tau, u) \leq \varepsilon. \tag{24.12b}$$

Remark 24.6 (The wave-variable energy estimates improve the bootstrap assumptions)**.** Note that when $\mathring{\epsilon}$ is sufficiently small, the estimates of Prop. 24.1 yield strict improvements over the bootstrap assumptions (24.12a)–(24.12b).

25. Preliminary below-top-order L^2 estimates for the acoustic geometry and a derivative-losing estimate

We continue to work under the assumptions of Sect. 13.2. In this short section, we derive preliminary L^2 estimates for the below-top-order derivatives of the eikonal function quantities μ, L^i, χ, and $\text{tr}_{\not{g}}\chi$. We state the bounds in terms of the wave L^2-controlling quantities $\mathbb{Q}_{[1,N]}(\tau, u)$ from Def. 20.10 and the initial data-size-parameter $\mathring{\epsilon}$. We also derive related L^2 estimates for top-order derivatives of χ and $\text{tr}_{\not{g}}\chi$ in the case that one L-differentiation is involved. We provide the main estimates in Lemma 25.1. The estimates are rather straightforward consequences of the transport inequalities provided by Prop. 13.7 and Lemma 16.3. In Cor. 25.2, we derive derivative-losing L^2 estimates for $\vec{\Psi}$ that do not involve any explicit singular powers of $|\tau|^{-1}$. The absence of such singular powers is important for our proof that the wave-variable energies become less and less singular with respect to powers of $|\tau|^{-1}$ as we descend below the top-order (see Prop. 24.1).

Most of the L^2 estimates we derive in this section lose one derivative. In Prop. 29.7, we will derive complementary L^2 estimates for the top-order derivatives of χ and $\text{tr}_{\not{g}}\chi$. Those estimates are much harder to prove because we cannot afford to lose any derivatives, which forces us to rely on the modified quantities from Sect. 19 and elliptic estimates for the top-order derivatives of χ.

25.1. **Preliminary below-top-order L^2 estimates for the eikonal function quantities.**

Lemma 25.1 (Preliminary below-top-order L^2 estimates for the eikonal function quantities)**.** *Let $N \leq N_{\text{top}}$, and recall the vectorfield differentiation conventions established in Def. 8.10. Then the following estimates hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times$*

$[-U_1, U_2]$:

$$\begin{pmatrix} \left\| L\mathcal{P}_*^{[1,N]}\mu \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| L\mathcal{P}^{\leq N} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| L\mathcal{P}^{\leq N-1} \mathrm{tr}_{\not{g}}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[U_1,u]}\right)} \\ \left\| \not{\mathcal{L}}_L \not{\mathcal{L}}_{\mathcal{P}}^{\leq N-1}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| L\mathcal{Z}^{\leq N;1} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| L\mathcal{Z}^{\leq N-1;1} \mathrm{tr}_{\not{g}}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| \not{\mathcal{L}}_L \not{\mathcal{L}}_{\mathcal{Z}}^{\leq N-1;1}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \end{pmatrix} \lesssim \mathring{\epsilon} + \frac{\mathbb{Q}_{[1,N]}^{1/2}(\tau,u)}{|\tau|^{1/2}}, \tag{25.1a}$$

$$\begin{pmatrix} \left\| \mathcal{P}_*^{[1,N]}\mu \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| \mathcal{P}^{[1,N]} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| \mathcal{P}^{\leq N-1} \mathrm{tr}_{\not{g}}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| \not{\mathcal{L}}_{\mathcal{P}}^{\leq N-1}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| \mathcal{Z}_*^{[1,N];1} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| \mathcal{Z}^{\leq N-1;1} \mathrm{tr}_{\not{g}}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ \left\| \not{\mathcal{L}}_{\mathcal{Z}}^{\leq N-1;1}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \end{pmatrix} \lesssim \mathring{\epsilon} + \int_{\tau'=\tau_0}^{\tau} \frac{\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)}{|\tau'|^{1/2}}\,d\tau'. \tag{25.1b}$$

Proof. We fix $u \in [-U_1, U_2]$ and define the following functions for $\tau \in [\tau_0, \tau_{\text{Boot}})$:

$$q_N(\tau) \stackrel{\text{def}}{=} \sum_{i=1}^{3} \left\| \mathcal{P}^{[1,N]} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} + \left\| \mathcal{P}^{\leq N-1}\mathrm{tr}_{\not{g}}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} + \left\| \not{\mathcal{L}}_{\mathcal{P}}^{\leq N-1}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}, \tag{25.2}$$

$$\begin{aligned} p_N(\tau) \stackrel{\text{def}}{=} & \sum_{i=1}^{3} \left\| \mathcal{Z}_*^{[1,N];1} L^i_{(\text{Small})} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} + \left\| \mathcal{Z}^{\leq N-1;1}\mathrm{tr}_{\not{g}}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\ & + \left\| \not{\mathcal{L}}_{\mathcal{Z}}^{\leq N-1;1}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} + \left\| \mathcal{P}_*^{[1,N]}\mu \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}. \end{aligned} \tag{25.3}$$

Next, we note the pointwise estimate $\left| L\left(|\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi|_{\not{g}} \right) \right| \lesssim \left| \not{\mathcal{L}}_L \not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi \right|_{\not{g}} + \varepsilon \left| \not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi \right|_{\not{g}}$, which follows from the Leibniz rule for $\ell_{t,u}$-projected Lie derivatives, the identity $\not{\mathcal{L}}_L \not{g}^{-1} = -2\chi^{\#\#}$ (which is a straightforward consequence of (3.42)), and the estimate $|\chi|_{\not{g}} \lesssim \varepsilon$, which follows from (3.49a), Prop. 9.1, the bootstrap assumptions, and Prop. 17.1. Multiplying (13.13c)–(13.13e) by $\frac{1}{L^{(\mathfrak{n})}\tau}$ and using (16.15) and Cor. 17.2, as well as the estimates (18.1), (18.9b), (20.53), (20.58), and the pointwise estimate noted above, we deduce:

$$q_N(\tau) \lesssim q_N(\tau_0) + \mathring{\epsilon} + \varepsilon \int_{\tau'=\tau_0}^{\tau} q_N(\tau')\,d\tau' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,d\tau'. \tag{25.4}$$

We next note that the L^2 assumptions on the data stated in Sect. 11.2.1 imply that $q_N(\tau_0) \lesssim \mathring{\epsilon}$. Inserting this estimate into RHS (25.4) and applying Grönwall's inequality, we find that $q_N(\tau) \lesssim \mathring{\epsilon} + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,d\tau'$, which yields (25.1b) for $\mathcal{P}^{[1,N]} L^i_{(\text{Small})}$, $\mathcal{P}^{\leq N-1}\mathrm{tr}_{\not{g}}\chi$, and $\not{\mathcal{L}}_{\mathcal{P}}^{\leq N-1}\chi$.

We now prove (25.1b) for the remaining terms $\mathcal{Z}_*^{[1,N];1} L^i_{(\text{Small})}$, $\mathcal{Z}^{\leq N-1;1}\mathrm{tr}_{\not{g}}\chi$, $\not{\mathcal{L}}_{\mathcal{Z}}^{\leq N-1;1}\chi$, and $\mathcal{P}_*^{[1,N]}\mu$. We begin by examining the first term on RHS (13.13b). Repeatedly using the commutator estimates (13.6a)–(13.6b), the bootstrap assumptions, and the estimates of Prop. 17.1, we deduce:

$$|\mathcal{Z}_*^{[1,N+1];1}\vec{\Psi}| \lesssim |\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}| + |\mathcal{P}^{[1,N+1]}\vec{\Psi}| + \varepsilon|\mathcal{Z}_*^{[1,N];1}\gamma| + \varepsilon|\mathcal{P}_*^{[1,N]}\underline{\gamma}|. \tag{25.5}$$

Arguing as in the proof of (25.4), but using (13.13b) and (13.13f)–(13.13h) in place of (13.13c)–(13.13e), and using (25.5), (20.53), and (20.58), we find that:

$$p_N(\tau) \lesssim p_N(\tau_0) + \mathring{\epsilon} + \int_{\tau'=\tau_0}^{\tau} p_N(\tau')\,d\tau' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,d\tau'. \tag{25.6}$$

As above, the L^2 assumptions on the data stated in Sect. 11.2.1 imply that $p(\tau_0) \lesssim \mathring{\epsilon}$, and we can use Grönwall's inequality to deduce $p_N(\tau) \lesssim \mathring{\epsilon} + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,d\tau'$, thereby concluding (25.1b) for $\mathcal{Z}_*^{[1,N];1}L^i_{(\text{Small})}$, $\mathcal{Z}^{\leq N-1;1}\text{tr}_{\not{g}}\chi$, $\not{\mathcal{L}}_{\mathcal{Z}}^{\leq N-1;1}\chi$, and $\mathcal{P}_*^{[1,N]}\mu$.

We now prove (25.1a). Taking the norm $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ of inequalities (13.13b)–(13.13h), and arguing as above using the already proven estimate (25.1b), we obtain the desired result. We clarify that this argument generates the time integrals $\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,d\tau'$, which we bound by $\lesssim \mathbb{Q}_{[1,N]}^{1/2}(\tau,u) \lesssim$ RHS (25.1a) by exploiting the monotonicity of $\mathbb{Q}_{[1,N]}(\tau,u)$ with respect to its arguments. □

25.2. L^2 **estimates for $\vec{\Psi}$ that lose one derivative.** In the next corollary, we prove the derivative-losing estimates for $\vec{\Psi}$ that we highlighted at the beginning of Sect. 25.

Corollary 25.2 (L^2 estimates for $\vec{\Psi}$ that lose one derivative)**.** *Let $1 \leq N \leq N_{\text{top}}$, and recall the vectorfield differentiation conventions established in Def. 8.10. Then the following estimates hold for $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$:*

$$\left\|\mathcal{Z}_*^{N;1}\vec{\Psi}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \lesssim \mathring{\epsilon} + \mathbb{Q}_{[1,N]}^{1/2}(\tau,u). \tag{25.7}$$

Proof. Using (25.5) with $N+1$ replaced by N, we deduce the following pointwise estimate:

$$\left|\mathcal{Z}_*^{N;1}\vec{\Psi}\right| \lesssim \left|\breve{X}\mathcal{P}^{[1,N-1]}\vec{\Psi}\right| + \left|\mathcal{P}^{[1,N]}\vec{\Psi}\right| + \varepsilon\left|\mathcal{Z}_*^{[1,N-1];1}\gamma\right| + \varepsilon\left|\mathcal{P}_*^{[1,N-1]}\underline{\gamma}\right|, \tag{25.8}$$

where when $N=1$, we must have $\mathcal{Z}_*^{1;1} = \mathcal{P}$ on LHS (25.8) and then only the second term on RHS (25.8) is present. Taking the norm $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ of (25.8) and using (20.53), (20.58), (25.1b), and the estimate $\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,d\tau' \lesssim \mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$, which follows from the fact that $\mathbb{Q}_{[1,N]}(\tau,u)$ is increasing in its arguments, we conclude (25.7). □

26. Below-top-order hyperbolic L^2 estimates for the specific vorticity and entropy gradient

We continue to work under the assumptions of Sect. 13.2. In this short section, we prove the below-top-order L^2 estimates for the specific vorticity and entropy gradient. Specifically, we prove (24.2a)–(24.2b), (24.3b)–(24.3c), (24.4b)–(24.4c), and (24.5b)–(24.5c). We derive a preliminary energy integral inequality in Sect. 26.1, and we prove the final estimates in Sects. 26.2–26.3. These estimates are relatively straightforward consequences of the transport energy identity (20.29) and various pointwise estimates we have already established, including the ones provided by Prop. 23.3. In Sect. 27, we will prove the top-order estimate (24.3a) and the related estimates (24.4a) and (24.5a). The proofs of these estimates are much more difficult because they rely on the intricate elliptic-hyperbolic integral identity (21.63).

In Sect. 29, we will use the estimates for Ω, S, $\mathcal{C}$, and $\mathcal{D}$ that we derive in this section in our proof of the wave a priori estimates, which we stated as Prop. 24.1. Hence, we highlight that for the logic of the paper, it is important that **the estimates we derive in this section do not rely on the wave estimates of Prop. 24.1**; our proofs of (24.2a)–(24.2b), (24.3b)–(24.3c), (24.4b)–(24.4c), and (24.5b)–(24.5c) instead rely on the bootstrap assumptions (24.12a)–(24.12b) for the wave energies, which are *weaker* than the estimates that we derive in Prop. 24.1.

26.1. **Integral inequalities for the below-top-order vorticity- and entropy gradient-controlling quantities.** We begin with the following preliminary lemma, which provides integral inequalities for the below-top-order vorticity- and entropy gradient-controlling quantities.

Lemma 26.1 (Integral inequalities for the below-top-order vorticity- and entropy gradient-controlling quantities)**.** *Let $0 \leq N \leq N_{\text{top}}$, and recall that the L^2-controlling quantities $\mathbb{V}_{\leq N}$, $\mathbb{S}_{\leq N}$, $\cdots$, are defined in Defs. 20.10 and 20.12. Then the following integral inequalities hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\mathbb{V}_{\leq N}(\tau, u) + \mathbb{S}_{\leq N}(\tau, u) \lesssim \mathring{\epsilon}^2 + \int_{u'=-U_1}^{u} \{\mathbb{V}_{\leq N}(\tau, u') + \mathbb{S}_{\leq N}(\tau, u')\}\, \mathrm{d}u' + \varepsilon^2 \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N]}(\tau', u)\, \mathrm{d}\tau'. \tag{26.1}$$

Moreover, the following estimates hold for $0 \leq N \leq N_{\text{top}} - 1$:

$$\begin{aligned}\mathbb{C}_{\leq N}(\tau, u) + \mathbb{D}_{\leq N}(\tau, u) \lesssim \mathring{\epsilon}^2 &+ \int_{u'=-U_1}^{u} \{\mathbb{C}_{\leq N}(\tau, u') + \mathbb{D}_{\leq N}(\tau, u)\}\, \mathrm{d}u' \\ &+ \int_{u'=-U_1}^{u} \{\mathbb{V}_{\leq N+1}(\tau, u') + \mathbb{S}_{\leq N+1}(\tau, u')\}\, \mathrm{d}u' \\ &+ \varepsilon^2 \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N]}(\tau', u)\, \mathrm{d}\tau' + \varepsilon^2 \int_{u'=-U_1}^{u} \mathbb{Q}_{[1,N]}(\tau, u')\, \mathrm{d}u' + \varepsilon^2 \mathbb{K}_{[1,N]}(\tau, u).\end{aligned} \tag{26.2}$$

Proof. We first prove (26.1). For $0 \leq N \leq N_{\text{top}}$, we consider the transport energy identity (20.29) with $(\mathcal{P}^N \Omega, \mathcal{P}^N S)$ in the role of f. We use the bootstrap assumptions and Prop. 17.1 to deduce the bound $|L\mu + \mu \mathrm{tr}_{\not g} \not k| \lesssim 1$ for the integrand factors in the last integral on RHS (20.29), and we use (18.9b) to deduce that the integrand factors $\frac{1}{L^{(\mathfrak{n})}\tau}$ in (20.29) verify $\frac{1}{L^{(\mathfrak{n})}\tau} \approx 1$. We also use Lemma 24.5 to bound the data-dependent terms $\mathbb{E}_{(\text{Transport})}[f](\tau_0, u) + \mathbb{F}_{(\text{Transport})}[f](\tau, -U_1)$ on RHS (20.29) by $\lesssim \mathring{\epsilon}^2$. Considering also Def. 20.10, the estimate (18.9b), and Lemma 20.14, and using Young's inequality, we deduce:

$$\begin{aligned}\mathbb{V}_{\leq N}(\tau, u) + \mathbb{S}_{\leq N}(\tau, u) \lesssim \mathring{\epsilon}^2 &+ \int_{u'=-U_1}^{u} \{\mathbb{V}_{\leq N}(\tau, u') + \mathbb{S}_{\leq N}(\tau, u')\}\, \mathrm{d}u' \\ &+ \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \left|\mu \mathbf{B} \mathcal{P}^{\leq N}(\Omega, S)\right|^2 \mathrm{d}\varpi.\end{aligned} \tag{26.3}$$

Next, we use the pointwise estimate (23.3) to bound the integrand factors of $\left|\mu \mathbf{B} \mathcal{P}^{\leq N}(\Omega, S)\right|$ on RHS (26.3). Again appealing to Def. 20.10 and Lemma 20.14, and also using (18.1), (18.9b), and (25.1b), we conclude (26.1), but with the additional double integral $\varepsilon^2 \int_{\tau'=\tau_0}^{\tau} \left\{\int_{\tau''=\tau_0}^{\tau'} \frac{\mathbb{Q}_{[1,N]}^{1/2}(\tau'', u)}{|\tau''|^{1/2}}\, \mathrm{d}\tau''\right\}^2 \mathrm{d}\tau'$ on the RHS arising from the estimate (25.1b), which we use to handle the terms $\varepsilon|\mathcal{P}_*^{[1,N]}\gamma|$ on RHS (23.3). By using that $\mathbb{Q}_{[1,N]}(\tau, u)$ is increasing in its arguments, we can bound this double integral by $\lesssim \varepsilon^2 \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N]}(\tau', u)\, \mathrm{d}\tau'$, which in turn is bounded by RHS (26.1) as desired. We have therefore proved (26.1).

The estimate (26.2) can be proved using similar arguments based on the pointwise estimates (23.4a)–(23.4b) for $0 \leq N \leq N_{\text{top}} - 1$. Although we do not provide full details, we point out three additional ingredients that play a role in the proof. Specifically, to bound the spacetime integrals $\varepsilon^2 \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} |\mathcal{P}^{N+1}\vec{\Psi}|^2\, \mathrm{d}\varpi$ generated by the terms $\varepsilon|\mathcal{P}^{N+1}\vec{\Psi}|$ on RHSs (23.4a)–(23.4b), we also use: **i)** in the case $\mathcal{P}^{N+1} = L\mathcal{P}^N$, we use the $L^2({}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau)})$-control guaranteed by (18.9b) and (20.53), which leads to the presence of the term $\varepsilon^2 \int_{u'=-U_1}^{u} \mathbb{Q}_{[1,N]}(\tau, u')\, \mathrm{d}u'$ on RHS (26.2); **iia)** in the case $\mathcal{P}^{N+1} = Y_{(A)}\mathcal{P}^N$, with $\mathbf{1}_{[-U_{☢},U_{☢}]} = \mathbf{1}_{[-U_{☢},U_{☢}]}(u')$ denoting the characteristic function of the interval $[-U_{☢}, U_{☢}]$, to bound $\varepsilon^2 \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \mathbf{1}_{[-U_{☢},U_{☢}]}|Y_{(A)}\mathcal{P}^N\vec{\Psi}|^2\, \mathrm{d}\varpi$, we use the spacetime integral coerciveness estimate (20.63a), which leads to the presence of the term $\varepsilon^2\mathbb{K}_{[1,N]}(\tau, u)$ on RHS (26.2); and **iib)** to bound the complementary integral $\varepsilon^2 \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \mathbf{1}_{[-U_{☢},U_{☢}]^c}|Y_{(A)}\mathcal{P}^N\vec{\Psi}|^2\, \mathrm{d}\varpi$, we use (18.2) and (20.53), which collectively allow us to bound the integral by $\lesssim \varepsilon^2 \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \mu|Y_{(A)}\mathcal{P}^N\vec{\Psi}|^2\, \mathrm{d}\varpi \lesssim \varepsilon^2 \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N]}(\tau', u)\, \mathrm{d}\tau'$ as desired. □

26.2. **Proof of** (24.2a)–(24.2b) **and** (24.3b)–(24.3c)**.** We first prove (24.2b). We set $\mathbb{T}(\tau, u) \stackrel{\text{def}}{=} \mathbb{V}_{\leq N_{\text{top}}-8}(\tau, u) + \mathbb{S}_{\leq N_{\text{top}}-8}(\tau, u)$. From (26.1), the wave energy bootstrap assumptions (24.12b), and (10.9b), we find that $\mathbb{T}(\tau, u) \leq C\mathring{\epsilon}^2 + C\int_{u'=-U_1}^{u} \mathbb{T}(\tau, u')\, \mathrm{d}u'$. Applying Grönwall's inequality, we conclude that $\mathbb{T}(\tau, u) \leq C\mathring{\epsilon}^2$, which yields (24.2b).

Similarly, to prove (24.3c), we set $\mathbb{T}(\tau,u) \stackrel{\text{def}}{=} \mathbb{C}_{\leq N_{\text{top}}-9}(\tau,u) + \mathbb{D}_{\leq N_{\text{top}}-9}(\tau,u)$. From (26.2), the wave energy bootstrap assumptions (24.12b), (10.9b), and the already proved estimates (24.2b), we find that $\mathbb{T}(\tau,u) \leq C\mathring{\epsilon}^2 + C\int_{u'=-U_1}^{u} \mathbb{T}(\tau,u')\,\mathrm{d}u'$. Applying Grönwall's inequality, we conclude that $\mathbb{T}(\tau,u) \leq C\mathring{\epsilon}^2$, which yields (24.3c).

The estimates (24.2a) and (24.3b) can be proved by combining similar arguments with the wave energy bootstrap assumptions (24.12a)–(24.12b).

□

26.3. **Proof of** (24.4b)–(24.4c) **and** (24.5b)–(24.5c)**.** We fix any integer N with $0 \leq N \leq N_{\text{top}} - 1$. Using (20.5), (18.9b), (20.55a)–(20.55b), and the data-estimates (24.10a)–(24.10b), we deduce that:

$$
\begin{aligned}
\left\|\mathcal{P}^N(\Omega,S)\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}^2 &\lesssim \left\|\mathcal{P}^N(\Omega,S)\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}\right)}^2 + \int_{{}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left|L\mathcal{P}^N(\Omega,S)\right|^2 \mathrm{d}\overline{\varpi} \\
&\lesssim \mathring{\epsilon}^2 + \left\|\frac{1}{\sqrt{L^{(\mathfrak{n})}\tau}}\mathcal{P}^{\leq N}(\Omega,S)\right\|_{L^2\left({}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau]}\right)}^2 \lesssim \mathring{\epsilon}^2 + \mathbb{V}_{N+1}(\tau,u) + \mathbb{S}_{N+1}(\tau,u).
\end{aligned}
\tag{26.4}
$$

From (26.4) and the already proven estimates (24.2a)–(24.2b) for $\mathbb{V}_{N+1}(\tau,u)$ and $\mathbb{S}_{N+1}(\tau,u)$, we conclude, in view of definitions (20.46a)–(20.46b), the desired estimates (24.4b)–(24.4c).

The estimates (24.5b)–(24.5c) can be proved via similar arguments based on the data-estimates (24.11a)–(24.11b), (20.56a)–(20.56b), the already proven estimates (24.3b)–(24.3c), and definitions (20.48a)–(20.48b).

□

27. Top-order elliptic-hyperbolic L^2 estimates for the specific vorticity and entropy gradient

We continue to work under the assumptions of Sect. 13.2. Our main goal in this section is to derive the top-order L^2 estimate (24.3a) for the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$. It turns out that due to the structure of the elliptic-hyperbolic integral identity (21.63), the proof of (24.3a) is coupled to the proof of the top-order energy estimates (24.4a) for Ω and S along the rough tori. Hence, we also prove (24.4a) in this section. Finally, as a simple consequence of (24.3a), we will also prove the top-order rough tori energy estimate (24.5a) for $\mathcal{C}$ and $\mathcal{D}$. In Sect. 29, we will use the estimates for $\mathcal{C}$ and $\mathcal{D}$ that we derive in this section in our proof of the wave a priori estimates, which we stated as Prop. 24.1. Hence, we highlight that for the logic of the paper, it is important that **the estimates we derive in this section do not rely on the wave estimates of Prop. 24.1**; our proofs of (24.3a) and (24.4a) instead rely on the bootstrap assumptions (24.12a)–(24.12b), for the wave energies, which are *weaker* than the estimates that we derive in Prop. 24.1.

To explain the main challenges in the analysis, we recall that the transport equations (2.24b) and (2.25a) satisfied by $\mathcal{C}$ and $\mathcal{D}$ feature some difficult source terms, denoted by $\mathfrak{M}_{(\mathcal{C})}^i$ and $\mathfrak{M}_{(\mathcal{D})}$, that depend on the general first-order derivatives of Ω and S. These source terms have the potential to cause the loss of a derivative at the top-order because they cannot be bounded using pure transport estimates. In the below-top-order estimates of Sect. 26, we allowed the loss of a derivative, as is signified by the terms $\int_{u'=-U_1}^{u} \{\mathbb{V}_{\leq N+1}(\tau,u') + \mathbb{S}_{\leq N+1}(\tau,u')\}\,\mathrm{d}u'$ on RHS (26.2). To avoid the loss at the top-order, we handle the difficult source terms in a different way, one that is based on combining the elliptic-hyperbolic integral identity provided by Prop. 21.14 with pointwise estimates that take into account the special structure of the equations of Theorem 2.15, sharp estimates for the acoustic geometry and the rough time function, and the below-top-order estimates that we already derived in Sect. 26.

We organize this section this as follows:

- In Sect. 27.1, we derive some preliminary energy integral inequalities for $\mathcal{C}$ and $\mathcal{D}$, which feature the difficult source terms; this part of the proof is not more difficult than the proof of the below-top-order energy integral inequalities we derived in Lemma 26.1.
- In Sect. 27.2, we use the preliminary energy integral inequalities to derive the main energy integral inequalities for the top-order derivatives of $\mathcal{C}$ and $\mathcal{D}$. These main energy integral inequalities are conditional on having L^2 estimates for the difficult source terms, which we prove independently as Prop. 27.5 in Sect. 27.6.
- In Sect. 27.3, we use the main energy integral inequalities to prove the top-order L^2 estimate (24.3a).
- In Sect. 27.4, to initiate the proof of Prop. 27.5, we derive estimates for the rough tori error integrals $\left\|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$ for $\tau \in [\tau_0,\tau_{\text{Boot}}]$, which appear on the right-hand side of the elliptic-hyperbolic integral identity (21.63) (with $\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)$ in the role of V) when we use the identity in our proof of Prop. 27.5. It might be tempting to think of these rough tori integrals as "data terms" since the rough tori $\widetilde{\ell}_{\tau,-U_1}$ of interest are contained in

the "data null hypersurface" portion ${}^{(\mathfrak{n})}\mathcal{P}_{-U_1}^{[\tau_0,\tau_{\text{Boot}}]}$. In particular, if the data on Σ_0 are compactly supported in $\Sigma_0 \cap \{-U_1 \le u \le U_2\}$, then standard domain of dependence considerations imply that (Ω, S) vanish along $\widetilde{\ell}_{\tau,-U_1}$. However, in general, the integrals $\left\|\mathcal{P}^{\le N_{\text{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$ are not true "data terms" because their size depends on various norms of the rough time function ${}^{(\mathfrak{n})}\tau$, which in turn depends on the behavior of the fluid near the singular boundary. Hence, to bound the integrals $\left\|\mathcal{P}^{\le N_{\text{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$, we combine slight extensions of the Cauchy stability results that we derive in Appendix B with suitable $C^{2,1}_{\text{geo}}$ estimates for ${}^{(\mathfrak{n})}\tau$ in a region that has slightly larger u-width compared to ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$.
- In Sect. 27.5, in service of the proof of Prop. 27.5, we derive estimates for the error integrals appearing in the elliptic-hyperbolic integral identity (21.63).
- In Sect. 27.6, we prove Prop. 27.5.
- Finally, in Sect. 27.7, we prove the top-order estimate (24.4a) for Ω and S along the rough tori as well as the top-order estimate (24.5a) for $\mathcal{C}$ and $\mathcal{D}$ along the rough tori.

27.1. **Preliminary energy integral inequalities for the top-order derivatives of $\mathcal{C}$ and $\mathcal{D}$.**

Lemma 27.1 (Preliminary energy integral inequalities for the top-order derivatives of $\mathcal{C}$ and $\mathcal{D}$)**.** *For any $\varsigma \in (0,1]$, the following integral inequalities hold for $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$, where the pointwise norm $|\cdot|_{\mathbf{h}}$ is defined in* (21.15) *and the implicit constants are **independent** of ς:*

$$\begin{aligned}
\mathbb{C}_{N_{\text{top}}}(\tau,u) + \mathbb{D}_{N_{\text{top}}}(\tau,u) \lesssim \mathring{\epsilon}^2 + \varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} & \left\{ \left|\partial \mathcal{P}^{N_{\text{top}}}\Omega\right|_{\mathbf{h}}^2 + \left|\partial \mathcal{P}^{N_{\text{top}}}S\right|_{\mathbf{h}}^2 \right\} \mathrm{d}\varpi \\
& + \left(1 + \frac{1}{\varsigma}\right) \int_{u'=-U_1}^{u} \left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u') + \mathbb{D}_{N_{\text{top}}}(\tau,u') \right\} \mathrm{d}u' \\
& + \int_{u'=-U_1}^{u} \left\{ \mathbb{C}_{\le N_{\text{top}}-1}(\tau,u') + \mathbb{D}_{\le N_{\text{top}}-1}(\tau,u') \right\} \mathrm{d}u' \\
& + \int_{u'=-U_1}^{u} \left\{ \mathbb{V}_{\le N_{\text{top}}}(\tau,u') + \mathbb{S}_{\le N_{\text{top}}}(\tau,u') \right\} \mathrm{d}u' \\
& + \epsilon^2 \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N_{\text{top}}]}(\tau',u)\,\mathrm{d}\tau' + \epsilon^2 \int_{u'=-U_1}^{u} \mathbb{Q}_{[1,N_{\text{top}}]}(\tau,u')\,\mathrm{d}u' + \epsilon^2 \mathbb{K}_{[1,N_{\text{top}}]}(\tau,u).
\end{aligned} \tag{27.1}$$

Proof. The proof is almost identical to the proof of (26.2) with N_{top} in the role of N, except we separate the spacetime error integrals generated by the top-order derivatives of (Ω,S). More precisely, with $N \stackrel{\text{def}}{=} N_{\text{top}}$ in the pointwise estimates (23.4a)–(23.4b) for $|\mu \mathbf{B}(\mathcal{P}^{N_{\text{top}}}\mathcal{C}, \mathcal{P}^{N_{\text{top}}}\mathcal{D})|$, we isolate the contribution of the error terms $|\mathcal{P}^{N_{\text{top}}+1}(\Omega,S)|$ on the RHSs. Since (schematically) $\mathcal{P}^{N_{\text{top}}+1}(\Omega^i,S^i) = P^\alpha \partial_\alpha \mathcal{P}^{N_{\text{top}}}(\Omega^i,S^i)$ for some $P \in \{L, Y_{(2)}, Y_{(3)}\}$, we can use the simple Cartesian component bound $|P^\alpha| \lesssim 1$ (which follows from Prop. 9.1 and the bootstrap assumptions) and (21.16b) to pointwise bound these error terms in magnitude by $\lesssim \left|\partial \mathcal{P}^{N_{\text{top}}}\Omega\right|_{\mathbf{h}} + \left|\partial \mathcal{P}^{N_{\text{top}}}S\right|_{\mathbf{h}}$. Thus, in the energy identity (that is, (20.29) with $f \stackrel{\text{def}}{=} (\mathcal{P}^{N_{\text{top}}}\mathcal{C}, \mathcal{P}^{N_{\text{top}}}\mathcal{D})$), the spacetime error integral corresponding to these terms is bounded by:

$$\lesssim \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \left\{ \left|\mathcal{P}^{N_{\text{top}}}\mathcal{C}\right|_{\mathbf{h}} + \left|\mathcal{P}^{N_{\text{top}}}\mathcal{D}\right|_{\mathbf{h}} \right\} \left\{ \left|\partial \mathcal{P}^{N_{\text{top}}}\Omega\right|_{\mathbf{h}} + \left|\partial \mathcal{P}^{N_{\text{top}}}S\right|_{\mathbf{h}} \right\} \mathrm{d}\varpi. \tag{27.2}$$

Using the estimate (18.9b), (20.56a)–(20.56b), and Young's inequality, for any $\varsigma \in (0,1]$, we find that RHS (27.2) $\lesssim \frac{1}{\varsigma}\int_{u'=-U_1}^{u} \left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u') + \mathbb{D}_{N_{\text{top}}}(\tau,u') \right\} \mathrm{d}u' + \varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \left\{ \left|\partial \mathcal{P}^{N_{\text{top}}}\Omega\right|_{\mathbf{h}}^2 + \left|\partial \mathcal{P}^{N_{\text{top}}}S\right|_{\mathbf{h}}^2 \right\} \mathrm{d}\varpi$, which is bounded by RHS (27.1) as desired. ☐

27.2. **The main energy integral inequalities for the top-order derivatives of $\mathcal{C}$ and $\mathcal{D}$, conditional on Prop. 27.5.**

Most of our effort in Sect. 27 is dedicated towards bounding the spacetime integrals:

$$\varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}} \left\{ \left|\partial \mathcal{P}^{N_{\text{top}}}\Omega\right|_{\mathbf{h}}^2 + \left|\partial \mathcal{P}^{N_{\text{top}}}S\right|_{\mathbf{h}}^2 \right\} \mathrm{d}\varpi,$$

which appear on RHS (27.1). We derive the needed estimates in Prop. 27.5. Given Prop. 27.5 and Lemma 27.1, it is easy to derive energy integral inequalities that can be used to obtain the desired top-order energy estimates for $\mathcal{C}$ and $\mathcal{D}$; we derive these integral inequalities in the next lemma.

Lemma 27.2 (The main energy integral inequalities for the top-order derivatives of $\mathcal{C}$ and $\mathcal{D}$)**.** *Assuming the results of Prop. 27.5, for any $\varsigma \in (0,1]$, the following integral inequalities hold for $(\tau,u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$, where the implicit constants are **independent** of ς:*

$$\begin{aligned} \mathbb{C}_{N_{\text{top}}}(\tau,u) + \mathbb{D}_{N_{\text{top}}}(\tau,u) \lesssim \varsigma \left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u) + \mathbb{D}_{N_{\text{top}}}(\tau,u) \right\} &+ \frac{\mathring{\epsilon}^2}{|\tau|^{5/2}} \\ &+ \left(1 + \frac{1}{\varsigma}\right) \int_{u'=-U_1}^{u} \left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u') + \mathbb{D}_{N_{\text{top}}}(\tau,u') \right\} \mathrm{d}u' \\ &+ \varepsilon^2 \frac{1}{|\tau|^{3/2}} \mathbb{Q}_{[1,N_{\text{top}}]}(\tau,u) + \varepsilon^2 \mathbb{K}_{[1,N_{\text{top}}]}(\tau,u) \\ &+ \frac{1}{|\tau|^2} \left\{ \mathbb{C}_{\leq N_{\text{top}}-1}(\tau,u) + \mathbb{D}_{\leq N_{\text{top}}-1}(\tau,u) \right\} + \frac{1}{|\tau|^{5/2}} \left\{ \mathbb{V}_{\leq N_{\text{top}}}(\tau,u) + \mathbb{S}_{\leq N_{\text{top}}}(\tau,u) \right\}. \end{aligned} \tag{27.3}$$

Proof. We start by considering inequality (27.1). Using Lemma 21.9 and the estimate $\frac{1}{L^{(\mathfrak{n})}\tau} \approx 1$ implied by (18.9b), we deduce the pointwise bounds $\left|\partial \mathcal{P}^{N_{\text{top}}} \Omega\right|_{\mathbf{h}}^2 \lesssim \frac{1}{L^{(\mathfrak{n})}\tau} \mathscr{Q}[\partial \mathcal{P}^{N_{\text{top}}} \Omega, \partial \mathcal{P}^{N_{\text{top}}} \Omega]$ and $\left|\partial \mathcal{P}^{N_{\text{top}}} S\right|_{\mathbf{h}}^2 \lesssim \frac{1}{L^{(\mathfrak{n})}\tau} \mathscr{Q}[\partial \mathcal{P}^{N_{\text{top}}} S, \partial \mathcal{P}^{N_{\text{top}}} S]$ for the error integrands on the first line of RHS (27.1). Hence, thanks to the estimates (27.12a)–(27.12b), which we prove independently in Sect. 27.6, the spacetime integral $\varsigma \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}} \left\{ \left|\partial \mathcal{P}^{N_{\text{top}}} \Omega\right|_{\mathbf{h}}^2 + \left|\partial \mathcal{P}^{N_{\text{top}}} S\right|_{\mathbf{h}}^2 \right\} \mathrm{d}\varpi$ on the first line of RHS (27.1) is bounded by $\varsigma\,\{\text{RHS (27.12a)} + \text{RHS (27.12b)}\}$. The RHS of the resulting inequality features the error integrals $\left(1+\frac{1}{\varsigma}\right) \int_{u'=-U_1}^{u} \left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u') + \mathbb{D}_{N_{\text{top}}}(\tau,u') \right\} \mathrm{d}u'$, which we place directly on RHS (27.3). Finally, we further bound the remaining error integrals on RHS (27.1) by using the monotonicity of the controlling quantities (with respect to their arguments τ, u) to pull the controlling quantities out of the integrals and to gain a power of $|\tau|$ upon integration with respect to $\mathrm{d}\tau'$, e.g., $\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^3} \mathbb{V}_{\leq N_{\text{top}}}(\tau',u)\,\mathrm{d}\tau' \lesssim \frac{1}{|\tau|^2} \mathbb{V}_{\leq N_{\text{top}}}(\tau,u)$ and $\int_{u'=-U_1}^{u} \mathbb{D}_{N_{\text{top}}}(\tau,u')\,\mathrm{d}u' \lesssim \mathbb{D}_{N_{\text{top}}}(\tau,u)$. In total, these arguments yield (27.3).

□

27.3. **Proof of the main top-order energy estimate** (24.3a)**.** Given Lemma 27.2, we are now ready to prove our main top-order a priori energy estimates (24.3a) for the modified fluid variables. We again emphasize that Lemma 27.2 is conditional on the estimates of Prop. 27.5, which we prove independently below.

To proceed, we choose and fix $\varsigma > 0$ sufficiently small such that the term $\varsigma\left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u) + \mathbb{D}_{N_{\text{top}}}(\tau,u) \right\}$ on RHS (27.3) can be absorbed back into the LHS at the expense of increasing the implicit constants. Next, we use the already proven estimates (24.2a)–(24.2b) and (24.3b)–(24.3c), the bootstrap assumptions (24.12a)–(24.12b), and (10.9b) to bound all terms on RHS (27.3) except for the integral $\int_{u'=-U_1}^{u} \left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u') + \mathbb{D}_{N_{\text{top}}}(\tau,u') \right\} \mathrm{d}u'$. In total, this leads to the following inequality, where C depends on the fixed value of ς:

$$\mathbb{C}_{N_{\text{top}}}(\tau,u) + \mathbb{D}_{N_{\text{top}}}(\tau,u) \leq C \mathring{\epsilon}^2 |\tau|^{-17.1} + C \int_{u'=-U_1}^{u} \left\{ \mathbb{C}_{N_{\text{top}}}(\tau,u') + \mathbb{D}_{N_{\text{top}}}(\tau,u') \right\} \mathrm{d}u'. \tag{27.4}$$

From (27.4) and Grönwall's inequality, we conclude that $\mathbb{C}_{N_{\text{top}}}(\tau,u) + \mathbb{D}_{N_{\text{top}}}(\tau,u) \leq C\mathring{\epsilon}^2|\tau|^{-17.1}$, which yields the desired bound (24.3a).

□

27.4. **Control of $\left\| \mathcal{P}^{\leq N_{\text{top}}}(\Omega, S) \right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$ for $\tau \in [\tau_0, \tau_{\text{Boot}})$.** Recall that our proof of the top-order L^2 estimates (24.3a) and (24.4a) relies on Prop. 27.5, whose proof relies on the integral identity (21.63). In order to exploit the identity (21.63) with $u_1 = -U_1$, we in particular have to first control the rough tori error integrals $\left\| \mathcal{P}^{\leq N_{\text{top}}}(\Omega, S) \right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$ for $\tau \in [\tau_0, \tau_{\text{Boot}})$; in view of (18.2) and (21.48) with $\mathcal{P}^{\leq N_{\text{top}}}(\Omega, S)$ in the role of V, we see that control of $\left\| \mathcal{P}^{\leq N_{\text{top}}}(\Omega, S) \right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$ is sufficient to bound the first integral $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u_1}} \cdots$ on RHS (21.63). As we explained at the beginning of Sect. 27, these integrals are not pure "data terms" because their size depends on various norms of the rough time function, which in turn depends on the behavior of the fluid near the singular boundary. Hence, in the next lemma, we derive estimates for $\left\|\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$. The proof relies on the Cauchy stability-type estimates proved in Appendix B, which also rely on various applications of the integral identity (21.63). We emphasize that our bounds for the top-order terms $\left\|\mathcal{P}^{N_{\mathrm{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$ cannot be proved by combining the data-estimates (11.12b) along the null hypersurface $\mathcal{P}_{-U_1}$ with trace estimates because of the usual loss of differentiability incurred by trace estimates. However, if we had assumed that the data are one degree more differentiable, more precisely that (11.12b) holds with N_{top} replaced by $N_{\mathrm{top}}+1$, then we could have used trace estimates to give a simpler proof of the desired estimates for $\left\|\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$. Though simpler, we avoided that approach because it would have led to sub-optimal estimates, i.e., to estimates such that the solution is less differentiable than the data.

Lemma 27.3 (Control of $\left\|\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}$ for $\tau\in[\tau_0,\tau_{\mathrm{Boot}})$)**.** *Let $\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ be the norm of the perturbation of the data away from the background solution, as defined in* (11.4)*. If $\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ is sufficiently small, then the following estimates hold for $\tau\in[\tau_0,\tau_{\mathrm{Boot}}]$:*

$$\left\|\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega,S)\right\|_{L^2(\widetilde{\ell}_{\tau,-U_1})}\leq\mathring{\epsilon},\tag{27.5}$$

where $\mathring{\epsilon}=\mathcal{O}\left(\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)$, and the implicit constants depend on the background solution.

Proof. The bootstrap assumption (**BA** $t-$**SIZE**) implies that for $\tau\in[\tau_0,\tau_{\mathrm{Boot}})$, the rough tori $\widetilde{\ell}_{\tau,-U_1}$ are contained in the "data null hypersurface" $\mathcal{P}^{[0,4\mathring{\delta}_*]}_{-U_1}$. For this reason, the proof of (27.5) relies on the smallness results we derived for the solution on $\mathcal{P}^{[0,4\mathring{\delta}_*]}_{-U_1}$ in Appendix B. In particular, in our proof here, we will refer to various steps in the proof of Prop. B.2. We clarify that although the proof of Prop. B.2 relies on ideas from the bulk of the paper, its proof is independent of the results of Lemma 27.3. In the rest of the proof, we will silently assume that $\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ is sufficiently small. Moreover, the number $T^{\mathrm{PS}}_{\mathrm{Shock}}>0$ defined in (A.42b) is the Cartesian blowup-time of the background solution.

There are two broad steps in the proof of (27.5): **I)** extend the rough time function ${}^{(\mathfrak{n})}\tau$ into a subset of the "smallness region" $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ from Prop. B.2 ($\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ is the subset of geometric coordinate space depicted in Fig. 16) and derive standard $C^{2,1}_{\mathrm{geo}}$ estimates for ${}^{(\mathfrak{n})}\tau$ in the extended region; and **II)** combine applications of the integral identity (21.63) in the extended region with the estimates for ${}^{(\mathfrak{n})}\tau$ from step **I)**, and use some results from the proof of Prop. B.2 to conclude (27.5).

Step I: Extending and controlling ${}^{(\mathfrak{n})}\tau$. In Steps 1 and 2 of the proof of Prop. B.2, we show that up to the top-order derivative level (i.e., the derivative level corresponding to the up-to-top-order energy estimates stated in Props. 24.1, 24.2, 24.3, and 24.4), the fluid variables, $\mu-1$, $L^i_{(\mathrm{Small})}$, and χ are bounded in L^2 by $\lesssim\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ on the region $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ defined in (A.95b), which, by (A.42b) and (B.2), contains $\mathcal{P}^{[0,4\mathring{\delta}_*]}_{-U_1}$. That is, in $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$, the solution is close to the trivial fluid solution with Euclidean acoustic geometry. Unlike in Props. 24.1, 24.2, 24.3, and 24.4, in Steps 1 and 2 of the proof of Prop. B.2 we derive the smallness in $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ with respect to foliations by portions of *Cartesian* time slices Σ_t, null hypersurfaces $\mathcal{P}_u$, and *acoustic tori* $\ell_{t,u}$. We will now explain how we can combine this smallness with the transport equation (4.4) and the data of ${}^{(\mathfrak{n})}\tau$ on ${}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau_{\mathrm{Boot}}]}_{-U_1}$ (the behavior of ${}^{(\mathfrak{n})}\tau$ on ${}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau_{\mathrm{Boot}}]}_{-U_1}$ has already been controlled by the last item in Lemma 15.6) to extend ${}^{(\mathfrak{n})}\tau$ to a larger domain with the following two properties: **a)** ${}^{(\mathfrak{n})}\tau={}^{(\mathfrak{n})}\tau(t,u,x^2,x^3)$ is defined on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_*,U_2]}$, where $U_*>U_1>0$ is the number in (B.16), and **b)** ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_*,-U_1]}\subset\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ and thus ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_*,U_2]}\subset{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}\cup\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$. To carry out this extension, we first note that (A.42b), (**BA** $t-$**SIZE**), (B.2), and (A.95b) imply that the data null hypersurface ${}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau_{\mathrm{Boot}}]}_{-U_1}$ is contained in $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ and that the distance (with respect to the standard Euclidean metric on geometric coordinate space) between ${}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau_{\mathrm{Boot}}]}_{-U_1}$ and the top boundary of $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ (which we denote by "$\Sigma^{[15T^{\mathrm{PS}}_{\mathrm{Shock}}-U_0,-U_1]}_{5T^{\mathrm{PS}}_{\mathrm{Shock}}}$" in Fig. 16) is at least $\frac{1}{2}T^{\mathrm{PS}}_{\mathrm{Shock}}$. We next note that Def. 4.1 implies that in $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ (a region in which $u<-U_{☢}$ and thus in which the cut-off function ϕ vanishes), the transport equation (4.4) takes the form $\breve{X}^{(\mathfrak{n})}\tau = 0$, where we recall that $\frac{\partial}{\partial u} = \breve{X} - \breve{X}^A \frac{\partial}{\partial x^A}$. The smallness provided by Steps 1 and 2 of the proof of Prop. B.2 implies that $\sum_{A=2,3} \|\breve{X}^A\|_{C^{2,1}_{\mathrm{geo}}\left(\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}\right)} \lesssim \mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$. Using this smallness and deriving standard $C^{2,1}_{\mathrm{geo}}$-estimates for solutions to $\breve{X}^{(\mathfrak{n})}\tau = 0$ starting from the data of ${}^{(\mathfrak{n})}\tau$ on the data null hypersurface portion ${}^{(\mathfrak{n})}\mathcal{P}^{[\tau_0,\tau_{\mathrm{Boot}}]}_{-U_1}$ (which is contained in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}$), we find that:

$$\|{}^{(\mathfrak{n})}\tau\|_{C^{2,1}_{\mathrm{geo}}\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_*,-U_1]}\right)} \leq \|{}^{(\mathfrak{n})}\tau\|_{C^{2,1}_{\mathrm{geo}}\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}\right)} \left\{1 + \mathcal{O}\left(\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)\right\} \leq C. \tag{27.6}$$

Step II: Using energy estimates and applications of the integral identity (21.63) **to finish the proof**. In Step 5 of the proof of Prop. B.2, we derived – independently of Lemma 27.3 (see Remark B.3) – geometric energy estimates in the region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_*,\frac{1}{2}\tau_0],[-U_*,U_2]}$, which is contained in the subset $\mathfrak{CS}^{T^{\mathrm{PS}}_{\mathrm{Shock}};\Delta^{\mathrm{PS}}}$ of geometric coordinate space defined in (A.96a). Here, $\tau_* \in [2\tau_0, (3/2)\tau_0]$ is the number from (B.10), $U_* > U_1 > 0$ is the number from (B.16), and we note that $\mathfrak{CS}^{T^{\mathrm{PS}}_{\mathrm{Shock}};\Delta^{\mathrm{PS}}}$ contains the set $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$ from above. The estimates from Step 5 of the proof of Prop. B.2 in particular yield the rough tori L^2 estimates stated in (B.19). The proof of (B.19) relies in particular on applying the integral identity[61] (21.63) on all the sub-regions ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_*,\tau_2],[-U_*,u_2]}$ with $(\tau_2, u_2) \in [\tau_*, \frac{1}{2}\tau_0] \times [-U_*, U_2]$, i.e., with $\tau_1 = \tau_*$ and $u_1 = -U_*$ in (21.63). We highlight two key ingredients that are needed to control the error terms in those applications of (21.63): **i)** bounds for the rough tori integrals arising from RHS (21.63) (see Remark B.3), i.e., the integrals $\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,-U_*}} \cdots$, $\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_*,u_2}} \cdots$, $\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_*,-U_*}} \cdots$, which we bound in the arguments given above (B.19); and **ii)** the bound $\|{}^{(\mathfrak{n})}\tau\|_{C^{2,1}_{\mathrm{geo}}\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_*,\frac{1}{2}\tau_0],[-U_*,U_2]}\right)} \leq C$, which is needed to bound various error terms on RHS (21.63) that arise when we apply it on the region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_*,\tau_2],[-U_*,u_2]}$ (in particular, the error term $\mathfrak{L}_{(\mathrm{Lower\text{-}order})}[V,V]$ defined in (21.47) depends on the second derivatives of ${}^{(\mathfrak{n})}\tau$). The key point is that the estimate (27.6) shows that the same bound for ${}^{(\mathfrak{n})}\tau$ holds on the subset ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_*,-U_1]}$ of $\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}$. With the help of this bound, the same arguments that yield (B.19) can be used to show that:

$$\sup_{(\tau,u)\in[\tau_*,\tau_{\mathrm{Boot}}]\times[-U_*,-U_1]} \int_{\widetilde{\ell}_{\tau,u}} |\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega,S)|^2 \,\mathrm{d}\varpi_{\widetilde{g}} \lesssim \left(\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2. \tag{27.7}$$

Finally, we note that since $[\tau_0, \tau_{\mathrm{Boot}}] \subset [\tau_*, \tau_{\mathrm{Boot}}]$, (27.7) implies (27.5) with $\mathring{\epsilon} = \mathcal{O}\left(\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)$.

□

27.5. L^2 **estimates for the error terms.** In the next lemma, we derive L^2 estimates for the error integrals in the elliptic-hyperbolic identities provided by Prop. 21.14.

Lemma 27.4 (L^2 estimates for the error terms in the elliptic-hyperbolic identities)**.** *Let* $\varsigma \in [0,1)$, *let* $(\tau, u) \in [\tau_0, \tau_{\mathrm{Boot}}) \times [-U_1, U_2]$, *let* $\mathfrak{M}[\cdot,\cdot]$ *be the error term defined by* (21.64), *and let* $\mathscr{Q}[\cdot,\cdot]$ *be the coercive quadratic form defined in* (21.23).

[61] In particular, the integral identity (21.63) yields control over the top-order terms on LHS (B.19).

Then the following spacetime integral estimates hold, where the implicit constants are ***independent*** *of* ς*:*

$$\begin{aligned}\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \mathfrak{M}[\mathcal{P}^{N_{\mathrm{top}}}\Omega, \partial\mathcal{P}^{N_{\mathrm{top}}}\Omega]\,\mathrm{d}\varpi\right| &\lesssim \varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,U_2]}} \frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\partial\mathcal{P}^{N_{\mathrm{top}}}\Omega, \partial\mathcal{P}^{N_{\mathrm{top}}}\Omega]\,\mathrm{d}\varpi \\ &+ \left(1+\frac{1}{\varsigma}\right)\frac{\mathring{\epsilon}^2}{|\tau|} \\ &+ \int_{u'=-U_1}^{u} \mathbb{C}_{N_{\mathrm{top}}}(\tau,u')\,\mathrm{d}u' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^3}\mathbb{C}_{\leq N_{\mathrm{top}}-1}(\tau',u)\,\mathrm{d}\tau' \\ &+ \left(1+\frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^3}\left\{\mathbb{V}_{\leq N_{\mathrm{top}}}(\tau',u) + \mathbb{S}_{\leq N_{\mathrm{top}}}(\tau',u)\right\}\mathrm{d}\tau' \\ &+ \varepsilon^2\left(1+\frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^2}\mathbb{Q}_{[1,N_{\mathrm{top}}]}(\tau',u)\,\mathrm{d}\tau',\end{aligned} \tag{27.8a}$$

$$\begin{aligned}\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \mathfrak{M}[\mathcal{P}^{N_{\mathrm{top}}}S, \partial\mathcal{P}^{N_{\mathrm{top}}}S]\,\mathrm{d}\varpi\right| &\lesssim \varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,U_2]}} \frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\partial\mathcal{P}^{N_{\mathrm{top}}}S, \partial\mathcal{P}^{N_{\mathrm{top}}}S]\,\mathrm{d}\varpi \\ &+ \left(1+\frac{1}{\varsigma}\right)\frac{\mathring{\epsilon}^2}{|\tau|} \\ &+ \int_{u'=-U_1}^{u} \mathbb{D}_{N_{\mathrm{top}}}(\tau,u')\,\mathrm{d}u' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^3}\mathbb{D}_{\leq N_{\mathrm{top}}-1}(\tau',u)\,\mathrm{d}\tau' \\ &+ \left(1+\frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^3}\left\{\mathbb{V}_{\leq N_{\mathrm{top}}}(\tau',u) + \mathbb{S}_{\leq N_{\mathrm{top}}}(\tau',u)\right\}\mathrm{d}\tau' \\ &+ \varepsilon^2\left(1+\frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^2}\mathbb{Q}_{[1,N_{\mathrm{top}}]}(\tau',u)\,\mathrm{d}\tau'.\end{aligned} \tag{27.8b}$$

Moreover, the error terms $\mathfrak{E}_{(\mathrm{Principal})}[\mathcal{P}^{N_{\mathrm{top}}}\Omega, \partial\mathcal{P}^{N_{\mathrm{top}}}\Omega], \cdots, \mathfrak{E}_{(\mathrm{Lower\text{-}order})}[\mathcal{P}^{N_{\mathrm{top}}}S, \mathcal{P}^{N_{\mathrm{top}}}S]$ *defined by* (21.46)–(21.47) *verify the following rough hypersurface integral estimates:*

$$\begin{aligned}&\left|\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mathfrak{E}_{(\mathrm{Principal})}[\mathcal{P}^{N_{\mathrm{top}}}\Omega, \partial\mathcal{P}^{N_{\mathrm{top}}}\Omega]\,\mathrm{d}\underline{\varpi}\right| \\ &\lesssim \frac{\mathring{\epsilon}^2}{|\tau|^{3/2}} + \mathbb{C}_{N_{\mathrm{top}}}(\tau,u) + \frac{1}{|\tau|^{3/2}}\mathbb{C}_{\leq N_{\mathrm{top}}-1}(\tau,u) \\ &\quad + \varepsilon^2\frac{1}{|\tau|^{3/2}}\mathbb{Q}_{[1,N_{\mathrm{top}}]}(\tau,u) + \frac{1}{|\tau|^{5/2}}\left\{\mathbb{V}_{\leq N_{\mathrm{top}}}(\tau,u) + \mathbb{S}_{\leq N_{\mathrm{top}}}(\tau,u)\right\},\end{aligned} \tag{27.9a}$$

$$\begin{aligned}&\left|\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mathfrak{E}_{(\mathrm{Principal})}[\mathcal{P}^{N_{\mathrm{top}}}S, \partial\mathcal{P}^{N_{\mathrm{top}}}S]\,\mathrm{d}\underline{\varpi}\right| \\ &\lesssim \frac{\mathring{\epsilon}^2}{|\tau|^{3/2}} + \mathbb{D}_{N_{\mathrm{top}}}(\tau,u) + \frac{1}{|\tau|^{3/2}}\mathbb{D}_{\leq N_{\mathrm{top}}-1}(\tau,u) \\ &\quad + \varepsilon^2\frac{1}{|\tau|^{3/2}}\mathbb{Q}_{[1,N_{\mathrm{top}}]}(\tau,u) + \frac{1}{|\tau|^{5/2}}\left\{\mathbb{V}_{\leq N_{\mathrm{top}}}(\tau,u) + \mathbb{S}_{\leq N_{\mathrm{top}}}(\tau,u)\right\},\end{aligned} \tag{27.9b}$$

$$\left|\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mathfrak{E}_{(\mathrm{Lower\text{-}order})}[\mathcal{P}^{N_{\mathrm{top}}}\Omega, \mathcal{P}^{N_{\mathrm{top}}}\Omega]\,\mathrm{d}\underline{\varpi}\right| \lesssim \frac{1}{|\tau|^{5/2}}\mathbb{V}_{N_{\mathrm{top}}}(\tau,u), \tag{27.10a}$$

$$\left|\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mathfrak{E}_{(\mathrm{Lower\text{-}order})}[\mathcal{P}^{N_{\mathrm{top}}}S, \mathcal{P}^{N_{\mathrm{top}}}S]\,\mathrm{d}\underline{\varpi}\right| \lesssim \frac{1}{|\tau|^{5/2}}\mathbb{S}_{N_{\mathrm{top}}}(\tau,u). \tag{27.10b}$$

Finally, the error terms $\mathfrak{P}[\mathcal{P}^{N_{\text{top}}}\Omega, \mathcal{P}^{N_{\text{top}}}\Omega], \cdots, \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}S, \mathcal{P}^{N_{\text{top}}}S]$ *defined by* (21.45) *verify the following rough tori integral estimates:*

$$\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,-U_1}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}\Omega, \mathcal{P}^{N_{\text{top}}}\Omega]\,d\varpi_{\widetilde{g}}, \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,-U_1}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}S, \mathcal{P}^{N_{\text{top}}}S]\,d\varpi_{\widetilde{g}} \lesssim \mathring{\epsilon}^2, \tag{27.11a}$$

$$\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}\Omega, \mathcal{P}^{N_{\text{top}}}\Omega]\,d\varpi_{\widetilde{g}}, \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}S, \mathcal{P}^{N_{\text{top}}}S]\,d\varpi_{\widetilde{g}} \lesssim \frac{\mathring{\epsilon}^2}{|\tau_0|}. \tag{27.11b}$$

Proof. We first prove (27.11a). We provide the details only for the $\mathcal{P}^{N_{\text{top}}}\Omega$-dependent integral on LHS (27.11a) since the $\mathcal{P}^{N_{\text{top}}}S$-dependent integral can be treated using identical arguments. To proceed, we first use (18.2) to deduce that μ is bounded from below by $\gtrsim 1$ on $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,-U_1}$. Hence, using (21.48), (18.8a) (which implies that $-L\mu \approx 1$ on the support of ϕ), (27.5), and (21.17a), we conclude that $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,-U_1}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}\Omega, \mathcal{P}^{N_{\text{top}}}\Omega]\,d\varpi_{\widetilde{g}} \lesssim \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,-U_1}} |\mathcal{P}^{N_{\text{top}}}\Omega|_g^2\,d\varpi_{\widetilde{g}} \lesssim \mathring{\epsilon}^2$ as desired. Similarly, to prove (27.11b) for the $\mathcal{P}^{N_{\text{top}}}\Omega$-dependent integral on the LHS, we use (18.1) with $\tau \overset{\text{def}}{=} \tau_0$, (21.48), (11.13b), and (21.17a) to conclude that $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}\Omega, \mathcal{P}^{N_{\text{top}}}\Omega]\,d\varpi_{\widetilde{g}} \lesssim \frac{1}{|\tau_0|}\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}} |\mathcal{P}^{N_{\text{top}}}\Omega|_g^2\,d\varpi_{\widetilde{g}} \lesssim \frac{1}{|\tau|_0}\mathring{\epsilon}^2$ as desired. The $\mathcal{P}^{N_{\text{top}}}S$-dependent integral on LHS (27.11b) can be bounded using identical arguments.

The remaining estimates (27.8a)–(27.10b) are straightforward consequences of the pointwise estimates provided by Prop. 23.4, the pointwise estimates $|\mu| \lesssim 1$ and $\left|\phi\frac{\mathfrak{n}}{L\mu}\right| \lesssim 1$ (which follow from the bootstrap assumptions), Def. 20.10 of the L^2-controlling quantities, the coerciveness guaranteed by Lemma 20.14 together with the estimates (18.1) and (18.9b), the already proven L^2 estimates (25.1b), and the fact that the L^2-controlling quantities $\mathbb{Q}_M(\tau, u)$, $\mathbb{C}_M(\tau, u)$, etc. are increasing in their arguments. □

27.6. **The main elliptic-hyperbolic integral inequalities.** Thanks to the availability of Lemma 27.4, we are now ready to prove Prop. 27.5.

Proposition 27.5 (The main elliptic-hyperbolic integral inequalities)**.** *Let* $\mathscr{Q}[\boldsymbol{\partial}V, \boldsymbol{\partial}V]$ *be the quadratic form from Def. 21.8, and let* $\mathfrak{P}[V, V]$ *be the quadratic form defined by* (21.45)*. Then the following spacetime integral estimates hold for* $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$*:*

$$\begin{aligned}
&\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega, \boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}\Omega]\,d\boldsymbol{\varpi} + \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}\Omega, \mathcal{P}^{N_{\text{top}}}\Omega]\,d\varpi_{\widetilde{g}} \\
&\lesssim \left(1 + \frac{1}{\varsigma}\right)\frac{\mathring{\epsilon}^2}{|\tau|^{5/2}} + \mathbb{C}_{N_{\text{top}}}(\tau, u) + \frac{1}{|\tau|^2}\mathbb{C}_{\leq N_{\text{top}}-1}(\tau, u) \\
&\quad + \varepsilon^2\frac{1}{|\tau|^{3/2}}\mathbb{Q}_{[1,N_{\text{top}}]}(\tau, u) + \frac{1}{|\tau|^{5/2}}\left\{\mathbb{V}_{\leq N_{\text{top}}}(\tau, u) + \mathbb{S}_{\leq N_{\text{top}}}(\tau, u)\right\} \\
&\quad + \int_{u'=-U_1}^{u} \mathbb{C}_{N_{\text{top}}}(\tau, u')\,du' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^3}\mathbb{C}_{\leq N_{\text{top}}-1}(\tau', u)\,d\tau' \\
&\quad + \left(1 + \frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^3}\left\{\mathbb{V}_{\leq N_{\text{top}}}(\tau', u) + \mathbb{S}_{\leq N_{\text{top}}}(\tau', u)\right\}d\tau' + \varepsilon^2\left(1 + \frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^2}\mathbb{Q}_{[1,N_{\text{top}}]}(\tau', u)\,d\tau',
\end{aligned} \tag{27.12a}$$

$$\begin{aligned}
&\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S, \boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}S]\,d\boldsymbol{\varpi} + \int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \mathfrak{P}[\mathcal{P}^{N_{\text{top}}}S, \mathcal{P}^{N_{\text{top}}}S]\,d\varpi_{\widetilde{g}} \\
&\lesssim \left(1 + \frac{1}{\varsigma}\right)\frac{\mathring{\epsilon}^2}{|\tau|^{5/2}} + \mathbb{D}_{N_{\text{top}}}(\tau, u) + \frac{1}{|\tau|^{3/2}}\mathbb{D}_{\leq N_{\text{top}}-1}(\tau, u) \\
&\quad + \varepsilon^2\frac{1}{|\tau|^{3/2}}\mathbb{Q}_{[1,N_{\text{top}}]}(\tau, u) + \frac{1}{|\tau|^{5/2}}\left\{\mathbb{V}_{\leq N_{\text{top}}}(\tau, u) + \mathbb{S}_{\leq N_{\text{top}}}(\tau, u)\right\} \\
&\quad + \int_{u'=-U_1}^{u} \mathbb{D}_{N_{\text{top}}}(\tau, u')\,du' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^3}\mathbb{D}_{\leq N_{\text{top}}-1}(\tau', u)\,d\tau' \\
&\quad + \left(1 + \frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^3}\left\{\mathbb{V}_{\leq N_{\text{top}}}(\tau', u) + \mathbb{S}_{\leq N_{\text{top}}}(\tau', u)\right\}d\tau' + \varepsilon^2\left(1 + \frac{1}{\varsigma}\right)\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^2}\mathbb{Q}_{[1,N_{\text{top}}]}(\tau', u)\,d\tau'.
\end{aligned} \tag{27.12b}$$

Proof. We consider the integral identity (21.63) with $\mathcal{P}^{N_{\mathrm{top}}}\Omega$ and $\mathcal{P}^{N_{\mathrm{top}}}S$ in the role of V. Using Lemma 27.4 with $\tau_1 \overset{\mathrm{def}}{=} \tau_0$, $\tau_2 \overset{\mathrm{def}}{=} \tau$, $u_1 \overset{\mathrm{def}}{=} -U_1$, and $u_2 \overset{\mathrm{def}}{=} u$, as well as the data-estimates (24.8a)–(24.8b), we bound the integrals on RHS (21.63) (the data-estimates are used to control the data-hypersurface integrals $\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,u]}} \cdots$), where we can discard the integrals $-\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,-U_1}} \mathfrak{P}[V,V]\,\mathrm{d}\varpi_{\widetilde{g\!\!/}}$ because they are non-positive in view of (21.48). Finally, by choosing and fixing $\varsigma > 0$ to be sufficiently small, we can absorb the first term $\varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,U_2]}} \frac{1}{L^{(\mathfrak{n})}\tau} \mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\mathrm{top}}}\Omega, \boldsymbol{\partial}\mathcal{P}^{N_{\mathrm{top}}}\Omega]\,\mathrm{d}\boldsymbol{\varpi}$ on RHS (27.8a) and the first term $\varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,U_2]}} \frac{1}{L^{(\mathfrak{n})}\tau} \mathscr{Q}[\boldsymbol{\partial}\mathcal{P}^{N_{\mathrm{top}}}S, \boldsymbol{\partial}\mathcal{P}^{N_{\mathrm{top}}}S]\,\mathrm{d}\boldsymbol{\varpi}$ on RHS (27.8b) into LHS (27.12a) and LHS (27.12b) respectively. This yields the desired estimates (27.12a)–(27.12b).

□

27.7. **Proof of the top-order rough tori energy estimates** (24.4a) **and** (24.5a)**.** We first prove the estimate (24.4a). We consider the estimates (27.12a)–(27.12b). In view of definitions (20.46a)–(20.46b), the quantitative positive definiteness estimate (21.48), the estimate (18.8a), the fact that the cut-off function ϕ on RHS (21.48) is supported in the u-interval $[-U_{☢}, U_{☢}]$ (see Def. 4.1), and the estimate $|\mu| \lesssim 1$ (which follows from the bootstrap assumptions), we see that up to $\mathcal{O}(1)$ factors, the rough tori integrals on LHSs (27.12a)–(27.12b) bound the terms on LHS (24.4a) from above. Moreover, using the already proven estimates (24.2a)–(24.3c), the wave energy bootstrap assumptions (24.12a)–(24.12b), and (10.9b), we see that all terms on RHSs (27.12a)–(27.12b) are bounded by $\lesssim \mathring{\epsilon}^2 |\tau|^{-17.1}$, which yields the desired result.

The estimate (24.5a) follows from combining the same arguments we used to prove (26.4) with the data-estimates (24.11a)–(24.11b), the already proven estimate (24.3a), (20.56a)–(20.56b), and definitions (20.48a)–(20.48b).

□

28. Elliptic estimates for the acoustic geometry on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$

We continue to work under the assumptions of Sect. 13.2. In this section, we derive elliptic L^2 estimates for symmetric $\binom{0}{2}$-type tensorfields that are tangent to the acoustic tori $\ell_{t,u}$. As we will explain, our analysis fundamentally relies on the *rough* tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, which, unlike the acoustic tori, are adapted to our foliations by level-sets of ${}^{(\mathfrak{n})}\tau$. We provide the main estimate in Prop. 28.1. In Sect. 29.3, we combine the elliptic estimates of Prop. 28.1 with hyperbolic L^2 estimates for the fully modified quantities defined in Sect. 19 to obtain top-order L^2 estimates for the null second fundamental form χ, which is tangent to $\ell_{t,u}$. We fundamentally need these top-order estimates for χ to avoid the loss of a derivative in the top-order commuted wave equations. Specifically, in the top-order case $N = N_{\mathrm{top}}$, we need these L^2 estimates to handle the terms on RHSs (22.3a)–(22.3b) that explicitly depend on the order N derivatives $\mathrm{tr}_{g\!\!/}\chi$. The point is that when $N = N_{\mathrm{top}}$, $\mathcal{Y}^N \mathrm{tr}_{g\!\!/}\chi$ cannot be controlled in L^2 through pure transport estimates; using only transport estimates at the top-order would result in the loss of one derivative due to the presence of the source term $|\chi|^2_{g\!\!/}$ on the RHS of the transport equation (19.9) satisfied by $\mathrm{tr}_{g\!\!/}\chi$, which depends not only on $\mathrm{tr}_{g\!\!/}\chi$, but also on its trace-free part $\hat{\chi}$. The strategy of avoiding derivative loss in χ via a combination of elliptic L^2 estimates on co-dimension 2 surfaces and hyperbolic L^2 estimates was originally employed in the context of Einstein's equations in [26]. Later, this strategy was used in many other works on wave and wave-like equations, for example, in the context of low regularity local well-posedness for quasilinear wave equations in [45], in the context of irrotational shock formation in [24], and in the context of shock formation in $3D$ with vorticity and entropy in [52]. What is new here compared to these works is our reliance on the rough tori to obtain the needed top-order estimates, even though the operators $\mathcal{Y}^N$ and χ are adapted to the *acoustic tori* $\ell_{t,u} = \Sigma_t \cap \mathcal{P}_u$.

To obtain the desired elliptic estimates, we decompose symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfields ξ into a main piece that is tangent to ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, which we control with elliptic estimates on the rough tori (see Lemma 28.10), and error terms, which we must control with separate (easier) arguments. Our primary application will be to apply the main elliptic estimate (28.1) with $\mathcal{L\!\!/}_{\mathcal{P}}^{N_{\mathrm{top}}-1}\chi$ in the role of ξ, which will yield L^2-control of $\mathcal{L\!\!/}_{\mathcal{P}}^{N_{\mathrm{top}}}\chi$; see the proof of (29.15b). There are many equivalent ways we could have carried out the decompositions and analysis of this section. We have chosen to use orthonormal frames on the acoustic tori and the rough tori and to quantitatively control the relationship between the two frames; see Sect. 28.2.

28.1. **Statement of the main elliptic estimates.** In this section, we state the proposition that yields the main elliptic estimates of interest. Its proof is located in Sect. 28.8.

Proposition 28.1 (The main elliptic estimates for symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfields)**.** *Let* ξ *be a symmetric type* $\binom{0}{2}$ $\ell_{t,u}$*-tangent tensorfield. Then the following estimate holds for* $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$*:*

$$\begin{aligned} \sum_{P \in \{L, Y_{(2)}, Y_{(3)}\}} \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\not{\mathcal{L}}_P \xi|_{\not{g}}^2 \, d\underline{\varpi} \leq C \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\not{\mathcal{L}}_L \xi|_{\not{g}}^2 \, d\underline{\varpi} + C \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\mathrm{d}\!\!/\mathrm{iv}\, \xi|_{\not{g}}^2 \, d\underline{\varpi} \\ + C \sum_{A=2,3} \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 (Y_{(A)} \mathrm{tr}_{\not{g}} \xi)^2 \, d\underline{\varpi} + C\varepsilon \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} |\xi|_{\not{g}}^2 \, d\underline{\varpi}. \end{aligned} \tag{28.1}$$

28.2. **Orthonormal frames on the acoustic tori and the rough tori.** For use throughout Sect. 28, we recall that $\not{g}$ denotes the first fundamental form of the acoustic tori $\ell_{t,u}$ and $\widetilde{\not{g}}$ denotes the first fundamental form of the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. In our ensuing analysis, we will use the pairs of orthonormal frames featured in the next definition.

Definition 28.2 (The frames $\{e_A\}_{A=2,3}$ and $\{f_A\}_{A=2,3}$)**.** $\{e_A\}_{A=2,3}$ is defined to be the orthonormal frame on the acoustic torus $\ell_{t,u}$ obtained from applying the Gram–Schmidt process to the geometric coordinate partial derivative vectorfields $\left\{\frac{\partial}{\partial x^A}\right\}_{A=2,3}$ with respect to $\not{g}$, starting with $e_2 \overset{\text{def}}{=} \frac{1}{\sqrt{\not{g}(\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^2})}} \frac{\partial}{\partial x^2}$. Similarly, $\{f_A\}_{A=2,3}$ is defined to be the orthonormal frame on the rough torus ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ obtained from applying the Gram–Schmidt process to the adapted rough coordinate partial derivative vectorfields $\left\{\frac{\widetilde{\partial}}{\partial x^A}\right\}_{A=2,3}$ with respect to $\widetilde{\not{g}}$, starting with $f_2 \overset{\text{def}}{=} \frac{1}{\sqrt{\widetilde{\not{g}}(\frac{\widetilde{\partial}}{\partial x^2},\frac{\widetilde{\partial}}{\partial x^2})}} \frac{\widetilde{\partial}}{\partial x^2}$.

In the next lemma, we provide standard expressions for $\not{g}^{-1}$ and $\widetilde{\not{g}}^{-1}$ relative to the orthonormal frames.

Lemma 28.3 (Expressions for $\not{g}^{-1}$ and $\widetilde{\not{g}}^{-1}$ relative to orthonormal frames)**.** *Let* $\not{g}^{-1}$ *be the inverse first fundamental form of* $\ell_{t,u}$ *from Def. 3.4, let* $\widetilde{\not{g}}^{-1}$ *be the inverse first fundamental form of* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ *from Def. 6.2, and let* $\{e_A\}_{A=2,3}$ *and* $\{f_A\}_{A=2,3}$ *be the orthonormal frames from Def. 28.2. Then the following identities hold, where* δ^{AB} *is the Kronecker delta:*

$$\not{g}^{-1} = \delta^{AB} e_A \otimes e_B, \tag{28.2a}$$

$$\widetilde{\not{g}}^{-1} = \delta^{AB} f_A \otimes f_B. \tag{28.2b}$$

Proof. (28.2a)–(28.2b) are standard identities for inverse metrics relative to orthonormal frames. ☐

In the next lemma, we exhibit the relationships between the two frames $\{e_A\}_{A=2,3}$ and $\{f_A\}_{A=2,3}$.

Lemma 28.4 (Relationship between $\{e_A\}_{A=2,3}$ and $\{f_A\}_{A=2,3}$)**.** *On* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$*, there exists a* 2×2 *orthogonal-matrix-valued function* $\mathscr{O}$ *with components* $\{\mathscr{O}_{AB}\}_{A,B=2,3}$ *and scalar functions* $\{\lambda_A\}_{A=2,3}$ *such that:*

$$e_A = \mathscr{O}_{AB} f_B + \lambda_A L = \mathscr{O}_{AB}\{f_B + \mathscr{O}_{CB}\lambda_C L\}, \tag{28.3a}$$

$$f_A = (\mathscr{O}^{-1})_{AB}\{e_B - \lambda_B L\} = \mathscr{O}_{BA}\{e_B - \lambda_B L\}. \tag{28.3b}$$

Moreover, $\widetilde{\not{g}}^{-1}$ *and* $\not{g}^{-1}$ *are related through the following identity:*

$$\widetilde{\not{g}}^{-1} = \not{g}^{-1} - \lambda_A e_A \otimes L - \lambda_A L \otimes e_A + \lambda_A \lambda_A L \otimes L. \tag{28.4}$$

Finally, the following estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$*:*

$$|\mathscr{O}_{AB}| \leq 1, \tag{28.5a}$$

$$|\lambda_A| \leq C\varepsilon. \tag{28.5b}$$

Proof. The existence of a matrix $\mathscr{O}$ and scalar functions $\{\lambda_A\}_{A=2,3}$ satisfying the first equality in (28.3a) follows from the fact that the frames $\{e_2,e_3,L\}$ and $\{f_2,f_3,L\}$ both span the tangent space of $\mathcal{P}_u$ at any of its points. To see that $\mathscr{O}$ is orthogonal, we use the fact that L is $\mathbf{g}$-orthogonal to the tangent space of $\mathcal{P}_u$, the fact that $\{e_A\}_{A=2,3}$ and $\{f_A\}_{A=2,3}$ are both $\mathbf{g}$-orthonormal, and the first equality in (28.3a) to deduce, with δ_{AB} denoting the Kronecker delta, that:

$$\delta_{AB} = \not{g}(e_A,e_B) = \mathbf{g}(e_A,e_B) = \mathscr{O}_{AC}\mathscr{O}_{BD}\mathbf{g}(f_C,f_D) = \mathscr{O}_{AC}\mathscr{O}_{BD}\widetilde{\not{g}}(f_C,f_D) = \mathscr{O}_{AC}\mathscr{O}_{BD}\delta_{CD} = \mathscr{O}_{AC}\mathscr{O}_{BC}. \tag{28.6}$$

From (28.6), we see that $(\mathscr{O}^{-1})_{AB} = \mathscr{O}_{BA}$ (i.e., $\mathscr{O}$ is an orthogonal matrix) and thus (28.3b) follows, as does the second equality in (28.3a).

(28.4) follows from substituting RHS (28.3b) for the frame vectorfields $\{f_A\}_{A=2,3}$ in (28.2b), using the orthogonality of the matrix $\mathscr{O}_{AB}$, and taking into account the identity (28.2a).

The estimate (28.5a) follows trivially since any orthogonal matrix has Euclidean-orthonormal rows and thus its entries are ≤ 1 in magnitude.

To derive the estimate (28.5b), we first use (28.3a) and Lemma 3.9 to deduce that $0 = \mathbf{g}(e_A, X) = \mathscr{O}_{AB}\mathbf{g}(f_B, X) - \lambda_A$. From this identity and (28.5a), we see that $|\lambda_A| \lesssim \sum_{B=2,3} |\mathbf{g}(f_B, X)|$. Hence, (28.5b) will follow once we show that:

$$\sum_{B=2,3} |\mathbf{g}(f_B, X)| \lesssim \varepsilon. \tag{28.7}$$

To prove (28.7), we first use (3.31a), Prop. 9.1, the bootstrap assumptions, and Cor. 17.2 to deduce that $g\!\!/_{AB} = c^{-2}\delta_{AB} + \mathcal{O}(\varepsilon) = \{1 + \mathcal{O}(\mathring{\alpha})\}\delta_{AB} + \mathcal{O}(\varepsilon)$, where δ_{AB} is the Kronecker delta. Also using (6.11) and the estimates of Lemma 15.5 for ${}^{(\mathfrak{n})}\tau$, we find that $\widetilde{g\!\!/}\left(\frac{\widetilde{\partial}}{\partial x^A}, \frac{\widetilde{\partial}}{\partial x^B}\right) = \{1 + \mathcal{O}(\mathring{\alpha})\}\delta_{AB} + \mathcal{O}(\varepsilon)$. Next, since $\left\{\frac{\widetilde{\partial}}{\partial x^2}, \frac{\widetilde{\partial}}{\partial x^3}\right\}$ spans the tangent space of the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, for $A = 2, 3$, there exist scalar functions α_A and β_A such that $f_A = \alpha_A \frac{\widetilde{\partial}}{\partial x^2} + \beta_A \frac{\widetilde{\partial}}{\partial x^3}$. Since $1 = \widetilde{g\!\!/}(f_A, f_A)$ (with no summation over A) by assumption, it follows from the estimate $\widetilde{g\!\!/}\left(\frac{\widetilde{\partial}}{\partial x^A}, \frac{\widetilde{\partial}}{\partial x^B}\right) = \{1 + \mathcal{O}(\mathring{\alpha})\}\delta_{AB} + \mathcal{O}(\varepsilon)$ that $|\alpha_A|, |\beta_A| \lesssim 1$, i.e., $f_A = \mathcal{O}(1)\frac{\widetilde{\partial}}{\partial x^2} + \mathcal{O}(1)\frac{\widetilde{\partial}}{\partial x^3}$. From this relation, (5.13c), Prop. 9.1, the bootstrap assumptions, Lemma 15.5, and Cor. 17.2, we deduce that $f_A = \mathcal{O}(1)\frac{\partial}{\partial x^2} + \mathcal{O}(1)\frac{\partial}{\partial x^3} + \mathcal{O}\left(\frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n})}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}\right)L = \mathcal{O}(1)\frac{\partial}{\partial x^2} + \mathcal{O}(1)\frac{\partial}{\partial x^3} + \mathcal{O}(\varepsilon)L$. From this relation and Lemma 3.9, we conclude (28.7).

□

28.3. **An alternate representation of g-orthogonal projection onto the rough tori.**

Definition 28.5 (**g**-orthogonal projection onto the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$). Let ξ be a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield. We define $\widetilde{\xi\!\!/}$ to be the symmetric type $\binom{0}{2}$ ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfield whose type $\binom{2}{0}$ **g**-dual, which we denote by $\widetilde{\xi\!\!/}^{\#\#}$, has the following expansion relative to the orthonormal frame $\{f_2, f_3\}$ on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ from Def. 28.2:

$$\widetilde{\xi\!\!/}^{\#\#} \stackrel{\text{def}}{=} \xi(f_A, f_B) f_A \otimes f_B. \tag{28.8}$$

Remark 28.6 ((28.8) is **g**-orthogonal projection onto ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$). With the help of (28.2b), one can check that the tensorfield $\widetilde{\xi\!\!/}$ defined by (28.8) is the **g**-orthogonal projection of ξ onto ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, i.e., $\widetilde{\xi\!\!/} = \widetilde{\Pi\!\!\!/}\xi$, where $\widetilde{\Pi\!\!\!/}\xi$ is defined by (6.27).

28.4. **Identities involving symmetric type $\binom{0}{2}$ tensorfields.** In the remainder of Sect. 28, we will work with $\mathcal{P}_u$-tangent tensorfields, as defined in Def. 21.1. It is straightforward to check that a tensorfield η is $\mathcal{P}_u$-tangent if and only if any contraction of it with L (which is **g**-orthogonal to $\mathcal{P}_u$) vanishes.

Lemma 28.7 (Identities involving symmetric type $\binom{0}{2}$ tensorfields). *Let ξ be a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, and let $\widetilde{\xi\!\!/}$ be the corresponding ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfield from Def. 28.5. Then the following identity holds:*

$$\widetilde{\xi}^{\#\#} = \xi^{\#\#} - \lambda_A \xi(e_A, e_B) L \otimes e_B - \lambda_B \xi(e_A, e_B) e_A \otimes L + \lambda_A \lambda_B \xi(e_A, e_B) L \otimes L. \tag{28.9}$$

Moreover, if η is any symmetric type $\binom{0}{2}$ $\mathcal{P}_u$-tangent tensorfield, then the following identities hold, where $\mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}$ and $\mathcal{L\!\!/}_L$ is $\ell_{t,u}$-projected Lie derivative operator from Def. 3.12:

$$[\mathbf{D}\eta](L, L) = 0, \tag{28.10a}$$

$$[\mathbf{D}_L \eta](L, \cdot) = 0, \tag{28.10b}$$

$$[\mathbf{D}_{e_A} \eta](L, e_B) = -\eta(e_C, e_B)\chi(e_A, e_C), \tag{28.10c}$$

$$[\mathbf{D}_L \eta](e_A, e_B) = [\mathcal{L\!\!/}_L \eta](e_A, e_B) - \eta(e_C, e_B)\chi(e_A, e_C) - \eta(e_A, e_C)\chi(e_B, e_C). \tag{28.11}$$

Proof. The identity (28.9) follows from using (28.3b) to substitute for the frame vectorfields $\{f_A\}_{A=2,3}$ in (28.8) and using the orthonormality of the matrix $\mathscr{O}$ and the fact that $\xi(L, \cdot) = 0$.

To prove (28.10a), we differentiate the identity $\eta(L, L) = 0$ with $\mathbf{D}$, use the Leibniz rule, and use that $\eta(L, \cdot) = 0$.

To prove (28.10b), we differentiate the identity $\eta(L,\cdot)=0$ with $\mathbf{D}_L$, use the Leibniz rule and the identity (3.19), and use that $\eta(L,\cdot)=0$.

To prove (28.10c), we differentiate the identity $\eta(L,\cdot)=0$ with $\mathbf{D}_{e_A}$, use the Leibniz rule and the identity (3.48), and use that $\eta(L,\cdot)=0$.

To prove (28.11), we contract the Lie differentiation identity $\mathcal{L}_L\eta_{\alpha\beta}=\mathbf{D}_L\eta_{\alpha\beta}+\eta_{\kappa\beta}\mathbf{D}_\alpha L^\kappa+\eta_{\alpha\kappa}\mathbf{D}_\beta L^\kappa$ against $e_A^\alpha e_B^\beta$, use that $\eta(L,\cdot)=0$, and use the identity $\mathbf{D}_{e_A}L=\chi(e_A,e_C)e_C-\zeta_A L$, which follows from Lemma 3.9, (3.43), and (3.48). □

28.5. **Identities connecting $g\!\!\!/$, $\widetilde{g\!\!\!/}$, and their corresponding differential operators.**

Lemma 28.8 (Identities involving contractions with $g\!\!\!/$ and $\widetilde{g\!\!\!/}$)**.** *Let η be a $\mathcal{P}_u$-tangent tensorfield. Recall that $|\cdot|_{g\!\!\!/}$ and $|\cdot|_{\widetilde{g\!\!\!/}}$ are defined in* (3.38b) *and* (6.24) *respectively. Then the following identity holds:*

$$|\eta|_{g\!\!\!/}=|\eta|_{\widetilde{g\!\!\!/}}. \tag{28.12}$$

Moreover, if ξ and $\widetilde{\xi}$ are as in Def. 28.5, then:

$$|\xi|_{g\!\!\!/}=|\widetilde{\xi}|_{\widetilde{g\!\!\!/}}. \tag{28.13}$$

In addition, if η is a symmetric type $\binom{0}{2}$ $\mathcal{P}_u$-tangent tensorfield, then with $\mathrm{tr}_{g\!\!\!/}$ and $\mathrm{tr}_{\widetilde{g\!\!\!/}}$ as defined in (3.37b) *and* (6.25) *respectively, the following identity holds:*

$$\mathrm{tr}_{g\!\!\!/}\eta=\mathrm{tr}_{\widetilde{g\!\!\!/}}\eta. \tag{28.14}$$

Finally, if ξ and $\widetilde{\xi}$ are as in Def. 28.5, then:

$$\mathrm{tr}_{g\!\!\!/}\xi=\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\xi}. \tag{28.15}$$

Proof. The lemma follows easily from the fact that any $\mathcal{P}_u$-tangent tensorfield (including $g\!\!\!/$ and $\widetilde{g\!\!\!/}$) vanishes upon any contraction with L, and the identities (28.4) and (28.9). □

28.6. **Differential operator pointwise comparison estimates used in the proof of the elliptic estimates.** In our proof of Prop. 28.1, we will use the following differential operator pointwise comparison estimates.

Lemma 28.9 (Differential operator pointwise comparison estimates needed for the elliptic estimates)**.** *Let φ be a scalar function, let $d\!\!\!/\varphi$ be the $\ell_{t,u}$-tangent one-form from Def. 3.10, and let $\widetilde{d\!\!\!/}\varphi$ be the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent one-form from Def. 6.12. Let ξ be a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, and let $\widetilde{\xi}$ be the corresponding symmetric type $\binom{0}{2}$ ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfield from Def. 28.5. Then the following pointwise estimates hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_1,U_2]}$:*

$$|\widetilde{d\!\!\!/}\varphi|_{\widetilde{g\!\!\!/}}=\{1+\mathcal{O}(\varepsilon)\}|d\!\!\!/\varphi|_{g\!\!\!/}+\mathcal{O}(\varepsilon)|L\varphi|, \tag{28.16a}$$

$$|\widetilde{\mathrm{d}\!\!\!/\mathrm{iv}}\,\widetilde{\xi}|_{\widetilde{g\!\!\!/}}=\{1+\mathcal{O}(\varepsilon)\}|\mathrm{d}\!\!\!/\mathrm{iv}\,\xi|_{g\!\!\!/}+\mathcal{O}(\varepsilon)|\mathcal{L}\!\!\!/_L\xi|_{g\!\!\!/}+\mathcal{O}(\varepsilon)|\xi|_{g\!\!\!/}, \tag{28.16b}$$

$$|\widetilde{\nabla\!\!\!/}\widetilde{\xi}|_{\widetilde{g\!\!\!/}}=\{1+\mathcal{O}(\varepsilon)\}|\nabla\!\!\!/\xi|_{g\!\!\!/}+\mathcal{O}(\varepsilon)|\mathcal{L}\!\!\!/_L\xi|_{g\!\!\!/}+\mathcal{O}(\varepsilon)|\xi|_{g\!\!\!/}. \tag{28.16c}$$

Proof. We prove only (28.16b) since (28.16a) and (28.16c) can be proved by similar arguments.

We start by noting the following identities, where $\mathscr{O}_{AB}$ is the orthogonal matrix from Lemma 28.4:

$$|\mathrm{d}\!\!\!/\mathrm{iv}\,\xi|_{g\!\!\!/}^2=[\mathbf{D}_{e_A}\xi](e_A,e_C)[\mathbf{D}_{e_B}\xi](e_B,e_C), \tag{28.17}$$

$$|\widetilde{\mathrm{d}\!\!\!/\mathrm{iv}}\,\widetilde{\xi}|_{\widetilde{g\!\!\!/}}^2=[\mathbf{D}_{f_A}\widetilde{\xi}](f_A,f_C)[\mathbf{D}_{f_B}\widetilde{\xi}](f_B,f_C)=\Big\{\mathscr{O}_{CD}[\mathbf{D}_{f_A}\widetilde{\xi}](f_A,f_D)\Big\}\Big\{\mathscr{O}_{CE}[\mathbf{D}_{f_B}\widetilde{\xi}](f_B,f_E)\Big\}. \tag{28.18}$$

The identity (28.17) follows from (28.2a) and the fact that $\nabla\!\!\!/\xi=\Pi\!\!\!/\mathbf{D}\xi$, where $\Pi\!\!\!/$ is the $\mathbf{g}$-orthogonal projection onto $\ell_{t,u}$ from Def. 3.3. Similarly, the first equality in (28.18) follows from (28.2b) and the fact that $\widetilde{\nabla\!\!\!/}\widetilde{\xi}=\widetilde{\Pi\!\!\!/}\mathbf{D}\widetilde{\xi}$, where $\widetilde{\Pi\!\!\!/}$ is the $\mathbf{g}$-orthogonal projection onto ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ from Def. 6.11. The second equality in (28.18) follows from the orthogonality of $\mathscr{O}$.

We now note the following identity, which we derive just below:

$$\begin{aligned}\mathscr{O}_{CD}[\mathbf{D}_{f_A}\widetilde{\xi}](f_A,f_D)&=[\mathbf{D}_{e_A}\xi](e_A,e_C)-\lambda_A[\mathcal{L}\!\!\!/_L\xi](e_A,e_C)\\&\quad-\lambda_D\xi(e_D,e_C)\mathrm{tr}_{g\!\!\!/}\chi+\lambda_C\xi(e_A,e_D)\chi(e_A,e_D)+2\lambda_A\xi(e_C,e_D)\chi(e_A,e_D).\end{aligned} \tag{28.19}$$

From (28.19), the identities (28.2a)–(28.2b) and (28.17)–(28.18), the orthogonality of the matrix $\mathscr{O}$, the Cauchy–Schwarz inequality with respect to $\mathscr{g}$, Lemma 28.8, the estimate (28.5b), and the pointwise estimate $|\chi|_{\mathscr{g}} \lesssim \varepsilon$ noted below (25.3), we conclude (28.16b).

It remains for us to prove (28.19). To proceed, we first use (3.19), (28.3a)–(28.3b), (28.9), (28.10a)–(28.10c), (28.11), the orthogonality of the matrix $\mathscr{O}_{AB}$, the fact that L is null and $\mathbf{g}$-orthogonal to $\mathcal{P}_u$, the fact that $\widetilde{\xi}$ and ξ are symmetric and satisfy $\widetilde{\xi}(L,\cdot) = \xi(L,\cdot) = 0$, and the fact that $\mathcal{L}_L L = 0$ to compute that:

$$\begin{aligned}
[\mathbf{D}_{e_A}\widetilde{\xi}](e_A,e_C) &= \mathscr{O}_{CD}[\mathbf{D}_{f_A}\widetilde{\xi}](f_A,f_D) \\
&\quad + \lambda_C[\mathbf{D}_{e_A}\widetilde{\xi}](e_A,L) + \lambda_A[\mathbf{D}_{e_A}\widetilde{\xi}](L,e_C) - \lambda_C\lambda_A[\mathbf{D}_{e_A}\widetilde{\xi}](L,L) \\
&\quad + \lambda_A[\mathbf{D}_L\widetilde{\xi}](e_A,e_C) - \lambda_A\lambda_A[\mathbf{D}_L\widetilde{\xi}](L,e_C) - \lambda_C\lambda_A[\mathbf{D}_L\widetilde{\xi}](e_A,L) + \lambda_C\lambda_A\lambda_A[\mathbf{D}_L\widetilde{\xi}](L,L) \\
&= \mathscr{O}_{CD}[\mathbf{D}_{f_A}\widetilde{\xi}](f_A,f_D) + \lambda_A[\mathcal{L}_L\xi](e_A,e_C) \\
&\quad - \lambda_C\xi(e_A,e_D)\chi(e_A,e_D) - 2\lambda_A\xi(e_C,e_D)\chi(e_A,e_D) - \lambda_A\xi(e_A,e_D)\chi(e_C,e_D).
\end{aligned} \tag{28.20}$$

Next, using (28.9), the fact that L is $\mathbf{g}$-orthogonal to $\mathcal{P}_u$, the fact that $\widetilde{\xi}(L,\cdot) = 0$, the identity $\mathbf{g}(\mathbf{D}_{e_A}L, e_B) = \chi(e_A,e_B)$ (which follows from (3.48)), and the identity $\chi(e_A,e_A) = \mathrm{tr}_{\mathscr{g}}\chi$, we deduce:

$$[\mathbf{D}_{e_A}\widetilde{\xi}](e_A,e_C) = [\mathbf{D}_{e_A}\xi](e_A,e_C) - \lambda_D\xi(e_D,e_C)\mathrm{tr}_{\mathscr{g}}\chi - \lambda_D\xi(e_A,e_D)\chi(e_A,e_C). \tag{28.21}$$

Using (28.21) to substitute for LHS (28.20) and rearranging terms, we conclude (28.19).

□

28.7. **Standard elliptic estimates for symmetric type $\binom{0}{2}$ tensorfields on the rough tori.** In the next lemma, we provide standard elliptic estimates for symmetric type $\binom{0}{2}$ ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfields. Its proof is located in Sect. 28.7.2.

Lemma 28.10 (Standard elliptic estimates for symmetric type $\binom{0}{2}$ tensorfields on the rough tori)**.** *Let Ξ be a symmetric type $\binom{0}{2}$ ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfield. Then the following estimate holds for $(\tau,u) \in [\tau_0,\tau_{\mathrm{Boot}}) \times [-U_1,U_2]$:*

$$\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \mu^2|\widetilde{\nabla}\Xi|^2_{\widetilde{\mathscr{g}}}\,d\varpi_{\widetilde{\mathscr{g}}} \leq 6\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \mu^2|\widetilde{\mathrm{div}}\,\Xi|^2_{\widetilde{\mathscr{g}}}\,d\varpi_{\widetilde{\mathscr{g}}} + 3\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} \mu^2|\widetilde{d}\,\mathrm{tr}_{\widetilde{\mathscr{g}}}\Xi|^2_{\widetilde{\mathscr{g}}}\,d\varpi_{\widetilde{\mathscr{g}}} + C\varepsilon\int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} |\Xi|^2_{\widetilde{\mathscr{g}}}\,d\varpi_{\widetilde{\mathscr{g}}}. \tag{28.22}$$

28.7.1. *The Gauss curvature of $\widetilde{\mathscr{g}}$.* Our proof of Lemma 28.10 relies on the next lemma, which provides an L^∞ estimate for the Gauss curvature $\widetilde{\mathfrak{K}}$ of $({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u},\widetilde{\mathscr{g}})$.

Lemma 28.11 (L^∞ estimate for the Gauss curvature of $\widetilde{\mathscr{g}}$)**.** *Recall that $\widetilde{\mathfrak{K}}$ denotes the Gauss curvature of $({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u},\widetilde{\mathscr{g}})$ (see Sect. 6.7). Then the following estimate holds for $(\tau,u) \in [\tau_0,\tau_{\mathrm{Boot}}] \times [-U_1,U_2]$:*

$$\left\|\widetilde{\mathfrak{K}}\right\|_{L^\infty({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})} \leq C\varepsilon. \tag{28.23}$$

Proof. Recall that $\widetilde{\mathfrak{K}}$ is equal to half the scalar curvature of $\widetilde{\mathscr{g}}$ (see (6.36)). Hence, at fixed (τ,u), relative to the coordinates (x^2,x^3) on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, using the standard expression for curvature in terms of the coordinate components of $\widetilde{\mathscr{g}}$ and their partial derivatives, we can schematically express $\widetilde{\mathfrak{K}}$ as follows, where $\widetilde{\mathscr{g}}^{-1}$ schematically denotes the component functions $\widetilde{\mathscr{g}}^{-1}(dx^A,dx^B)$ and $\widetilde{\mathscr{g}}$ schematically denotes the component functions $\widetilde{\mathscr{g}}\left(\frac{\widetilde{\partial}}{\partial x^A},\frac{\widetilde{\partial}}{\partial x^B}\right)$:

$$\widetilde{\mathfrak{K}} = \widetilde{\mathscr{g}}^{-1}\cdot\widetilde{\mathscr{g}}^{-1}\cdot\frac{\widetilde{\partial}}{\widetilde{\partial}x^A}\frac{\widetilde{\partial}}{\widetilde{\partial}x^B}\widetilde{\mathscr{g}} + \widetilde{\mathscr{g}}^{-1}\cdot\widetilde{\mathscr{g}}^{-1}\cdot\widetilde{\mathscr{g}}^{-1}\cdot\frac{\widetilde{\partial}}{\widetilde{\partial}x^A}\widetilde{\mathscr{g}}\cdot\frac{\widetilde{\partial}}{\widetilde{\partial}x^B}\widetilde{\mathscr{g}}. \tag{28.24}$$

From (28.24) and arguments similar to the ones we used to prove (18.35), we further deduce that there is a smooth function f such that schematically, we have:

$$\widetilde{\mathfrak{K}} = \mathrm{f}\left(\mathcal{P}^{\leq 2}\gamma, \frac{1}{L^{(\mathfrak{n})}\tau}, \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}, \mathcal{P}^{[1,3]}\tau\right)\cdot\left(\mathcal{P}^{[1,2]}\gamma, \mathcal{P}^{\leq 2}\mathcal{Y}^{(\mathfrak{n})}\tau\right). \tag{28.25}$$

From (28.25), Prop. 9.1, the results of Lemma 15.5, including the estimates (15.20), (15.22), and (15.24), Lemma 15.6, the bootstrap assumptions, Cor. 17.2, (18.9b), and Rademacher's theorem, we arrive at the desired estimate (28.23).

□

28.7.2. *Proof of Lemma 28.10.* We now prove Lemma 28.10. In this proof only, we will use capital Latin indices to denote the components of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfields with respect to the frame $\left\{\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\right\}_{A=2,3}$ and co-frame $\{\widetilde{d\!\!/} x^A\}_{A=2,3}$ on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, and we raise and lower indices with $\widetilde{g\!\!/}^{-1}$ and $\widetilde{g\!\!/}$. In particular, $\Xi = \Xi\left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial} x^B}\right)\widetilde{d\!\!/} x^A \otimes \widetilde{d\!\!/} x^B$. We start by defining $\widetilde{I} = \widetilde{I}^A \frac{\widetilde{\partial}}{\widetilde{\partial} x^A}$ to be the ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent vectorfield with the following components relative to the coordinates (x^2, x^3) on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$:

$$\widetilde{I}^A \overset{\text{def}}{=} \mu^2 \Xi_{\widetilde{BC}} \widetilde{\nabla\!\!\!/}^B \Xi^{AC} - \mu^2 \Xi^{AB} (\widetilde{\mathrm{d\!\!/iv}}\,\Xi)_B. \tag{28.26}$$

Next, with $\widetilde{\mathfrak{K}}$ denoting the Gauss curvature of the (two-dimensional) rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, we note the following standard identity, which follows from the symmetry of Ξ (see [69, Lemma 18.9] for the main ideas of the proof, where we note that only trace-free tensorfields were handled in [69, Lemma 18.9] and thus RHSs (28.22) and (28.27) feature additional $\mathrm{tr}_{\widetilde{g\!\!/}}\Xi$-dependent terms compared to [69, Lemma 18.9]):

$$\begin{aligned} \mu^2 |\widetilde{\nabla\!\!\!/}\Xi|^2_{\widetilde{g\!\!/}} + 2\mu^2 \widetilde{\mathfrak{K}} |\Xi|^2_{\widetilde{g\!\!/}} = {} & 2\mu^2 |\widetilde{\mathrm{d\!\!/iv}}\,\Xi|^2_{\widetilde{g\!\!/}} + \mu^2 \widetilde{\mathfrak{K}} (\mathrm{tr}_{\widetilde{g\!\!/}}\Xi)^2 + \mu^2 |\widetilde{d\!\!/} \mathrm{tr}_{\widetilde{g\!\!/}}\Xi|^2_{\widetilde{g\!\!/}} \\ & - 2\mu^2 \widetilde{g\!\!/}^{-1} (\widetilde{\mathrm{d\!\!/iv}}\,\Xi, \widetilde{d\!\!/} \mathrm{tr}_{\widetilde{g\!\!/}}\Xi) + 2\mu \Xi^{AB} \left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\mu\right) (\widetilde{\mathrm{d\!\!/iv}}\,\Xi)_{\widetilde{B}} \\ & - 2\mu \Xi_{BC} \left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\mu\right) \widetilde{\nabla\!\!\!/}^B \Xi^{AC} + \widetilde{\mathrm{d\!\!/iv}}\,\widetilde{I}. \end{aligned} \tag{28.27}$$

We then integrate (28.27) over ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ with respect to the area form $\mathrm{d}\varpi_{\widetilde{g\!\!/}}$ defined in (8.8) and note that the integral of the perfect divergence term $\widetilde{\mathrm{d\!\!/iv}}\,\widetilde{I}$ vanishes. Next, we use the $\widetilde{g\!\!/}$-Cauchy–Schwarz inequality and Young's inequality to pointwise bound the three cross-terms on RHS (28.27) as follows:

$$2\left|\mu^2 \widetilde{g\!\!/}^{-1} (\widetilde{\mathrm{d\!\!/iv}}\,\Xi, \widetilde{d\!\!/} \mathrm{tr}_{\widetilde{g\!\!/}}\Xi)\right| \leq \mu^2 \left|\widetilde{\mathrm{d\!\!/iv}}\,\Xi\right|^2_{\widetilde{g\!\!/}} + \mu^2 \left|\widetilde{d\!\!/} \mathrm{tr}_{\widetilde{g\!\!/}}\Xi\right|^2_{\widetilde{g\!\!/}}, \tag{28.28}$$

$$2\left|\mu \Xi^{AB} \left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\mu\right) (\widetilde{\mathrm{d\!\!/iv}}\,\Xi)_{\widetilde{B}}\right| \leq \mu^2 \left|\widetilde{\mathrm{d\!\!/iv}}\,\Xi\right|^2_{\widetilde{g\!\!/}} + \left|\widetilde{d\!\!/}\mu\right|^2_{\widetilde{g\!\!/}} |\Xi|^2_{\widetilde{g\!\!/}}, \tag{28.29}$$

$$2\left|\mu \Xi_{BC} \left(\frac{\widetilde{\partial}}{\widetilde{\partial} x^A}\mu\right) \widetilde{\nabla\!\!\!/}^B \Xi^{AC}\right| \leq \frac{1}{3}\mu^2 |\widetilde{\nabla\!\!\!/}\Xi|^2_{\widetilde{g\!\!/}} + 3\left|\widetilde{d\!\!/}\mu\right|^2_{\widetilde{g\!\!/}} |\Xi|^2_{\widetilde{g\!\!/}}. \tag{28.30}$$

Just below, we will show that:

$$\left|\widetilde{d\!\!/}\mu\right|_{\widetilde{g\!\!/}} \lesssim |\mathrm{d}\mu|_{g\!\!/} + \varepsilon |L\mu| \lesssim |\mathcal{Y}\mu| + \varepsilon |{}^{(\mathfrak{n})}\widetilde{L}\mu| \lesssim \varepsilon. \tag{28.31}$$

Using (28.31) to control the relevant factors on RHSs (28.29)–(28.30), using the Gauss curvature estimate (28.23) to control the factors of $\widetilde{\mathfrak{K}}$ in (28.27), using the elementary inequality $|\mathrm{tr}_{\widetilde{g\!\!/}}\Xi| \lesssim |\Xi|_{\widetilde{g\!\!/}}$, and using the estimate $|\mu| \lesssim 1$ (which follows from the bootstrap assumptions), we conclude (28.22).

To prove (28.31), we use (28.16a), the bootstrap assumptions, and Cor. 17.2 to deduce that $\left|\widetilde{d\!\!/}\mu\right|_{\widetilde{g\!\!/}} \lesssim |\mathrm{d}\mu|_{g\!\!/} + \varepsilon|L\mu| \lesssim |\mathcal{Y}\mu| + \varepsilon|{}^{(\mathfrak{n})}\widetilde{L}\mu| \lesssim \varepsilon$ as desired.

□

28.8. **Proof of Prop. 28.1.** We now prove Prop. 28.1. Let ξ be a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, and let $\widetilde{\xi}$ be the corresponding symmetric type $\binom{0}{2}$ ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfield from Def. 28.5. Since the term $\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\mathcal{L}\!\!\!/_L \xi|^2_{g\!\!/} \, \mathrm{d}\underline{\varpi}$ on LHS (28.1) is manifestly bounded by RHS (28.1), we only have to show that for $A = 2, 3$, the term $\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\mathcal{L}\!\!\!/_{Y_{(A)}} \xi|^2_{g\!\!/} \, \mathrm{d}\underline{\varpi}$ on LHS (28.1) is $\leq$ RHS (28.1). To proceed, we consider the inequality (28.22) with $\widetilde{\xi}$ in the role of Ξ. Integrating the inequality with respect to u' and using Lemmas 28.8 and 28.9, we deduce, in view of definition (8.9), that:

$$\begin{aligned}\int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\nabla\!\!\!/ \xi|_{g\!\!\!/}^2 \, d\underline{\varpi} \lesssim \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\mathrm{d}\!\!\!/\mathrm{iv}\, \xi|_{g\!\!\!/}^2 \, d\underline{\varpi} + \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/} \xi|_{g\!\!\!/}^2 \, d\underline{\varpi} \\ + \varepsilon \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |L \mathrm{tr}_{g\!\!\!/} \xi|^2 \, d\underline{\varpi} + \varepsilon \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\mathcal{L}\!\!\!/_L \xi|_{g\!\!\!/}^2 \, d\underline{\varpi} + \varepsilon \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} |\xi|_{g\!\!\!/}^2 \, d\underline{\varpi}.\end{aligned} \tag{28.32}$$

Next, using the Leibniz rule, (13.4a), and the bootstrap assumptions, we find that $|L\mathrm{tr}_{g\!\!\!/}\xi| \lesssim |\mathcal{L}\!\!\!/_L \xi|_{g\!\!\!/} + |\mathcal{L}\!\!\!/_L g\!\!\!/^{-1}|_{g\!\!\!/} |\xi|_{g\!\!\!/} \lesssim |\mathcal{L}\!\!\!/_L \xi|_{g\!\!\!/} + |\xi|_{g\!\!\!/}$ and thus the third integral $\varepsilon \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |L\mathrm{tr}_{g\!\!\!/}\xi|^2 \, d\underline{\varpi}$ on RHS (28.32) is bounded by the last two integrals on RHS (28.32). Next, we use the torsion-free property of the connection $\nabla\!\!\!/$ and the $g\!\!\!/$-Cauchy–Schwarz inequality to deduce the pointwise estimate $|\mathcal{L}\!\!\!/_{Y_{(A)}} \xi|_{g\!\!\!/} \leq |\nabla\!\!\!/_{Y_{(A)}} \xi|_{g\!\!\!/} + 2|\xi|_{g\!\!\!/} |\nabla\!\!\!/ Y_{(A)}|_{g\!\!\!/}$. Also using (13.1), (13.2), the estimate for $|\mathbf{g}(\mathbf{D}_{Y_{(A)}} Y_{(B)}, Y_{(C)})|$ given in the proof of (13.11b), and Cor. 17.2, we find that $|\mathcal{L}\!\!\!/_{Y_{(A)}} \xi|_{g\!\!\!/} \lesssim |\nabla\!\!\!/ \xi|_{g\!\!\!/} + \varepsilon |\xi|_{g\!\!\!/}$. From this bound, (28.32), the estimates proved above, and the pointwise bound $|\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/}\xi|_{g\!\!\!/} \lesssim \sum_{A=2,3} |Y_{(A)} \mathrm{tr}_{g\!\!\!/}\xi|_{g\!\!\!/}$, which follows from (13.2), we conclude that $\int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2 |\mathcal{L}\!\!\!/_{Y_{(A)}} \xi|_{g\!\!\!/}^2 \, d\underline{\varpi} \lesssim$ RHS (28.1). We have therefore proved (28.1), which completes the proof of Prop. 28.1.

□

29. Proof of the L^2 estimates for the wave-variables and the acoustic geometry

We continue to work under the assumptions of Sect. 13.2. In this section, we prove Props. 24.1 and 24.4, which provide the main a priori energy estimates for the wave-variables and the acoustic geometry along the rough foliations. We accomplish this via a bootstrap argument that relies on the energy estimates of Prop. 24.2 for the transport-variables, which we already proved in Sects. 26 and 27.

We recall that the fundamental L^2-controlling quantities, such as $\mathbb{W}_{[1,N]}$ and $\mathbb{W}_{[1,N]}^{(\mathrm{Partial})}$, are defined in Sect. 20.5 (see in particular Def. 20.12).

29.1. **Statement of the integral inequalities used in proving a priori L^2 estimates for the wave-variables.**

In this section, we state Prop. 29.1, which provides a coupled system of integral inequalities for the wave energies $\mathbb{W}_N$ and the partial wave energies $\mathbb{W}_N^{(\mathrm{Partial})}$. As we will see in Sect. 29.7.1, these integral inequalities are the main ingredients in our proof of the L^2 a priori estimates of Prop. 24.1. Most of our effort in Sect. 29 is dedicated towards proving preliminary estimates that we will use in proving Prop. 29.1.

We now state the proposition. Its proof is located in Sect. 29.7.

Proposition 29.1 (The system of integral inequalities satisfied by the $\mathbb{W}_N$). *Let $\varsigma \in (0,1]$, and let $^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}$ denote the set from* (18.12). *For $\vec{\Psi} = (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4) = (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$, let $\vec{\mathfrak{G}} \overset{def}{=} (\mathfrak{G}_0, \cdots, \mathfrak{G}_4)$ be the vector array of the inhomogeneous terms in the covariant wave equations $\mu \square_{\mathbf{g}} \Psi_\iota = \mathfrak{G}_\iota$, i.e., $\mu\vec{\mathfrak{G}}$ is equal to $\mu \times$ RHS* (2.22). *Similarly, we define $\vec{\mathfrak{G}}_{(\mathrm{Partial})} \overset{def}{=} (\mathfrak{G}_1, \cdots, \mathfrak{G}_4)$ to be μ-weighted inhomogeneous terms in the covariant wave equations satisfied by $\vec{\Psi}_{(\mathrm{Partial})} = (\mathcal{R}_{(-)}, v^2, v^3, s)$. Then there exist constants $C > 0$ and $C_* > 0$ (see Remark 22.7) that are* ***independent*** *of ς such that the following estimates hold for $(\tau, u) \in [\tau_0, \tau_{\mathrm{Boot}}) \times [-U_1, U_2]$:*

Top-order integral inequalities for $\vec{\Psi}$. *In the case* $N = N_{\text{top}}$, *we have the following estimates for the* L^2*-controlling quantity* $\mathbb{W}_N$ *defined in* (20.43c)*:*

$$
\begin{aligned}
\mathbb{W}_N(\tau,u) \leq\ & \boxed{\left\{\frac{4\times 1.01}{1.99} + 4.13\right\}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_N(\tau',u)\,d\tau' \\
& + \boxed{\frac{8\times(1.01)^2}{1.99}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_N^{1/2}(\tau',u) \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|} \mathbb{Q}_N^{1/2}(\tau'',u)\,d\tau''\,d\tau' \\
& + \boxed{4.13} \frac{1}{|\tau|^{1/2}} \mathbb{Q}_N^{1/2}(\tau,u) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_N^{1/2}(\tau',u)\,d\tau' \\
& + C_* \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_N^{1/2}(\tau',u) \left(\mathbb{Q}_N^{(\text{Partial})}\right)^{1/2}(\tau',u)\,d\tau' \\
& + C_* \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_N^{1/2}(\tau',u) \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|} \left(\mathbb{Q}_N^{(\text{Partial})}\right)^{1/2}(\tau'',u)\,d\tau''\,d\tau' \\
& + C_* \frac{1}{|\tau|^{1/2}} \mathbb{Q}_N^{1/2}(\tau,u) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left(\mathbb{Q}_N^{(\text{Partial})}\right)^{1/2}(\tau',u)\,d\tau' \\
& + \text{Error}_N^{(\text{Top})}(\tau,u),
\end{aligned}
\tag{29.1}
$$

where $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ *satisfies the following estimate, in which the implicit constants are* ***independent*** *of* $\varsigma \in (0,1]$:

$$
\begin{aligned}
|\mathrm{Error}_N^{(\mathrm{Top})}|(\tau,u) \lesssim \left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2 \frac{1}{|\tau|^{3/2}} & \\
& + \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_N^{1/2}(\tau',u)\left\{\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right\}(\tau',u)\,\mathrm{d}\tau' \\
& + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\tau_0}^{\tau'}\left[\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2 \mathrm{d}\tau' \\
& + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\left[\mathbb{C}_{\leq N-1}^{1/2}+\mathbb{D}_{\leq N-1}^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2 \mathrm{d}\tau' \\
& + \int_{u'=-U_1}^{u}\{\mathbb{C}_{\leq N-1}+\mathbb{D}_{\leq N-1}\}(\tau,u')\,\mathrm{d}u' \\
& + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\left[\mathbb{V}_{\leq N}^{1/2}+\mathbb{S}_{\leq N}^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2 \mathrm{d}\tau' \\
& + \int_{u'=-U_1}^{u}\{\mathbb{V}_{\leq N}+\mathbb{S}_{\leq N}\}(\tau,u')\,\mathrm{d}u' \\
& + \varepsilon\mathbb{Q}_N(\tau,u)+\varsigma\mathbb{Q}_N(\tau,u)+\varsigma\mathbb{W}_N(\tau,u) \\
& + \varepsilon\frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + \frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + \mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + \frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + \varepsilon\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau' \\
& + \left(1+\varsigma^{-1}\right)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{2/3}}\mathbb{Q}_{[1,N]}(\tau',u)\,\mathrm{d}\tau' \\
& + \left(1+\varsigma^{-1}\right)\int_{u'=-U_1}^{u}\mathbb{Q}_N(\tau,u')\,\mathrm{d}u' \\
& + \varepsilon\int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_N^{1/2}(\tau',u)\frac{1}{|\tau'|}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\mathbb{Q}_{[1,N]}^{1/2}(\tau'',u)\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
& + \int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_N^{1/2}(\tau',u)\frac{1}{|\tau'|}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau'',u)\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
& + \int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_N^{1/2}(\tau',u)\frac{1}{|\tau'|}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\int_{\tau'''=\tau_0}^{\tau''}\frac{1}{|\tau'''|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau''',u)\,\mathrm{d}\tau'''\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
& + \int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{5/2}}\mathbb{Q}_{[1,N-1]}(\tau',u)\,\mathrm{d}\tau'.
\end{aligned}
\tag{29.2}
$$

Top-order integral inequalities for $\vec{\Psi}_{(\text{Partial})}$. *In the case $N = N_{\text{top}}$, we have the following estimates for the L^2-controlling quantity $\mathbb{W}_N^{(\text{Partial})}$ defined in* (20.44c)*:*

$$\mathbb{W}_N^{(\text{Partial})}(\tau, u) \leq \text{Error}_N^{(\text{Top})}(\tau, u), \tag{29.3}$$

where $\text{Error}_N^{(\text{Top})}(\tau, u)$ *satisfies* (29.2).

Below-top-order integral inequalities for $\vec{\Psi}$. *Finally, if $2 \leq N \leq N_{\text{top}}$, then we have the following estimates for the L^2-controlling quantity $\mathbb{W}_{[1,N-1]}$ defined by* (20.43c) *and Def. 20.12:*

$$\begin{aligned} \mathbb{W}_{[1,N-1]}(\tau, u) \leq C \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N-1]}^{1/2}(\tau', u) \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \mathbb{Q}_N^{1/2}(\tau'', u)\, \mathrm{d}\tau''\, \mathrm{d}\tau' \\ + \text{Error}_{N-1}^{(\text{Sub-critical})}(\tau, u), \end{aligned} \tag{29.4}$$

where for any integer $M \geq 1$, $\text{Error}_M^{(\text{Sub-critical})}(\tau, u)$ *is defined to be any term that satisfies the following estimate, in which the implicit constants are **independent** of $\varsigma \in (0, 1]$:*

$$\begin{aligned} \left|\text{Error}_M^{(\text{Sub-critical})}\right|(\tau, u) \lesssim \mathring{\epsilon}^2 + \left(1 + \varsigma^{-1}\right) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,M]}(\tau', u)\, \mathrm{d}\tau' \\ + \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_M^{1/2}(\tau', u) \left\{\mathbb{C}_M^{1/2} + \mathbb{D}_M^{1/2}\right\}(\tau', u)\, \mathrm{d}\tau' \\ + \int_{u'=-U_1}^{u} \{\mathbb{C}_{\leq M-1} + \mathbb{D}_{\leq M-1}\}(\tau, u')\, \mathrm{d}u' \\ + \int_{u'=-U_1}^{u} \{\mathbb{V}_{\leq M} + \mathbb{S}_{\leq M}\}(\tau, u')\, \mathrm{d}u' \\ + \left(1 + \varsigma^{-1}\right) \int_{u'=-U_1}^{u} \mathbb{Q}_{[1,M]}(\tau, u')\, \mathrm{d}u' \\ + \varsigma \mathbb{K}_{[1,M]}(\tau, u). \end{aligned} \tag{29.5}$$

Remark 29.2 (Non-optimality of some estimates involving $\text{Error}_N^{(\text{Top})}(\tau, u)$)**.** The reader might notice that some of the error terms we bound in the forthcoming Sects. 29.2–(29.5) are *strictly* less singular with respect to powers of $|\tau|^{-1}$ compared to the terms featured on RHS (29.2), i.e., our estimates are not always optimal. For example, RHS (29.11) features the error integral $\left(1 + \varsigma^{-1}\right) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_N(\tau', u)\, \mathrm{d}\tau'$, whereas RHS (29.2) (which defines a term of type $\text{Error}_N^{(\text{Top})}(\tau, u)$) features $\left(1 + \varsigma^{-1}\right) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{2/3}} \mathbb{Q}_N(\tau', u)\, \mathrm{d}\tau'$. We chose to sometimes use non-optimal estimates because the proof of the a priori L^2 estimates in Sect. 29.7.1 is based on a Grönwall argument involving *all* of the error integrals found in Sects. 29.2–(29.5), and the terms on RHS (29.2) represent the worst terms we encounter in the entire top-order wave energy estimate analysis.

Remark 29.3 (Redundancies in $\text{Error}_N^{(\text{Top})}(\tau, u)$)**.** Despite Remark 29.2, we acknowledge that there are some redundancies in the terms on RHS (29.2). For example, since $\mathbb{Q}_M(\tau, u)$ is increasing in its arguments, we have the estimate $\varepsilon \frac{1}{|\tau|^{1/2}} \mathbb{Q}_N^{1/2}(\tau, u) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau', u)\, \mathrm{d}\tau' \lesssim \varepsilon \mathbb{Q}_N(\tau', u)$, and yet we have included the terms on the LHS *and* the RHS of this estimate on RHS (29.2). We chose to keep these redundancies for the sake of the reader because we anticipate it might make some of the forthcoming proofs in Sects. 29.2–29.5 easier to read.

29.2. **Estimates for the easiest error integrals.**

29.2.1. *Estimates for the error integrals generated by the error terms* $\text{Harmless}_{(\text{Wave})}^{[1,N]}$. In the following lemma, we derive bounds for all the wave equation error integrals that involve the $\text{Harmless}_{(\text{Wave})}^{[1,N]}$ terms defined in (22.1).

Lemma 29.4 (Bounds for error integrals involving $\text{Harmless}_{(\text{Wave})}^{[1,N]}$ terms)**.** *Let $1 \leq N \leq N_{\text{top}}$, let $\Psi \in \vec{\Psi} = \{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$, and let $\varsigma \in (0, 1]$. Let $\mathcal{P}^N \in \mathfrak{P}^{(N)}$, where $\mathfrak{P}^{(N)}$ is the set of order N $\mathcal{P}_u$-tangential commutator operators from*

Def. 8.10. Recall that terms of type $\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N]}$ *are defined in Def. 22.1. Then the following estimates hold for* $(\tau,u) \in [\tau_0,\tau_{\mathrm{Boot}}) \times [-U_1,U_2]$, *where the implicit constants are* ***independent*** *of* ς:

$$\begin{aligned}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left| \begin{pmatrix} (1+2\mu)L\mathcal{P}^N\Psi \\ 2\breve{X}\mathcal{P}^N\Psi \end{pmatrix} \right| \left| \mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N]} \right| \mathrm{d}\varpi &\lesssim \left(1+\varsigma^{-1}\right) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}(\tau',u)\,\mathrm{d}\tau' \\ &\quad + \left(1+\varsigma^{-1}\right) \int_{u'=-U_1}^{u} \mathbb{Q}_{[1,N]}(\tau,u')\,\mathrm{d}u' \\ &\quad + \varsigma \mathbb{K}_{[1,N]}(\tau,u) + \mathring{\epsilon}^2. \end{aligned} \tag{29.6}$$

In particular, RHS (29.6) *is of type* $\mathrm{Error}_N^{(\text{Sub-critical})}(\tau,u)$, *where* $\mathrm{Error}_N^{(\text{Sub-critical})}(\tau,u)$ *satisfies* (29.5) *(hence, in the case* $N = N_{\mathrm{top}}$, *it is also is of type* $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$, *i.e., it satisfies the weaker estimate* (29.2)*).*

Proof. We will give a detailed proof for terms of type $\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left|L\mathcal{P}^N\Psi\right| \cdot \left|Y_{(A)}\mathcal{P}^{\leq N}\Psi\right| \mathrm{d}\varpi$, which are the most difficult terms generated by the LHS (29.6). By using the pointwise bound $\left|Y_{(A)}\mathcal{P}^N\Psi\right| \leq |Y_{(A)}|_{g\!\!\!/} \left|{d\!\!\!/}\mathcal{P}^N\Psi\right|_{g\!\!\!/} \lesssim \left|{d\!\!\!/}\mathcal{P}^N\Psi\right|_{g\!\!\!/}$ (see (13.1)), the estimates (18.9b), (20.53), (20.58), and (20.63a), and Young's inequality, and noting that (18.2) implies that $1 = \mathbf{1}_{[-U_{☢},U_{☢}]}(u') + \mathbf{1}_{[-U_{☢},U_{☢}]^c}(u') \leq \mathbf{1}_{[-U_{☢},U_{☢}]}(u') + C\mu$, we deduce the following estimate for any $\varsigma \in (0,1]$, where the implicit constants are **independent** of ς:

$$\begin{aligned} &\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left|L\mathcal{P}^N\Psi\right| \cdot \left|Y_{(A)}\mathcal{P}^{\leq N}\Psi\right| \mathrm{d}\varpi \\ &\lesssim \left(1+\varsigma^{-1}\right) \int_{u'=-U_1}^{u} \int_{{}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left|L\mathcal{P}^N\Psi\right|^2 \mathrm{d}\overline{\varpi}\,\mathrm{d}u' \\ &\quad + \int_{u'=-U_1}^{u} \int_{{}^{(\mathfrak{n})}\mathcal{P}_{u'}^{[\tau_0,\tau]}} \frac{1}{L^{(\mathfrak{n})}\tau} \mu \left|{d\!\!\!/}\mathcal{P}^N\Psi\right|_{g\!\!\!/}^2 \mathrm{d}\overline{\varpi}\,\mathrm{d}u' \\ &\quad + \varsigma \int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \mathbf{1}_{[-U_{☢},U_{☢}]}(u') \frac{1}{L^{(\mathfrak{n})}\tau} \left|{d\!\!\!/}\mathcal{P}^N\Psi\right|_{g\!\!\!/}^2 \mathrm{d}\varpi \\ &\quad + \int_{\tau'=\tau_0}^{\tau} \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}} \left|\mathcal{P}^{[1,N]}\Psi\right|^2 \mathrm{d}\underline{\varpi}\,\mathrm{d}\tau' \\ &\lesssim \left(1+\varsigma^{-1}\right) \int_{u'=-U_1}^{u} \mathbb{Q}_{[1,N]}(\tau,u)\,\mathrm{d}u' + \varsigma\mathbb{K}_{[1,N]}(\tau,u) + \mathring{\epsilon}^2 + \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N]}(\tau',u)\,\mathrm{d}\tau', \end{aligned} \tag{29.7}$$

which is $\lesssim$ RHS (29.6) as desired. The remaining terms on LHS (29.6) can be bounded by combining similar arguments with the estimate $|\mu| \lesssim 1$ (which follows from the bootstrap assumptions), the estimates of Prop. 18.1 (especially (18.1)), the coerciveness estimates of Lemmas 20.14 and 20.15, the estimates (25.1b) and (25.7), the Cauchy–Schwarz inequality for integrals, the pointwise commutator-type estimate (25.5), and Young's inequality; we refer to the proof of [50, Lemma 14.12] for further details. □

29.2.2. *Estimates for the error integrals generated by the inhomogeneous terms in the covariant wave equations.*

Lemma 29.5 (Estimates for the error integrals generated by the inhomogeneous terms in the covariant wave equations)**.** *Let* $\vec{\Psi} = (\Psi_0,\Psi_1,\Psi_2,\Psi_3,\Psi_4) = (\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s)$ *be the array of wave-variables, let* $\vec{\mathfrak{G}} = (\mathfrak{G}_0,\cdots,\mathfrak{G}_4)$ *be the corresponding array of the inhomogeneous terms in the covariant wave equations* $\mu\Box_{\mathbf{g}}\Psi_\iota = \mathfrak{G}_\iota$ *(see* (2.22)*). Let* $\mathcal{P}^N \in \mathfrak{P}^{(N)}$, *where* $\mathfrak{P}^{(N)}$ *is the set of order* N $\mathcal{P}_u$*-tangential commutator operators from Def. 8.10. Recall that* $\breve{T}$ *is the multiplier vectorfield defined in* (20.22)*. Then the following estimates hold for* $(\tau,u) \in [\tau_0,\tau_{\mathrm{Boot}}) \times [-U_1,U_2]$.

Top-order estimates. *If* $N = N_{\mathrm{top}}$, *then we have:*

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left|\breve{T}\mathcal{P}^N\vec{\Psi}\right| \left|\mathcal{P}^N\vec{\mathfrak{G}}\right| \mathrm{d}\varpi \leq \mathrm{Error}_N^{(\mathrm{Top})}(\tau,u), \tag{29.8}$$

where $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ *satisfies* (29.2).

Below-top-order estimates. *If $2 \leq N \leq N_{\text{top}}$, then we have:*

$$\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left|\breve{T}\mathcal{P}^{N-1}\vec{\Psi}\right| \left|\mathcal{P}^{N-1}\vec{\mathfrak{G}}\right| \, \mathrm{d}\varpi \leq \mathrm{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u), \tag{29.9}$$

where $\mathrm{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u)$ *satisfies* (29.5) *(with $N-1$ in the role of M in* (29.5)*).*

Proof. Throughout the proof, we will silently use the estimate $\frac{1}{L^{(\mathfrak{n})}\tau} \approx 1$ implied by (18.9b).

We first prove (29.8), i.e, we handle the case $N = N_{\text{top}}$. We pointwise bound the term $|\mathcal{P}^N\vec{\mathfrak{G}}|$ on LHS (29.8). Specifically, using the pointwise estimate (13.19), the bootstrap assumptions, (13.1), Young's inequality, and noting (as in the proof of (29.7)) that (18.2) implies that $1 = \mathbf{1}_{[-U_{☢},U_{☢}]}(u') + \mathbf{1}_{[-U_{☢},U_{☢}]^c}(u') \leq \mathbf{1}_{[-U_{☢},U_{☢}]}(u') + C\mu$, we deduce the following pointwise estimate for the integrand on LHS (29.8), valid for any $\varsigma \in (0,1]$, with implicit constants that are **independent** of ς:

$$\begin{aligned} \frac{1}{L^{(\mathfrak{n})}\tau}\left|\breve{T}\mathcal{P}^N\vec{\Psi}\right|\left|\mathcal{P}^N\vec{\mathfrak{G}}\right| \lesssim & \left|\breve{X}\mathcal{P}^N\vec{\Psi}\right| \cdot \left|\mu^{1/2}\mathcal{P}^N(\mathcal{C},\mathcal{D})\right| + \left|\mu^{1/2}L\mathcal{P}^N\vec{\Psi}\right| \cdot \left|\mu^{1/2}\mathcal{P}^N(\mathcal{C},\mathcal{D})\right| \\ & + \left|\mathcal{P}^{\leq N-1}(\mathcal{C},\mathcal{D})\right|^2 \\ & + \left(1+\varsigma^{-1}\right)\left|\breve{X}\mathcal{P}^{[1,N]}\vec{\Psi}\right|^2 + \left(1+\varsigma^{-1}\right)\left|L\mathcal{P}^{[1,N]}\vec{\Psi}\right|^2 \\ & + \varsigma\mathbf{1}_{[-U_{☢},U_{☢}]}\left|{d\!\!/}\mathcal{P}^{[1,N]}\vec{\Psi}\right|_{g\!\!/}^2 + \left|\mu^{1/2}{d\!\!/}\mathcal{P}^{[1,N]}\vec{\Psi}\right|_{g\!\!/}^2 \\ & + \left|\mathcal{P}^{\leq N}(\Omega,S)\right|^2 + \varepsilon\left|\mathcal{P}_*^{[1,N]}\underline{\gamma}\right|^2. \end{aligned} \tag{29.10}$$

We now integrate RHS (29.10) over ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}$. Using (20.53), (20.56a)–(20.56b), and the Cauchy–Schwarz inequality, we find that the integrals of the first two terms on RHS (29.10) are $\lesssim \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_N^{1/2}(\tau',u)\left\{\mathbb{C}_N^{1/2} + \mathbb{D}_N^{1/2}\right\}(\tau',u)\,\mathrm{d}\tau'$, which in turn is manifestly bounded by RHS (29.2) as desired. Moreover, using (20.56a)–(20.56b), we see that the integral of $\left|\mathcal{P}^{\leq N-1}(\mathcal{C},\mathcal{D})\right|^2$ is $\lesssim \int_{u'=-U_1}^{u} \{\mathbb{C}_{\leq N-1} + \mathbb{D}_{\leq N-1}\}(\tau,u')\,\mathrm{d}u'$, which is manifestly bounded by RHS (29.2) as desired. Finally, using (18.1), Lemma 20.14, Lemma 20.15, the estimate (25.1b), Young's inequality, and the fact that the L^2-controlling quantities $\mathbb{Q}_M(\tau,u)$, $\mathbb{C}_M(\tau,u)$, etc. are increasing in their arguments, it is straightforward to check that the integrals of the remaining terms on RHS (29.10) are of type $\mathrm{Error}_N^{(\text{Top})}(\tau,u)$, i.e., they satisfy (29.2).

To prove (29.9), we repeat the above arguments with $N-1$ in the role of N and simply note that the same arguments imply something stronger than what was claimed above: they imply that the error integrals generated by LHS (29.9) are of type $\mathrm{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u)$, i.e., they are all bounded by RHS (29.5) (with $N-1$ in the role of M in (29.5)). ☐

29.2.3. *Estimates for the error integrals generated by the multiplier vectorfield.* In the next lemma, we bound the error integrals that are generated by the multiplier vectorfield.

Lemma 29.6 (Estimates for the error integrals generated by the multiplier vectorfield)**.** *Assume that $1 \leq N \leq N_{\text{top}}$, $\Psi \in \vec{\Psi} = \{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^2,s\}$, and $\varsigma \in (0,1]$. Let $\mathcal{P}^N \in \mathfrak{P}^{(N)}$, where $\mathfrak{P}^{(N)}$ is the set of order N $\mathcal{P}_u$-tangential commutator operators from Def. 8.10. Recall that the multiplier vectorfield $\breve{T}$ is defined in* (20.22) *and that ${}^{(\breve{T})}\mathfrak{B}[\mathcal{P}^N\Psi]$ is the error term defined by* (20.27) *and* (20.28a)–(20.28f) *and appearing on RHS* (20.26)*. Then the following estimates hold for $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$, where the implicit constants are **independent** of ς:*

$$\begin{aligned} \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \left|\frac{1}{L^{(\mathfrak{n})}\tau}{}^{(\breve{T})}\mathfrak{B}[\mathcal{P}^N\Psi]\right| \mathrm{d}\varpi \lesssim & \left(1+\varsigma^{-1}\right)\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau' \\ & + \left(1+\varsigma^{-1}\right)\int_{u'=-U_1}^{u} \mathbb{Q}_N(\tau,u')\,\mathrm{d}u' \\ & + \varsigma\mathbb{K}_N(\tau,u). \end{aligned} \tag{29.11}$$

In particular, RHS (29.11) *is of type* $\mathrm{Error}_N^{(\text{Sub-critical})}(\tau,u)$*, where* $\mathrm{Error}_N^{(\text{Sub-critical})}(\tau,u)$ *satisfies* (29.5) *(hence, in the case $N = N_{\text{top}}$, it is also is of type* $\mathrm{Error}_N^{(\text{Top})}(\tau,u)$*, i.e., it satisfies the weaker estimate* (29.2)*).*

Proof. Throughout this proof, we will silently use the estimate (18.9b), which implies that the factor $\frac{1}{L^{(\mathfrak{n})}\tau}$ on LHS (29.11) is ≈ 1.

We first bound the error integral generated by the first term on RHS (20.27), i.e., in view of the factor $\frac{1}{L^{(\mathfrak{n})}\tau}$ on LHS (29.11) and definition (6.3), the integral of $\frac{1}{2}\mathbf{1}_{[-U_{\ast},U_{\ast}]^c}({}^{(\mathfrak{n})}\widetilde{L}\mu)\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2$. Using (18.2), we find that the integrand satisfies the pointwise bound $\frac{1}{L^{(\mathfrak{n})}\tau}\mathbf{1}_{[-U_{\ast},U_{\ast}]^c}|L\mu|\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2 \lesssim \mu\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2$. Hence, by (20.53), we deduce:

$$\frac{1}{2}\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\left|\frac{1}{L^{(\mathfrak{n})}\tau}\mathbf{1}_{[-U_{\ast},U_{\ast}]^c}(L\mu)\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2\right|\mathrm{d}\varpi \lesssim \int_{u'=-U_1}^{u}\mathbb{Q}_N(\tau,u')\,\mathrm{d}u', \tag{29.12}$$

which is $\lesssim$ RHS (29.11) as desired.

Next, we bound the integral of the term ${}^{(\check{T})}\mathfrak{B}_{(3)}[\mathcal{P}^N\Psi]$ defined in (20.28c). First, using the crucial pointwise bound (18.7), we bound the term $({}^{(\mathfrak{n})}\breve{R}\mu)\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2$ from (20.28c) in magnitude by $\lesssim \sqrt{\mu}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2$. Hence, using the coerciveness estimate (20.53) and (18.1), we can bound the integral of this term over the region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}$ by $\lesssim \int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau'$ as desired.

We now handle the remaining terms in the definition (20.28c) of ${}^{(\check{T})}\mathfrak{B}_{(3)}[\mathcal{P}^N\Psi]$ as well as the remaining bulk terms ${}^{(\check{T})}\mathfrak{B}_{(i)}[\mathcal{P}^N\Psi]$ with $i\in\{1,2,4,5,6\}$, i.e., the terms defined in (20.28a), (20.28b), (20.28d), (20.28e), and (20.28f). To this end, we first use Prop. 9.1, (18.26), the bootstrap assumptions, and Young's inequality, and we split $1 = \mathbf{1}_{[-U_{\ast},U_{\ast}]}(u) + \mathbf{1}_{[-U_{\ast},U_{\ast}]^c}(u)$, to deduce the following pointwise estimates, valid for any $\varsigma\in(0,1]$ with implicit constants that are **independent** of ς:

$$\begin{aligned}
&\left|\mu^{(\mathfrak{n})}U\mu + 2\mu L\mu + \frac{1}{2}\mu\mathrm{tr}_{\not{g}}\chi + \mu^2\mathrm{tr}_{\not{g}}\not{k}^{(\mathrm{Tan}-\vec{\Psi})} + \mu\mathrm{tr}_{\not{g}}\not{k}^{(\mathrm{Trans}-\vec{\Psi})}\right|\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2, \\
&\max_{i\in\{1,2,4,5,6\}}\left|{}^{(\check{T})}\mathfrak{B}_{(i)}[\mathcal{P}^N\Psi]\right| \\
&\lesssim \left(1+\varsigma^{-1}\right)(L\mathcal{P}^N\Psi)^2 + \left(1+\varsigma^{-1}\right)(\check{X}\mathcal{P}^N\Psi)^2 \\
&\quad + \mu\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2 + \mathbf{1}_{[-U_{\ast},U_{\ast}]^c}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2 + \varsigma\mathbf{1}_{[-U_{\ast},U_{\ast}]}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2.
\end{aligned} \tag{29.13}$$

Next, we use (18.2) to deduce the following pointwise estimate for the next-to-last term on RHS (29.13): $\mathbf{1}_{[-U_{\ast},U_{\ast}]^c}\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2 \lesssim \mu\left|\not{d}\mathcal{P}^N\Psi\right|_{\not{g}}^2$. Combining this estimate with the coerciveness guaranteed by (20.53) and (20.63a), we find that the integral of RHS (29.13) over the spacetime region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}$ is $\lesssim$ RHS (29.11) as desired.

□

29.3. **Top-order L^2 estimates for χ.** In this section, we derive L^2 estimates for the top-order terms $\mu\mathcal{P}^{N_{\mathrm{top}}}\mathrm{tr}_{\not{g}}\chi$ and $\mu\not{\mathcal{L}}_{\mathcal{P}}^{N_{\mathrm{top}}}\chi$ in terms of the L^2-controlling quantities. In the most difficult case, which is $\mathcal{P}^{N_{\mathrm{top}}} = \mathcal{Y}^{N_{\mathrm{top}}}$, the proofs rely on the pointwise estimate (22.20) satisfied by the fully-modified quantity ${}^{(\mathcal{Y}^N)}\mathscr{X}$ as well as the elliptic estimate (28.1) with $\not{\mathcal{L}}_{\mathcal{Y}}^{N_{\mathrm{top}}-1}\chi$ in the role of ξ.

29.3.1. *Statement of the top-order L^2 estimates for χ.* In the next proposition, we state the estimates. Its proof is located in Sect. 29.3.3.

Proposition 29.7 (Top-order L^2 estimates for χ)**.** *Let $N = N_{\text{top}}$, and let $\mathfrak{P}^{(N)}$ and $\mathscr{L}_{\mathfrak{P}}^{(N)}$ denote the sets of order N $\mathcal{P}_u$-tangential operators from Def. 8.10. Then the following estimates hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\begin{aligned}
&\max_{\mathcal{P}^N \in \mathfrak{P}^{(N)}} \left\| \mu \mathcal{P}^N \mathrm{tr}_{\not{g}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}, \max_{\mathscr{L}_{\mathcal{P}}^N \in \mathscr{L}_{\mathfrak{P}}^{(N)}} \left\| \mu \mathscr{L}_{\mathcal{P}}^N \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \\
&\lesssim \mathring{\epsilon} \ln\left(|\tau|^{-1}\right) + \mathbb{Q}_{[1,N]}^{1/2}(\tau, u) + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_{[1,N]}^{1/2}(\tau', u) \, d\tau' \\
&+ \int_{\tau'=\tau_0}^{\tau} \left\{ \mathbb{C}_N^{1/2} + \mathbb{D}_N^{1/2} \right\}(\tau', u) \, d\tau' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\{ \mathbb{C}_{\leq N-1}^{1/2} + \mathbb{D}_{\leq N-1}^{1/2} \right\}(\tau', u) \, d\tau' \\
&+ \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\{ \mathbb{V}_{\leq N}^{1/2} + \mathbb{S}_{\leq N}^{1/2} \right\}(\tau', u) \, d\tau'.
\end{aligned} \tag{29.14}$$

29.3.2. *Preliminary estimates.* In the following lemma, we derive preliminary L^2 estimates that we will use in the proof of Prop. 29.7.

Lemma 29.8 (Preliminary top-order L^2 estimates for χ)**.** *Let $N = N_{\text{top}}$, and let $\mathfrak{P}^{(N)}$, $\mathscr{L}_{\mathfrak{P}}^{(N)}$, and $\mathscr{L}_{\mathfrak{Y}}^{(N)}$ be the sets of order N $\mathcal{P}_u$-tangential commutator operators from Def. 8.10. Then the following estimates hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\begin{aligned}
\max_{\mathcal{P}^N \in \mathfrak{P}^{(N)}} \left\| \mu \mathcal{P}^N \mathrm{tr}_{\not{g}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} &\lesssim \mathring{\epsilon} \ln\left(|\tau|^{-1}\right) + \mathbb{Q}_{[1,N]}^{1/2}(\tau, u) + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_{[1,N]}^{1/2}(\tau', u) \, d\tau' \\
&+ \varepsilon \int_{\tau'=\tau_0}^{\tau} \left\| \mu \max_{\mathscr{L}_{\mathcal{Y}}^N \in \mathscr{L}_{\mathfrak{Y}}^{(N)}} \mathscr{L}_{\mathcal{Y}}^N \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} d\tau' \\
&+ \int_{\tau'=\tau_0}^{\tau} \left\{ \mathbb{C}_N^{1/2} + \mathbb{D}_N^{1/2} \right\}(\tau', u) \, d\tau' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\{ \mathbb{C}_{\leq N-1}^{1/2} + \mathbb{D}_{\leq N-1}^{1/2} \right\}(\tau', u) \, d\tau' \\
&+ \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\{ \mathbb{V}_{\leq N}^{1/2} + \mathbb{S}_{\leq N}^{1/2} \right\}(\tau', u) \, d\tau',
\end{aligned} \tag{29.15a}$$

$$\max_{\mathscr{L}_{\mathcal{P}}^N \in \mathscr{L}_{\mathfrak{P}}^{(N)}} \left\| \mu \mathscr{L}_{\mathcal{P}}^N \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \lesssim \mathring{\epsilon} + \mathbb{Q}_{[1,N]}^{1/2}(\tau, u) + \max_{\mathcal{P}^N \in \mathfrak{P}^{(N)}} \left\| \mu \mathcal{P}^N \mathrm{tr}_{\not{g}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}. \tag{29.15b}$$

Proof.

Proof of (29.15a): It suffices to show that for any $\mathcal{P}^N \in \mathfrak{P}^{(N)}$, we have: $\left\| \mu \mathcal{P}^N \mathrm{tr}_{\not{g}} \chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \lesssim$ RHS (29.15a). We first consider the most difficult case, which is $\mathcal{P}^N = \mathcal{Y}^N \in \mathfrak{Y}^{(N)}$. We take the $\| \cdot \|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the estimate (22.20). In the rest of the proof, we sometimes silently use (16.16) and Cor. 17.2, which together imply that the flow map factors ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ in (22.20) distort $\| \cdot \|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norms only by overall factors of $1 + \mathcal{O}(\varepsilon)$. For this reason, in this proof, we often suppress the factors of ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ to simplify the notation.

To proceed, we use (18.9b), the estimate $|\mu| \lesssim 1$ (which follows from the bootstrap assumptions), (16.9), and Lemma 20.14 to deduce that the norm $\| \cdot \|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ of the terms $\mu|\mathcal{P}^{N+1}\vec{\Psi}|$, $|\breve{X}\mathcal{P}^N\vec{\Psi}|$, and $\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} |\breve{X}\mathcal{P}^N\vec{\Psi}| \, d\tau'$ on RHS (22.20) are $\lesssim$ the sum of the second and third terms on RHS (29.15a). To bound the norm of the term $|\mathcal{Z}_*^{N;1}\vec{\Psi}|$ on RHS (22.20), we simply use the already proven estimate (25.7), while to handle the terms $\left| \begin{pmatrix} \mathcal{P}^{[1,N]}\gamma \\ \mathcal{P}_*^{[1,N]}\underline{\gamma} \end{pmatrix} \right|$ on RHS (22.20), we use (25.7) and the already proven estimate (25.1b). To bound the norms of the terms $\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \left| \mathcal{Z}_*^{[1,N];1}\vec{\Psi} \right| d\tau'$ and $\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \left| \begin{pmatrix} \mathcal{P}^{[1,N]}\gamma \\ \mathcal{P}_*^{[1,N]}\underline{\gamma} \end{pmatrix} \right| d\tau'$ on RHS (22.20), we use similar arguments, where we note that (25.7) and (25.1b) generate the error term $C \int_{\tau_0}^{\tau} \frac{\mathring{\epsilon}}{|\tau'|} \, d\tau'$, which is $\lesssim$ the term $\mathring{\epsilon} \ln\left(|\tau|^{-1}\right)$ on RHS (29.15a). To bound the norm $\| \cdot \|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ of the first term $\left|{}^{(\mathcal{Y}^N)}\mathscr{X}\right|(\tau_0,u,x^2,x^3)$ on RHS (22.20), we first use (16.7a) to deduce that: $\left\|{}^{(\mathcal{Y}^N)}\mathscr{X}(\tau_0,\cdot)\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \lesssim \left\|{}^{(\mathcal{Y}^N)}\mathscr{X}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,u]}\right)}$. Next, using Def. 19.2 and the data-assumptions stated in Sect. 11.2.1, we find that $\left\|{}^{(\mathcal{Y}^N)}\mathscr{X}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,u]}\right)} \lesssim \mathring{\epsilon}$, which is $\lesssim$ RHS (29.15a) as desired. The L^2 norm of the remaining time integrals on RHS (22.20) can be bounded using similar arguments, the estimate $|\mu| \lesssim 1$ (which follows from the bootstrap assumptions), and Lemma 20.14, which we use to control the $(\mathcal{C},\mathcal{D},\Omega,S)$-involving terms. We have therefore proved (29.15a) in the case $\mathcal{P}^N = \mathcal{Y}^N$.

We now prove (29.15a) in the case that the operator $\mathcal{P}^N$ on the LHS is not of type $\mathcal{Y}^N$, i.e., the case in which $\mathcal{P}^N$ contains at least one factor of L. In this case, we can use (13.6a) and the bootstrap assumptions to commute the factor of L so that it acts last and then use the pointwise estimates (13.4a) and (13.13d) to deduce: $|\mathcal{P}^N \mathrm{tr}_{\not{g}}\chi| \lesssim |L\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi| + |\mathcal{P}^{\leq N-1}\mathrm{tr}_{\not{g}}\chi| + \left|\mathcal{P}^{[1,N-1]}\gamma\right| \lesssim |\mathcal{P}^{[1,N+1]}\vec{\Psi}| + |\mathcal{P}^{[1,N]}\gamma|$. Multiplying this inequality by μ and using the arguments given above, including (25.1b) and the fact that $\mathbb{Q}_{[1,N]}(\tau,u)$ is increasing in its arguments, we find that $\|\mu\mathcal{P}^N\mathrm{tr}_{\not{g}}\chi\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \lesssim \mathring{\epsilon} + \mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$, which is $\lesssim$ RHS (29.15a) as desired. We have therefore proved (29.15a).

Proof of (29.15b): It suffices to show that for any $\not{\mathcal{L}}_{\mathcal{P}}^N \in \not{\mathcal{L}}_{\mathfrak{P}}^{(N)}$, we have: $\left\|\mu\not{\mathcal{L}}_{\mathcal{P}}^N\chi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \lesssim$ RHS (29.15b). To this end, we first apply the elliptic estimate (28.1) with $\xi \stackrel{\text{def}}{=} \not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi$ to deduce:

$$\begin{aligned}\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2|\not{\mathcal{L}}_{\mathcal{P}}^N\chi|_{\not{g}}^2\,d\underline{\varpi} &\lesssim \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2|\not{\mathcal{L}}_L\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi|_{\not{g}}^2\,d\underline{\varpi} + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2|\mathrm{d}\!\!\!/\mathrm{iv}\,\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi|_{\not{g}}^2\,d\underline{\varpi}\\ &\quad + \sum_{A=2,3}\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \mu^2(Y_{(A)}\mathrm{tr}_{\not{g}}\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi)^2\,d\underline{\varpi} + \int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} |\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi|_{\not{g}}^2\,d\underline{\varpi}.\end{aligned} \tag{29.16}$$

The same arguments given in the previous paragraph, starting from the pointwise estimate (13.13e), imply that the first term on RHS (29.16) is $\lesssim \mathring{\epsilon} + \mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$ as desired. Those arguments also imply, based on (25.1b), that $\|\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \lesssim \mathring{\epsilon} + \mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$, which yields the desired bound for the last term on RHS (29.16). To handle the third term on RHS (29.16), we start with the following pointwise triangle inequality estimate:

$$\mu\left|Y_{(A)}\mathrm{tr}_{\not{g}}\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi\right| \lesssim \mu\left|Y_{(A)}\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi\right| + \mu\left|Y_{(A)}\left(\mathrm{tr}_{\not{g}}\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi - \mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi\right)\right|. \tag{29.17}$$

The norm $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}$ of the first term $\mu\left|Y_{(A)}\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi\right|$ on RHS (29.17) is $\lesssim$ the last term on RHS (29.15b). To handle the second term on RHS (29.17), we first note the following pointwise commutator estimate, which follows easily from the Leibniz rule, the bootstrap assumptions, and (13.4a):

$$\left|Y_{(A)}\left(\mathrm{tr}_{\not{g}}\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi - \mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi\right)\right| \lesssim \left|\mathcal{P}^{[1,N]}\gamma\right|. \tag{29.18}$$

Multiplying (29.18) by μ and arguing as in the proof of (29.15a), using in particular Lemma 20.14 and (25.1b), we bound the norm $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}$ of the resulting RHS by $\lesssim \mathring{\epsilon} + \mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$, which is $\lesssim$ RHS (29.15b) as desired. It remains for us to bound the $\mathrm{d}\!\!\!/\mathrm{iv}\,\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi$-involving integral on RHS (29.16). We start with the following pointwise triangle inequality estimate:

$$\mu\left|\mathrm{d}\!\!\!/\mathrm{iv}\,\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi\right|_{\not{g}} \lesssim \mu\left|\not{d}\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi\right|_{\not{g}} + \mu\left|\mathrm{d}\!\!\!/\mathrm{iv}\,\not{\mathcal{L}}_{\mathcal{P}}^{N-1}\chi - \not{d}\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi\right|_{\not{g}}. \tag{29.19}$$

The norm $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}$ of the first term $\mu\left|\not{d}\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi\right|_{\not{g}}$ on RHS (29.19) is $\lesssim$ the last term on RHS (29.15b). To handle the second term on RHS (29.19), we simply multiply the pointwise estimate (13.16) by μ, take the norm $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}$ of the resulting inequality, and use the same arguments we used to control $\mu \times$ RHS (29.18). We have therefore proved (29.15b), which finishes the proof of the lemma.

□

29.3.3. *Proof of Prop. 29.7.* The estimate (29.14) for $\max_{\mathcal{P}^N\in\mathfrak{P}^{(N)}}\left\|\mu\mathcal{P}^N\mathrm{tr}_{\not g}\chi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ follows from inserting the estimate (29.15b) for $\max_{\not{\mathcal{L}}_{\mathcal{P}}^N\in\not{\mathcal{L}}_{\mathfrak{P}}^{(N)}}\left\|\mu\not{\mathcal{L}}_{\mathcal{P}}^N\chi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ into RHS (29.15a) and applying Grönwall's inequality. We then insert the already proved estimate (29.14) for $\max_{\mathcal{P}^N\in\mathfrak{P}^{(N)}}\left\|\mu\mathcal{P}^N\mathrm{tr}_{\not g}\chi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ into RHS (29.15b), thereby obtaining the desired estimate (29.14) for $\max_{\not{\mathcal{L}}_{\mathcal{P}}^N\in\not{\mathcal{L}}_{\mathfrak{P}}^{(N)}}\left\|\mu\not{\mathcal{L}}_{\mathcal{P}}^N\chi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ and completing the proof of the proposition.

□

29.4. **Estimates for the easy top-order eikonal function-involving error integrals.** In Prop. 29.7, we derived preliminary top-order L^2 estimates for χ. With the help of these estimates, we are now ready to control the wave equation error integrals that depend on these terms. More precisely, in the next lemma, we use these preliminary estimates to control "easy" error integrals, which are generated by the first product $\not{d}^\sharp\Psi_\iota\cdot\mu\not{d}\mathcal{Y}^{N-1}\mathrm{tr}_{\not g}\chi$ on RHS (22.3a) and the second product $(c^{-2}X^A)\not{d}^\sharp\Psi_\iota\cdot\mu\not{d}\mathcal{Y}^{N-1}\mathrm{tr}_{\not g}\chi$ on RHS (22.3b). The corresponding error integrals are easy in the sense that the integrands contain a helpful factor of μ. In Sect. 29.5, we will control the analogous – but much more difficult – error integral generated by the first product $(\check{X}\Psi_\iota)\mathcal{Y}^{N-1}Y_{(A)}\mathrm{tr}_{\not g}\chi$ on RHS (22.3b), which lacks the factor of μ.

Lemma 29.9 (Estimates for the easy top-order eikonal function-involving error integrals)**.** *Let $N=N_{\mathrm{top}}$, let $\Psi\in\vec{\Psi}=\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$, and let $\mathcal{P}^N\in\mathfrak{P}^{(N)}$, where $\mathfrak{P}^{(N)}$ is the set of order N $\mathcal{P}_u$-tangential commutator operators from Def. 8.10. Then the following estimates hold for $(\tau,u)\in[\tau_0,\tau_{\mathrm{Boot}})\times[-U_1,U_2]$:*

$$
\begin{aligned}
&\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}\left|\begin{pmatrix}(1+2\mu)L\mathcal{P}^N\Psi\\ 2\check{X}\mathcal{P}^N\Psi\end{pmatrix}\right|\left|\begin{pmatrix}(\not{d}^\sharp\Psi)\cdot\mu\not{d}\mathcal{Y}^{N-1}\mathrm{tr}_{\not g}\chi\\ (c^{-2}X^A)\not{d}^\sharp\Psi_\iota\cdot\mu\not{d}\mathcal{Y}^{N-1}\mathrm{tr}_{\not g}\chi\end{pmatrix}\right|\,\mathrm{d}\varpi\\
&=\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u),
\end{aligned}
\tag{29.20}
$$

where $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ *satisfies the estimate* (29.2).

Proof. The bootstrap assumptions and (18.1) imply that the integrand on LHS (29.20) is pointwise bounded by $\lesssim\left(|L\mathcal{P}^N\Psi|+|\check{X}\mathcal{P}^N\Psi|\right)\cdot\mu|\mathcal{Y}^N\mathrm{tr}_{\not g}\chi|$. Hence, integrating over ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}$ and using (20.53) and Young's inequality, we bound LHS (29.20) by:

$$
\begin{aligned}
&\lesssim\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\left\{(L\mathcal{P}^N\Psi)^2+(\check{X}\mathcal{P}^N\Psi)^2\right\}\mathrm{d}\varpi+\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}(\mu\mathcal{Y}^N\mathrm{tr}_{\not g}\chi)^2,\mathrm{d}\varpi\\
&\lesssim\int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau'+\int_{u'=-U_1}^{u}\mathbb{Q}_N(\tau,u')\,\mathrm{d}u'+\int_{\tau'=\tau_0}^{\tau}\left\|\mu\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\right\|^2_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)}\mathrm{d}\tau'.
\end{aligned}
\tag{29.21}
$$

The first two integrals on RHS (29.21) are $\lesssim$ RHS (29.2). Using the estimate (29.14) and straightforward applications of Young's inequality, we find that the last time integral on RHS (29.21), is bounded by:

$$
\begin{aligned}
&\lesssim\mathring{\epsilon}^2+\int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau'+\int_{\tau'=\tau_0}^{\tau}\left\{\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\mathbb{Q}^{1/2}_{[1,N]}(\tau'',u)\,\mathrm{d}\tau''\right\}^2\mathrm{d}\tau'\\
&+\int_{\tau'=\tau_0}^{\tau}\left\{\int_{\tau''=\tau_0}^{\tau'}\left[\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2\mathrm{d}\tau'\\
&+\int_{\tau'=\tau_0}^{\tau}\left\{\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\left[\mathbb{C}^{1/2}_{\leq N-1}+\mathbb{D}^{1/2}_{\leq N-1}+\mathbb{V}^{1/2}_{\leq N}+\mathbb{S}^{1/2}_{\leq N}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2\mathrm{d}\tau'.
\end{aligned}
\tag{29.22}
$$

We clarify that in deriving (29.22), we used that $\ln\left(|\tau|^{-1}\right)$ is square integrable on $\tau\in[\tau_0,0]$. The first integral on RHS (29.22) is $\lesssim$ RHS (29.2). Next, since the controlling quantities $\mathbb{Q}_M(\tau,u)$, $\mathbb{C}_M(\tau,u)$, etc. are increasing in their arguments, we deduce that the last two integrals on RHS (29.22) are $\lesssim$ RHS (29.2).

It remains for us to control the second integral on RHS (29.22). Again using the monotonicity of $\mathbb{Q}_M(\tau,u)$, we see that this integral is $\lesssim\int_{\tau'=\tau_0}^{\tau}\ln^2\left(|\tau'|^{-1}\right)\mathbb{Q}_{[1,N]}(\tau',u)\,\mathrm{d}\tau'\lesssim\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau|^{2/3}}\mathbb{Q}_{[1,N]}(\tau',u)\,\mathrm{d}\tau'$. This final integral is $\lesssim$ RHS (29.2), as desired. □

29.5. **Estimates for the most difficult top-order eikonal function-involving error integrals.** In this section, we control the most difficult eikonal function-involving terms appearing in the commuted wave equations of Prop. 22.3. More precisely, the most difficult product is the first one $(\breve{X}\Psi_\iota)\mathcal{Y}^{N-1}Y_{(A)}\mathrm{tr}_{\not g}\chi$ on RHS (22.3b). When we derive energy estimates using the fundamental energy–null-flux identity (20.26), these difficult terms are multiplied by $\breve{T}\mathcal{Y}^N\Psi_\iota$, where the multiplier vectorfield $\breve{T}$ is defined in (20.22). This leads to the following difficult error integrals:

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left\{2\breve{X}\mathcal{Y}^N\Psi_\iota\right\}\left\{(\breve{X}\Psi_\iota)\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\right\} \mathrm{d}\varpi, \tag{29.23a}$$

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left\{(1+2\mu)L\mathcal{Y}^N\Psi_\iota\right\}\left\{(\breve{X}\Psi_\iota)\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\right\} \mathrm{d}\varpi. \tag{29.23b}$$

We bound the integral (29.23a) in Sect. 29.5.1 and the integral (29.23b), which we control via a further integration by parts with respect to L, in Sect. 29.5.2. It turns out that these two integrals are the main ones driving top-order wave energy blowup-rate, i.e., RHS (24.1a) with $K = 0$. We provide the main estimates for these two integrals in Lemmas 29.11 and 29.13.

29.5.1. *Estimates that do not involve integration by parts.* We begin our analysis of the integral (29.23a) with the following lemma, which provides L^2 estimates for the difficult product $(\breve{X}\Psi_\iota)\mathcal{Y}^N\mathrm{tr}_{\not g}\chi$ on RHS (22.3b). More precisely, in the lemma, we handle the most difficult case, which is $\Psi_\iota = \mathcal{R}_{(+)}$. The products corresponding to remaining wave-variables $\{\mathcal{R}_{(-)}, v^2, v^3, s\}$ are much easier to handle in the energy estimates because in these cases, we gain a smallness factor of ε from the factor $\breve{X}\Psi_\iota$; see (17.11).

Lemma 29.10 (L^2 estimates for the most difficult product)**.** *Let $N = N_{\mathrm{top}}$, and let $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$, where $\mathfrak{Y}^{(N)}$ is the set of order N $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Then the following estimates hold for* $(\tau, u) \in [\tau_0, \tau_{\mathrm{Boot}}) \times$

$[-U_1, U_2]$:

$$
\begin{aligned}
&\left\| \frac{1}{L^{(\mathfrak{n})}\tau} (\check{X}\mathcal{R}_{(+)}) \mathcal{Y}^N \mathrm{tr}_{\not g}\chi \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \\
&\leq \boxed{\frac{2\times 1.01}{\sqrt{1.99}}} \frac{1}{|\tau|} \mathbb{Q}_N^{1/2}(\tau,u) \\
&+ \boxed{\frac{4\times (1.01)^2}{\sqrt{1.99}}} \frac{1}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_N^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&+ \frac{C_*}{|\tau|} \left(\mathbb{Q}_N^{(\mathrm{Partial})}\right)^{1/2}(\tau,u) + \frac{C_*}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \left(\mathbb{Q}_N^{(\mathrm{Partial})}\right)^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&+ \frac{C\varepsilon}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&+ \frac{C\varepsilon}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \int_{\tau''=\tau_0}^{\tau'} \left\{ \frac{1}{|\tau''|} \mathbb{Q}_{[1,N]}^{1/2} + \mathbb{C}_N^{1/2} + \mathbb{D}_N^{1/2} \right\}(\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau' \\
&+ \frac{C\varepsilon}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \left\{ \mathbb{C}_{\leq N-1}^{1/2} + \mathbb{D}_{\leq N-1}^{1/2} \right\}(\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau' \\
&+ \frac{C\varepsilon}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \left\{ \mathbb{V}_{\leq N}^{1/2} + \mathbb{S}_{\leq N}^{1/2} \right\}(\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau' \\
&+ \frac{C}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \left\{ \mathbb{C}_N^{1/2} + \mathbb{D}_N^{1/2} \right\}(\tau',u)\,\mathrm{d}\tau' + \frac{C}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\{ \mathbb{C}_{\leq N-1}^{1/2} + \mathbb{D}_{\leq N-1}^{1/2} \right\}(\tau',u)\,\mathrm{d}\tau' \\
&+ \frac{C}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\{ \mathbb{V}_{\leq N}^{1/2} + \mathbb{S}_{\leq N}^{1/2} \right\}(\tau',u)\,\mathrm{d}\tau' \\
&+ \frac{C\varepsilon}{|\tau|} \mathbb{Q}_N^{1/2}(\tau,u) + \frac{C}{|\tau|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau,u) + \frac{C}{|\tau|^{3/2}} \mathbb{Q}_{[1,N-1]}^{1/2}(\tau,u) \\
&+ \frac{C}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&+ \frac{C}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau' \\
&+ \frac{C\varepsilon}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' + \frac{C\mathring{\varepsilon}}{|\tau|^{3/2}}.
\end{aligned}
\tag{29.24}
$$

Proof. We consider the pointwise estimate (22.18). In this proof, we sometimes silently use (16.16) and Cor. 17.2, which together imply that the flow map factors ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ in (22.18) distort $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}$ norms only by overall factors of $1+\mathcal{O}(\varepsilon)$; the $\mathcal{O}(\varepsilon)$ factors lead to small error terms on RHS (29.24).

We now use (18.1) and (18.15) to bound $\left| \frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu} \mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \cap {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \right| \leq \frac{1.01}{|\tau|}$ everywhere it appears on RHS (22.18). In particular, we bound the first and third terms on RHS (22.18) (which are multiplied by boxed constants) by $\frac{2(1.01)}{|\tau|} \left|\check{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$ and $\frac{4(1.01)^2}{|\tau|} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|} \left|\check{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'$ respectively. We now take the norm $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}$ of the resulting pointwise inequality and use (16.18) and Cor. 17.2. Also using the sharpened coerciveness estimate (20.59), we see that these two terms lead, respectively, to the presence of the two boxed-constant-involving products $\boxed{\frac{2\times 1.01}{\sqrt{1.99}}}\cdots$, $\boxed{\frac{4\times (1.01)^2}{\sqrt{1.99}}}\cdots$ on RHS (29.24) plus some error terms with $C\varepsilon$ factors. Similarly, since the C_*-multiplied terms on RHS (22.18) involve $\breve{X}\mathcal{Y}^N\vec{\Psi}_{(\text{Partial})}$, we can use Lemma 20.14 (specifically (20.54)) to bound their $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norms by the C_*-multiplied terms on the third line of RHS (29.24).

Next, we use (16.18) and (29.14) to bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the term $\frac{C\varepsilon}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\mu|\mathcal{L}_{\mathcal{Y}}^N\chi|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'$ on RHS (22.18). We find that this term is bounded by the sum of the double time integrals on the fifth through seventh lines of RHS (29.24), plus as a few other terms on RHS (29.24).

Next, using (16.18), (17.12), (18.1), and Lemma 20.14, we see that the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norms of the time-integrals of $\mu|\mathcal{Y}^N(\mathcal{C},\mathcal{D})|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}$, $|\mathcal{Y}^{\leq N-1}(\mathcal{C},\mathcal{D})|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}$, and $|\mathcal{Y}^{\leq N}(\Omega,S)|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}$ on RHS (22.18) are bounded by the eighth and ninth lines of RHS (29.24).

It remains for us to bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the terms $\mathrm{Error}\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)$ on RHS (22.18), which satisfy the pointwise bound (22.19). To handle the first term $\frac{1}{|\tau|}\left|{}^{(\mathcal{Y}^N)}\mathscr{X}\right|(\tau_0,u,x^2,x^3)$ on RHS (22.19), we note that in our proof of (29.15a), we showed that $\left\|{}^{(\mathcal{Y}^N)}\mathscr{X}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \lesssim \left\|{}^{(\mathcal{Y}^N)}\mathscr{X}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,u]}\right)} \lesssim \mathring{\epsilon}$. Hence, the first term on RHS (22.19) is $\lesssim \frac{\mathring{\epsilon}}{|\tau|} \lesssim$ RHS (29.24) as desired. With the help of (18.1), Lemma 20.14, and the estimate (25.1b), we can bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the remaining terms on RHS (22.18) by $\lesssim$ RHS (29.24) by using a subset of the ideas we used above; we refer to the proofs of [73, Lemma 14.8] and [50, Lemma 14.14] for more details. □

With the help of Lemma 29.10, we now establish the following lemma, which is the main result of Sect. 29.5.1.

Lemma 29.11 (Bounds for the most difficult error integrals in the wave equation energy estimates)**.** *Let* $N = N_{\text{top}}$*, and let* $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$*, where* $\mathfrak{Y}^{(N)}$ *is the set of order* N $\ell_{t,u}$*-tangential commutator operators from Def. 8.10. Then the following estimates hold for* $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$*:*

$$\begin{aligned}
&2\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}(\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)})(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\,\mathrm{d}\varpi\right| \\
&\leq \boxed{\frac{4\times 1.01}{1.99}}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau' \\
&\quad+\boxed{\frac{8\times(1.01)^2}{1.99}}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',u)\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\mathbb{Q}_N^{1/2}(\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau' \\
&\quad+C_*\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',u)\left(\mathbb{Q}_N^{(\text{Partial})}\right)^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&\quad+C_*\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',u)\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\left(\mathbb{Q}_N^{(\text{Partial})}\right)^{1/2}(\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau' \\
&\quad+\mathrm{Error}_N^{(\text{Top})}(\tau,u),
\end{aligned} \tag{29.25}$$

where $\mathrm{Error}_N^{(\text{Top})}$ *satisfies the estimate* (29.2).

Moreover, for $\Psi\in\vec{\Psi}_{(\text{Partial})}=\{\mathcal{R}_{(-)},v^2,v^3,s\}$*, we have the following less degenerate estimates:*

$$2\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}(\breve{X}\mathcal{Y}^N\Psi)(\breve{X}\Psi)\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\,\mathrm{d}\varpi\right|\lesssim\mathrm{Error}_N^{(\text{Top})}(\tau,u), \tag{29.26}$$

where $\mathrm{Error}_N^{(\text{Top})}$ *satisfies the estimate* (29.2).

Proof. We first prove (29.25). By Hölder's inequality, we have:

$$
\begin{aligned}
&2\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}(\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)})(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not g}\chi \,\mathrm{d}\varpi\right| \\
&\leq 2\int_{\tau'=\tau_0}^{\tau} \left\|\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} \left\|\frac{1}{L^{(\mathfrak{n})}\tau}(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not g}\chi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} \mathrm{d}\tau'.
\end{aligned}
\tag{29.27}
$$

Using the sharpened coerciveness bound (20.59), we find that:

$$
\mathrm{RHS}\ (29.27) \leq \frac{2}{\sqrt{1.99}}\int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_N^{1/2}(\tau',u)\left\|\frac{1}{L^{(\mathfrak{n})}\tau}(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^N \mathrm{tr}_{\not g}\chi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} \mathrm{d}\tau'. \tag{29.28}
$$

We now insert the estimate (29.24) into RHS (29.28). The desired estimate (29.25) then follows from a series of standard applications of Young's inequality, as we now explain. We will control several representative terms in detail and leave the remaining details to the reader. First, the τ'-integrals of the product of $\frac{2}{\sqrt{1.99}}\mathbb{Q}_N^{1/2}(\tau',u)$ and the first four terms on RHS (29.24) are clearly bounded by the first four terms on RHS (29.25) as desired. Next, we observe that the τ'-integrals of the product of $\frac{2}{\sqrt{1.99}}\mathbb{Q}_N^{1/2}(\tau',u)$ and the τ'-integrals on the eighth and ninth lines of RHS (29.24) are bounded in magnitude by:

$$
\begin{aligned}
&\lesssim \int_{\tau'=\tau_0}^{\tau}\left\{\frac{1}{|\tau'|^{1/3}}\mathbb{Q}_N^{1/2}(\tau',u)\right\}\left\{\frac{1}{|\tau'|^{2/3}}\int_{\tau''=\tau_0}^{\tau'}\left[\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}\mathrm{d}\tau' \\
&\quad+\int_{\tau'=\tau_0}^{\tau}\left\{\frac{1}{|\tau'|^{1/3}}\mathbb{Q}_N^{1/2}(\tau',u)\right\}\left\{\frac{1}{|\tau'|^{2/3}}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\left[\mathbb{C}_{\leq N-1}^{1/2}+\mathbb{D}_{\leq N-1}^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}\mathrm{d}\tau' \\
&\quad+\int_{\tau'=\tau_0}^{\tau}\left\{\frac{1}{|\tau'|^{1/3}}\mathbb{Q}_N^{1/2}(\tau',u)\right\}\left\{\frac{1}{|\tau'|^{2/3}}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\left[\mathbb{V}_{\leq N}^{1/2}+\mathbb{S}_{\leq N}^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}\mathrm{d}\tau' \\
&\lesssim \int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{2/3}}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau'+\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\tau_0}^{\tau'}\left[\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2\mathrm{d}\tau' \\
&\quad+\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\left[\mathbb{C}_{\leq N-1}^{1/2}+\mathbb{D}_{\leq N-1}^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2\mathrm{d}\tau' \\
&\quad+\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\left[\mathbb{V}_{\leq N}^{1/2}+\mathbb{S}_{\leq N}^{1/2}\right](\tau'',u)\,\mathrm{d}\tau''\right\}^2\mathrm{d}\tau'.
\end{aligned}
\tag{29.29}
$$

Accounting for the term $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ on RHS (29.25), we conclude that RHS (29.29) $\lesssim$ RHS (29.25) as desired. Moreover, since $\mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$, $\mathbb{C}_N^{1/2}(\tau,u)$, $\mathbb{D}_N^{1/2}(\tau,u)$, $\mathbb{V}_N^{1/2}(\tau,u)$, and $\mathbb{S}_N^{1/2}(\tau,u)$ are increasing in their arguments, the terms involving double τ-integrals on the fifth through seventh lines of RHS (29.24) are bounded by the single τ-integrals on the eighth and ninth lines of RHS (29.24) plus $\frac{C}{|\tau|}\int_{\tau'=\tau_0}^{\tau}\ln\left(|\tau'|^{-1}\right)\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau'$. Hence, the τ'-integral of the product of $\frac{2}{\sqrt{1.99}}\mathbb{Q}_N^{1/2}(\tau',u)$ and the double τ-integrals on the fifth through seventh lines of RHS (29.24) are bounded by RHS (29.29) plus:

$$
\int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_N^{1/2}(\tau',u)\frac{1}{|\tau'|}\int_{\tau''=\tau_0}^{\tau'}\ln\left(|\tau''|^{-1}\right)\mathbb{Q}_{[1,N]}(\tau'',u)\,\mathrm{d}\tau''\,\mathrm{d}\tau'. \tag{29.30}
$$

Using the trivial bound $\ln\left(|\tau''|^{-1}\right)\lesssim\frac{1}{|\tau''|^{1/2}}$, we bound (29.30) by $\lesssim\int_{\tau'=\tau_0}^{\tau}\mathbb{Q}_N^{1/2}(\tau',u)\frac{1}{|\tau'|}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau'',u)\,\mathrm{d}\tau''\,\mathrm{d}\tau'$, which in turn is bounded by the term $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ on RHS (29.25) (more precisely, by the third-from-last term on RHS (29.2)). As our last representative term, we note that the τ'-integral of the product of $\frac{2}{\sqrt{1.99}}\mathbb{Q}_N^{1/2}(\tau',u)$ and the term

$\frac{C\mathring{\epsilon}}{|\tau|^{3/2}}$ term on RHS (29.24) is:

$$\begin{aligned}
&\lesssim \int_{\tau'=\tau_0}^{\tau} \left\{ \frac{\mathring{\epsilon}}{|\tau'|^{5/4}} \right\} \left\{ \frac{1}{|\tau'|^{1/4}} \mathbb{Q}_N^{1/2}(\tau', u) \right\} d\tau' \\
&\lesssim \mathring{\epsilon}^2 \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{5/2}} d\tau' + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_N \, d\tau' \\
&\lesssim \frac{\mathring{\epsilon}^2}{|\tau|^{3/2}} + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{2/3}} \mathbb{Q}_N \, d\tau' \lesssim \mathrm{Error}_N^{(\mathrm{Top})}(\tau, u),
\end{aligned} \tag{29.31}$$

as desired.

To prove (29.26) for $\Psi \in \vec{\Psi}_{(\mathrm{Partial})} = \{\mathcal{R}_{(-)}, v^2, v^3, s\}$, we first use (17.11) to deduce that the magnitude of the integrand on the LHS of (29.26) is $\lesssim \varepsilon \left|\mathcal{Y}^N \Psi\right| \left|\mathcal{Y}^N \mathrm{tr}_{\not g}\chi\right|$. We can now argue as in the proof of (29.25), except that due to the smallness factor ε, we do not have to carefully track any boxed constant-involving terms (we can relegate such terms to the ε-multiplied terms in $\mathrm{Error}_N^{(\mathrm{Top})}(\tau, u)$ on RHS (29.26)), and on RHS (29.26), we can bound all wave error terms in terms of the full wave energies $\mathbb{Q}$, i.e., without reference to the partial wave energies $\mathbb{Q}^{(\mathrm{Partial})}$.

□

29.5.2. *Estimates involving integration by parts with respect to* ${}^{(\mathfrak{n})}\widetilde{L}$. In this section, we bound the difficult top-order integrals highlighted in (29.23b). The proof involves several rather technical steps, and we therefore provide some preliminary lemmas before proving the main estimates in Lemma 29.16. We will control the error integrals (29.23b) by integrating by parts with respect to the rough null vectorfield ${}^{(\mathfrak{n})}\widetilde{L}$ defined in (6.3). As we will see in the proof of Lemma 29.16, the analysis fundamentally relies on the decomposition $\mathcal{Y}^N \mathrm{tr}_{\not g}\chi = Y_{(A)}\mathcal{Y}^{N-1}\mathrm{tr}_{\not g}\chi = Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}} - Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}$, where ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ is the partially modified quantity from Def. 19.2. The integrals involving ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ are the most difficult to estimate and are the ones that we treat via integration by parts with respect to ${}^{(\mathfrak{n})}\widetilde{L}$, specifically by invoking the identity (20.15).

Before bounding the error integrals on RHS (20.15), we first establish a preliminary lemma in which we handle the most difficult part of the analysis.

Lemma 29.12 (Difficult top-order hypersurface L^2 estimates related to integration by parts with respect to ${}^{(\mathfrak{n})}\widetilde{L}$)**.** *Let $N = N_{\mathrm{top}}$, and let $\mathcal{Y}^{N-1} \in \mathfrak{Y}^{(N-1)}$, where $\mathfrak{Y}^{(N-1)}$ is the set of order $N-1$ $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Let ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ be the partially modified quantity defined by* (19.7a)*. The following estimates hold for $(\tau, u) \in$*

$[\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:

$$\begin{aligned}
\left\| \frac{1}{\sqrt{\mu}} (\breve{X}\mathcal{R}_{(+)})^{(\mathfrak{n})}\widetilde{L}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \leq & \boxed{2.89} \frac{1}{|\tau|} \mathbb{Q}_N^{1/2}(\tau,u) \\
& + \frac{C_*}{|\tau|} (\mathbb{Q}_N^{(\text{Partial})})^{1/2}(\tau,u) + \frac{C\varepsilon}{|\tau|} \mathbb{Q}_{[1,N]}^{1/2}(\tau,u) + \frac{C}{|\tau|^{1/2}} \mathbb{Q}_N^{1/2}(\tau,u) \\
& + \frac{C}{|\tau|} \mathbb{Q}_{[1,N-1]}^{1/2}(\tau,u) + \frac{C\mathring{\varepsilon}}{|\tau|^{1/2}},
\end{aligned} \tag{29.32a}$$

$$\begin{aligned}
\left\| \frac{1}{\sqrt{\mu}} (\breve{X}\mathcal{R}_{(+)})^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \leq & \boxed{2.89} \frac{1}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_N^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + \frac{C_*}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} (\mathbb{Q}_N^{(\text{Partial})})^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + \frac{C\varepsilon}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + C \frac{1}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + C \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
& + \frac{C}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N-1]}^{1/2}(\tau',u)\,\mathrm{d}\tau' + \frac{C\mathring{\varepsilon}}{|\tau|^{1/2}}.
\end{aligned} \tag{29.32b}$$

Moreover, we have the following less precise estimates:

$$\left\| {}^{(\mathfrak{n})}\widetilde{L}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \lesssim \frac{1}{|\tau|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau,u) + \mathring{\varepsilon}, \tag{29.33a}$$

$$\left\| {}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}} \right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)} \lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau' + \mathring{\varepsilon} \lesssim \mathbb{Q}_{[1,N]}^{1/2}(\tau,u) + \mathring{\varepsilon}. \tag{29.33b}$$

Proof. In the proof, we sometimes silently use (16.16) and Cor. 17.2, which together imply that the flow map factors ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ in, for example, (22.31a), distort $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}\right)}$ norms only by overall factors of $1+\mathcal{O}(\varepsilon)$; the $\mathcal{O}(\varepsilon)$ factors lead to small error terms on the RHS of our estimates.

Proof of (29.32a)**:** In view of the above remarks, to simplify the notation, in the following discussion, we will sometimes suppress the factors of ${}^{(\mathfrak{n})}\widetilde{\Lambda}$ in (22.31a). We start by multiplying (22.31a) by $\frac{1}{\sqrt{\mu}}\breve{X}\mathcal{R}_{(+)}$. We now consider the product generated by the first term on RHS (22.31a). Multiplying both sides of (22.28) by $\frac{\sqrt{\mu}}{2}$, we deduce:

$$\frac{1}{2L^{(\mathfrak{n})}\tau} \frac{1}{\sqrt{\mu}} G_{LL}^0 \breve{X}\mathcal{R}_{(+)} = \frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\sqrt{\mu}} + \mathcal{O}(\varepsilon)\frac{\sqrt{\mu}}{|\tau|}. \tag{29.34}$$

Using (29.34) to substitute for the product generated by $\frac{1}{\sqrt{\mu}}\breve{X}\mathcal{R}_{(+)}$ times the first term on RHS (22.31a), we pointwise bound this difficult product as follows, where ${}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}$ is the set from (18.12):

$$\begin{aligned}
\leq & \left| \frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu} \mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \cap {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}} \right| \cdot \left| \sqrt{\mu} \not{\Delta} \mathcal{Y}^{N-1}\mathcal{R}_{(+)} \right| \\
& + \left| \frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu} \mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \setminus {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}} \right| \cdot \left| \sqrt{\mu} \not{\Delta} \mathcal{Y}^{N-1}\mathcal{R}_{(+)} \right| + \mathcal{O}(\varepsilon)\frac{1}{|\tau|} \left| \sqrt{\mu} \not{\Delta} \mathcal{Y}^{N-1}\mathcal{R}_{(+)} \right|.
\end{aligned} \tag{29.35}$$

Using (18.16) and the crude estimate $|{}^{(\mathfrak{n})}\widetilde{L}\mu| \lesssim 1$ (see (17.13)), we deduce the pointwise bound $\left| \frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu} \mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]} \setminus {}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\text{Boot}}]}} \right| \lesssim 1$. Hence, using the pointwise comparison estimate (13.11b), we see that the second product on RHS (29.35) is pointwise bounded by $\lesssim \left|\sqrt{\mu}\not{d}\mathcal{Y}^{[N-1,N]}\mathcal{R}_{(+)}\right|$. Similarly, we find that the third product on RHS (29.35) is pointwise bounded by $\lesssim \frac{\varepsilon}{|\tau|}\left|\sqrt{\mu}\not{d}\mathcal{Y}^{[N-1,N]}\mathcal{R}_{(+)}\right|$. From these pointwise bounds and (20.53), we find that the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norms of the second and third terms in (29.35) are $\leq$ the sum of the third, fourth, and fifth terms on RHS (29.32a), as desired. Next, we use (18.1), (18.15), (13.11b), and Cor. 17.2 to pointwise bound the first term in (29.35) as follows:

$$\begin{aligned}\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\mu}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right|\cdot\left|\sqrt{\mu}\not{\Delta}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| &\leq \sqrt{2}\{1+\mathcal{O}_{\blacklozenge}(\mathring{\alpha})\}\frac{1.01}{|\tau|}\sqrt{\sum_{A=2}^{3}\left|\sqrt{\mu}\not{d}Y_{(A)}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right|_{\not{g}}^2} \\ &\quad+\frac{C}{|\tau|}\left|\sqrt{\mu}\not{d}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right|_{\not{g}}.\end{aligned} \tag{29.36}$$

From (29.36), (20.53) and in particular its implication:

$$\left\|\sqrt{\sum_{A=2}^{3}\left|\sqrt{\mu}\not{d}Y_{(A)}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right|_{\not{g}}^2}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \leq \sqrt{\frac{2}{0.49}}\mathbb{Q}_N^{1/2}(\tau,u), \tag{29.37}$$

our assumption that $\mathring{\alpha}$ and ε are sufficiently small, and the inequality $\sqrt{2}\times 1.01\times\sqrt{\frac{2}{0.49}} < 2.886$, we deduce that the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the first term in (29.35) is $\leq$ the sum of the $\boxed{2.89}$-multiplied term on RHS (29.32a) and the $\frac{C}{|\tau|}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau,u)$ term. We have therefore obtained the desired estimates for the product of $\frac{1}{\sqrt{\mu}}\breve{X}\mathcal{R}_{(+)}$ and the first term on RHS (22.31a). Combining similar but simpler arguments with (20.58) and the pointwise bound $\frac{1}{\mu^{1/2}}|\breve{X}\mathcal{R}_{(+)}| \lesssim \frac{1}{|\tau|^{1/2}}$ implied by (17.10) and (18.1), we find that the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norms of the product of $\frac{1}{\sqrt{\mu}}\breve{X}\mathcal{R}_{(+)}$ and the terms $C_*\left|\not{\Delta}\mathcal{Y}^{N-1}\vec{\Psi}_{(\mathrm{Partial})}\right|$ and $C\varepsilon\left|\mathcal{P}^{[1,N+1]}\vec{\Psi}\right|$ on RHS (22.31a) are bounded by the sum of the last five terms on RHS (29.32a). Finally, using the pointwise bound $\frac{1}{\mu^{1/2}}|\breve{X}\mathcal{R}_{(+)}| \lesssim \frac{1}{|\tau|^{1/2}}$ noted above, (20.58), (25.1b), (18.1), and the fact that the $\mathbb{Q}_M(\tau,u)$ are increasing in their arguments, we bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the product of $\frac{1}{\sqrt{\mu}}\breve{X}\mathcal{R}_{(+)}$ and the last term $C\left|\mathscr{P}^{[1,N]}\gamma\right|$ on RHS (22.31a), by $\lesssim \mathring{\epsilon}\frac{1}{|\tau|^{1/2}}+\frac{1}{|\tau|^{1/2}}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,d\tau' \lesssim \mathring{\epsilon}\frac{1}{|\tau|^{1/2}}+\frac{1}{|\tau|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$. We have therefore proved (29.32a).

Proof of (29.32b)**:** We start by multiplying (22.31b) by $\frac{1}{\sqrt{\mu}}\breve{X}\mathcal{R}_{(+)}$. We now focus on the most difficult product, which is generated by the second term on RHS (22.31b), i.e.,

$$\frac{1}{2}\left\{\left|\frac{1}{L^{(\mathfrak{n})}\tau}\frac{1}{\sqrt{\mu}}G_{LL}^0\breve{X}\mathcal{R}_{(+)}\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3)\right\}\int_{\tau'=\tau_0}^{\tau}\left|\not{\Delta}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right|\circ{}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,d\tau'. \tag{29.38}$$

Next, we note that the same reasoning we used to prove (22.28) also yields the following bound:

$$\frac{1}{2}\frac{1}{L^{(\mathfrak{n})}\tau}\frac{1}{\sqrt{\mu}}G_{LL}^0\breve{X}\mathcal{R}_{(+)} = \frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\sqrt{\mu}}+\mathcal{O}(\varepsilon)\frac{1}{|\tau|^{1/2}}. \tag{29.39}$$

Using (29.39), we substitute $\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\sqrt{\mu}}\right|+\mathcal{O}(\varepsilon)\frac{1}{|\tau|^{1/2}}$ for the first product $\left|\frac{1}{2L^{(\mathfrak{n})}\tau}\frac{1}{\sqrt{\mu}}G_{LL}^0\breve{X}\mathcal{R}_{(+)}\right|$ in (29.38). Next, we bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the product generated by the factor $\mathcal{O}(\varepsilon)\frac{1}{|\tau|^{1/2}}$ by using (13.11b), (16.18), (18.1), and (20.53). We find that these error terms are $\leq$ the $C\varepsilon$-multiplied term on the third line of RHS (29.32b). Next, we use the triangle inequality, (18.1), (18.15), (18.16), and the crude estimate $|{}^{(\mathfrak{n})}\widetilde{L}\mu| \lesssim 1$ noted earlier to deduce the following pointwise bound:

$$\begin{aligned}\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\sqrt{\mu}}\right| &\leq \left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\sqrt{\mu}}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\cap{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right|+\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\sqrt{\mu}}\mathbf{1}_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\setminus{}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0,\tau_{\mathrm{Boot}}]}}\right| \\ &\leq \frac{1.01}{|\tau|^{1/2}}+C.\end{aligned} \tag{29.40}$$

From (29.40) and (18.1), it follows that $\left|\frac{{}^{(\mathfrak{n})}\widetilde{L}\mu}{\sqrt{\mu}}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau,u,x^2,x^3) \times \int_{\tau'=\tau_0}^{\tau} \left|\not\Delta \mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'$ is pointwise bounded by:

$$\begin{aligned} &\leq \frac{1.01}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left|\sqrt{\mu}\not\Delta \mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\ &\quad + C \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left|\sqrt{\mu}\not\Delta \mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'. \end{aligned} \tag{29.41}$$

Using (13.11b), (16.18), and (20.53), we find that the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the last term on RHS (29.41) is bounded by the terms on the next-to-last line of RHS (29.32b). Next, we use (13.11b) and Cor. 17.2 to deduce the following pointwise bound for the first term on RHS (29.41):

$$\begin{aligned} &\frac{1.01}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left|\sqrt{\mu}\not\Delta \mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\ &\leq \frac{1.01}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\{ \sqrt{2\left[1+\mathcal{O}_{\blacklozenge}(\mathring{\alpha})\right] \sum_{A=2}^{3} \left|\sqrt{\mu}\not{d} Y_{(A)}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right|_{\not{g}}^2} \right\} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau' \\ &\quad + \frac{C}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left|\sqrt{\mu}\not{d}\mathcal{Y}^{N-1}\mathcal{R}_{(+)}\right|_{\not{g}} \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'. \end{aligned} \tag{29.42}$$

By combining the same arguments we used to bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of RHS (29.36) with (16.18) and Cor. 17.2, we find that when $\mathring{\alpha}$ and ε are sufficiently small, the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of RHS (29.42) is $\leq$ the sum of the $\boxed{2.89}$-multiplied term on RHS (29.32b) and the next-to-last term on RHS (29.32b). We have therefore obtained the desired bound for the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the product of $\frac{1}{\sqrt{\mu}}\breve{X}\mathcal{R}_{(+)}$ and the first term on RHS (22.31b). Similarly, by combining the pointwise bound $\frac{1}{\mu^{1/2}}|\breve{X}\mathcal{R}_{(+)}| \lesssim \frac{1}{|\tau|^{1/2}}$ noted above with (16.18), (18.1), and (20.54), we bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norm of the product of $\frac{1}{\mu^{1/2}}\breve{X}\mathcal{R}_{(+)}$ and the term $C_* \int_{\tau'=\tau_0}^{\tau} \left|\not\Delta \mathcal{Y}^{N-1}\vec{\Psi}_{(\mathrm{Partial})}\right| \circ {}^{(\mathfrak{n})}\widetilde{\Lambda}(\tau',u,x^2,x^3)\,\mathrm{d}\tau'$ from RHS (22.31b) by $\leq$ the sum of the C_*-multiplied term on RHS (29.32b) and the next-to-last term on RHS (29.32b). Finally, using the bound $\frac{1}{\mu^{1/2}}|\breve{X}\mathcal{R}_{(+)}| \lesssim \frac{1}{|\tau|^{1/2}}$ noted above, (16.18), and (18.1), we bound the $\|\cdot\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)}$ norms of the product of $\frac{1}{\mu^{1/2}}\breve{X}\mathcal{R}_{(+)}$ and the terms on the last line of RHS (22.31b) by $\leq \frac{C\varepsilon}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\|\sqrt{\mu}\mathcal{P}^{N+1}\vec{\Psi}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \mathrm{d}\tau'$ $+ \frac{C}{|\tau|^{1/2}} \int_{\tau'=\tau_0}^{\tau} \left\|\mathcal{P}^{[1,N]}\gamma\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}\right)} \mathrm{d}\tau'$, and using (20.53), (20.58), (25.1b), and the fact that the $\mathbb{Q}_M(\tau,u)$ are increasing in their arguments, we conclude that these time integrals are $\leq$ the sum of the non-boxed-constant-multiplied terms on RHS (29.32b) as desired.

Proof of (29.33a)–(29.33b)**:** These estimates can be proved using only a subset of the arguments we gave above; we omit the details, which are simpler since they *do not* involve sharp constants or delicate decompositions as in (29.34), and the LHSs of the estimates are less degenerate by a factor of $\sqrt{\mu}$ compared to (29.32a)–(29.32b). □

With the help of the preliminary estimates provided by Lemma 29.12, we are now ready to bound the most difficult error integral integrals that arise when we integrate by parts with respect to ${}^{(\mathfrak{n})}\widetilde{L}$ using the identity (20.15). Specifically, we bound the first two error integrals on RHS (20.15).

Lemma 29.13 (Estimates for difficult top-order error integrals related to integration by parts with respect to ${}^{(\mathfrak{n})}\widetilde{L}$)**.** *Let $N = N_{\mathrm{top}}$, and let $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$, where $\mathfrak{Y}^{(N)}$ is the set of order N $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Let $\mathcal{Y}^{N-1} \in \mathfrak{Y}^{(N-1)}$ be such that $\mathcal{Y}^N = Y_{(A)}\mathcal{Y}^{N-1}$ for some $Y_{(A)} \in \mathscr{Y}$, and let ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ be the corresponding partially modified* *quantity defined by* (19.7a). *Then the following estimates hold for* $(\tau,u)\in[\tau_0,\tau_{\text{Boot}})\times[-U_1,U_2]$:

$$\begin{aligned}
&\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}(1+2\mu)(Y_{(A)}\mathcal{Y}^N\mathcal{R}_{(+)})(\breve{X}\mathcal{R}_{(+)})^{(\mathfrak{n})}\widetilde{L}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\boldsymbol{\varpi}\right| \\
&\leq \boxed{4.13}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau' \\
&\quad + C_*\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',u)\left(\mathbb{Q}_N^{(\text{Partial})}\right)^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&\quad + \text{Error}_N^{(\text{Top})}(\tau,u),
\end{aligned} \tag{29.43}$$

$$\begin{aligned}
&\left|\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}}(1+2\mu)(Y_{(A)}\mathcal{Y}^N\mathcal{R}_{(+)})(\breve{X}\mathcal{R}_{(+)})^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\underline{\varpi}\right| \\
&\leq \boxed{4.13}\frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_N^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&\quad + C_*\frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\left(\mathbb{Q}_N^{(\text{Partial})}\right)^{1/2}(\tau',u)\,\mathrm{d}\tau' \\
&\quad + \text{Error}_N^{(\text{Top})}(\tau,u),
\end{aligned} \tag{29.44}$$

where $\text{Error}_N^{(\text{Top})}(\tau,u)$ *satisfies* (29.2).

Moreover, for every $\Psi\in\vec{\Psi}_{(\text{Partial})}=\{\mathcal{R}_{(-)},v^2,v^3,s\}$, *we have the following less degenerate estimates:*

$$\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}(1+2\mu)(Y_{(A)}\mathcal{Y}^N\Psi)(\breve{X}\Psi)^{(\mathfrak{n})}\widetilde{L}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\boldsymbol{\varpi}\right|\lesssim\text{Error}_N^{(\text{Top})}(\tau,u), \tag{29.45}$$

$$\left|\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}}(1+2\mu)(Y_{(A)}\mathcal{Y}^N\Psi)(\breve{X}\Psi)^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\underline{\varpi}\right|\lesssim\text{Error}_N^{(\text{Top})}(\tau,u). \tag{29.46}$$

Proof. We first prove (29.43). We start by noting the following estimates, which follow from the bootstrap assumptions: $|\breve{X}\mathcal{R}_{(+)}|\lesssim 1$, $|\mu|\lesssim 1$. Also using (13.1), Cor. 17.2, (18.1), and the Cauchy–Schwarz inequality for integrals, we bound LHS (29.43) by:

$$\begin{aligned}
&\leq(1+C_{\blacklozenge}\mathring{\alpha})\int_{\tau'=\tau_0}^{\tau}\left\|\sqrt{\mu}\,d\!\!/\,\mathcal{Y}^N\mathcal{R}_{(+)}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)}\left\|\frac{1}{\sqrt{\mu}}(\breve{X}\mathcal{R}_{(+)})^{(\mathfrak{n})}\widetilde{L}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)}\mathrm{d}\tau' \\
&\quad + C\int_{\tau'=\tau_0}^{\tau}\left\|\sqrt{\mu}\,d\!\!/\,\mathcal{Y}^N\mathcal{R}_{(+)}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)}\left\|{}^{(\mathfrak{n})}\widetilde{L}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)}\mathrm{d}\tau'.
\end{aligned} \tag{29.47}$$

The desired estimate (29.43) now follows from inserting (29.32a) and (29.33a) into the relevant factors in the integrals on RHS (29.47) and using Young's inequality in the form $ab\leq\frac{1}{f}a^2+fb^2$ (for appropriately chosen f) as well as the coerciveness estimate $\left\|\sqrt{\mu}\,d\!\!/\,\mathcal{Y}^N\mathcal{R}_{(+)}\right\|_{L^2({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]})}\leq\frac{1}{\sqrt{0.49}}\mathbb{Q}_N^{1/2}(\tau',u)$ (see (20.53)). We clarify that the factor $\boxed{4.13}>\frac{2.89}{\sqrt{0.49}}$ stems from the factor $\boxed{2.89}$ in the first term on RHS (29.32a), the factor $\frac{1}{\sqrt{0.49}}$ in the previous sentence, and our assumed smallness of $\mathring{\alpha}$. We further clarify that the integral $C\int_{\tau'=\tau_0}^{\tau}\mathring{\epsilon}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_N^{1/2}(\tau',u)\,\mathrm{d}\tau'$, which is generated by the last terms on RHSs (29.32a) and (29.32b), is $\lesssim\int_{\tau'=\tau_0}^{\tau}\left(\frac{\mathring{\epsilon}^2}{|\tau'|^{1/3}}+\frac{1}{|\tau|^{2/3}}\mathbb{Q}_N(\tau',u)\right)\mathrm{d}\tau'\lesssim\mathring{\epsilon}^2+\text{Error}_N^{(\text{Top})}(\tau,u)\lesssim\text{Error}_N^{(\text{Top})}(\tau,u)$.

The estimate (29.44) follows from arguments similar to the ones we used in proving (29.43), but we now rely on (29.32b) and (29.33b) in place of (29.32a) and (29.33a). We clarify that the last term on RHS (29.32b) generates the error term $C\mathring{\epsilon}\frac{1}{|\tau|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau,u)$, which, for any $\varsigma\in(0,1]$, by Young's inequality, we can bound by $\leq C\varsigma^{-1}\mathring{\epsilon}^2\frac{1}{|\tau|}+C\varsigma\mathbb{Q}_{[1,N]}(\tau,u)\lesssim\text{Error}_N^{(\text{Top})}(\tau,u)$. We omit the remaining details.

The estimates (29.45)–(29.46) follow from applying similar arguments that rely on the bounds (29.33a)–(29.33b). The desired estimates are in fact much simpler to deduce since there is an overall gain in smallness stemming from the bound $|\breve{X}\Psi| \lesssim \varepsilon$ for $\Psi \in \vec{\Psi}_{(\text{Partial})}$, which we proved in (17.11); we omit the details. □

Before proving the main estimates of this section, we first bound the remaining error integrals on the right-hand side of the integration by parts identity (20.15). The estimates are much easier to derive compared to the ones we established in Lemma 29.13. We split the analysis into two lemmas. In the next lemma, we bound the error integral involving the term Error on RHS (20.15).

Lemma 29.14 (Estimates for easy error integrals that arise during integration by parts with respect to ${}^{(\mathfrak{n})}\widetilde{L}$)**.** *Let $N = N_{\text{top}}$, and let $\mathcal{Y}^N \in \mathfrak{Y}^{(N)}$, where $\mathfrak{Y}^{(N)}$ is the set of order N $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Let $\mathcal{Y}^{N-1} \in \mathfrak{Y}^{(N-1)}$ be such that $\mathcal{Y}^N = Y_{(A)}\mathcal{Y}^{N-1}$ for some $Y_{(A)} \in \mathscr{Y}$. Let $\Psi \in \{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$, let ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ be the partially modified quantity defined by* (19.7a)*, and let* $\text{Error}[\Psi; {}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}; \mathcal{Y}^N; Y_{(A)}]$ *be the error term defined in* (20.16) *(with Ψ in the role of φ and ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ in the role of η). Then the following estimate holds for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \left|\text{Error}[\Psi; {}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}; \mathcal{Y}^N; Y_{(A)}]\right| \, d\varpi \lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}(\tau', u) \, d\tau' + \mathring{\varepsilon}^2. \tag{29.48}$$

In particular, the error integral on LHS (29.48) *is of type* $\text{Error}_N^{(\text{Top})}$*, i.e., it satisfies the bound* (29.2).

Proof. First, using the identity $\text{tr}_{g\!\!/}{}^{(Y_{(A)})}\pi\!\!\!/ = (g\!\!/^{-1})^{\alpha\beta} Y_{(A)} \mathbf{g}_{\alpha\beta} + 2\Pi\!\!\!/_{\kappa}{}^{\lambda} \partial_\lambda Y_{(A)}^{\kappa}$, Cor. 5.7, Prop. 9.1, the commutator estimate (13.7a), (18.1), (18.31), and the bootstrap assumptions, we deduce the following pointwise bound for the error term defined in (20.16):

$$\left|\text{Error}[\Psi; {}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}; \mathcal{Y}^N; Y_{(A)}]\right| \lesssim \frac{1}{|\tau|^{1/2}} \left|\sqrt{\mu}\mathcal{P}^{N+1}\Psi\right| \left|{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right|. \tag{29.49}$$

From (29.49) and the Cauchy–Schwarz inequality, we deduce that:

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} \left|\text{Error}[\Psi; {}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}; \mathcal{Y}^N; Y_{(A)}]\right| \, d\varpi \tag{29.50}$$

$$\lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \left\|\sqrt{\mu}\mathcal{P}^{N+1}\Psi\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} \left\|{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} d\tau'. \tag{29.51}$$

From (20.53), (29.33b), and Young's inequality, we conclude that RHS (29.50) $\lesssim$ RHS (29.48) as desired. □

The hypersurface error integrals that we treat in the next lemma appear on RHS (20.15). The integrals involve the $Y_{(A)}$-derivatives of the rough time function and are therefore new compared to earlier works on shocks, such as [24, 52, 73].

Lemma 29.15 (Estimates for additional hypersurface error terms related to integration by parts with respect to ${}^{(\mathfrak{n})}\widetilde{L}$)**.** *Assume that $N = N_{\text{top}}$, and let $\varsigma \in (0,1]$. Let $\Psi \in \{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$, and let ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ be the partially modified quantity defined by* (19.7a)*. Then the following estimates hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\begin{aligned}\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,u]}} \left|(Y_{(A)}{}^{(\mathfrak{n})}\tau)(1+2\mu)\breve{X}\Psi({}^{(\mathfrak{n})}\widetilde{L}\mathcal{Y}^N\Psi){}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right| d\underline{\varpi} &\lesssim \varepsilon \frac{1}{|\tau|^{1/2}} \mathbb{Q}_N^{1/2}(\tau, u) \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N]}(\tau', u) \, d\tau' \\ &\quad + \varepsilon \mathbb{Q}_N(\tau, u) + \mathring{\varepsilon}^2 \frac{1}{|\tau|},\end{aligned} \tag{29.52}$$

$$\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,u]}} \left|(Y_{(A)}{}^{(\mathfrak{n})}\tau)(1+2\mu)\breve{X}\Psi({}^{(\mathfrak{n})}\widetilde{L}\mathcal{Y}^N\vec{\Psi}){}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right| d\underline{\varpi} \lesssim \mathring{\varepsilon}^2 \frac{1}{|\tau_0|}. \tag{29.53}$$

In particular, the error integrals on LHSs (29.52)–(29.53) *are of type* $\text{Error}_N^{(\text{Top})}$*, i.e., they satisfy the bound* (29.2).

Proof. We first prove (29.52). We first use the bootstrap assumptions, Lemma 13.1, (15.11b), Cor. 17.2, and (18.1) to pointwise bound the integrand on LHS (29.52) by $\leq \varepsilon\frac{1}{|\tau|^{1/2}}\left|\sqrt{\mu}L\mathcal{P}^N\mathcal{R}_{(+)}\right|\left|{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right|$. From this estimate, the Cauchy–Schwarz inequality, (20.53), the first inequality in (29.33b), and Young's inequality, it follows that $|\text{LHS (29.52)}| \lesssim \mathring{\varepsilon}^2\frac{1}{|\tau|} +$

$\varepsilon\mathbb{Q}_N(\tau,u)+\varepsilon\frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N]}^{1/2}(\tau',u)\,\mathrm{d}\tau'$, which is in turn bounded by RHS (29.2), i.e., this term is of type $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ as desired.

To prove (29.53), we note that the integral on LHS (29.53) is a data integral that, by virtue of the arguments we used to prove (29.52), but now with τ_0 in the role of τ, can be bounded by $\lesssim \mathring{\epsilon}^2\frac{1}{|\tau_0|}+\mathbb{Q}_{[1,N]}(\tau_0,u)$. Using (24.7), we see that the RHS of the previous expression is $\lesssim \mathring{\epsilon}^2\frac{1}{|\tau_0|}+\mathring{\epsilon}^2\lesssim\mathring{\epsilon}^2\frac{1}{|\tau_0|}$, which, in view of the fact that $|\tau_0|\geq|\tau|$ for $\tau\in[\tau_0,\tau_{\mathrm{Boot}})$, is bounded by RHS (29.2) as desired.

□

We are now ready to combine the results of the lemmas established above to obtain the main estimates for the top-order error integrals highlighted in (29.23b).

Lemma 29.16 (The main estimates for difficult top-order spacetime error integrals requiring integration by parts in ${}^{(\mathfrak{n})}\widetilde{L}$)**.** *Let $N=N_{\mathrm{top}}$, and let $\mathcal{Y}^N\in\mathfrak{Y}^{(N)}$, where $\mathfrak{Y}^{(N)}$ is the set of order N $\ell_{t,u}$-tangential commutator operators from Def. 8.10. Then the following estimates hold for $(\tau,u)\in[\tau_0,\tau_{\mathrm{Boot}})\times[-U_1,U_2]$:*

$$\begin{aligned}
&\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}(1+2\mu)(L\mathcal{Y}^N\mathcal{R}_{(+)})(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\,\mathrm{d}\varpi\right|\\
&\leq\boxed{4.13}\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N(\tau',u)\,\mathrm{d}\tau'\\
&\quad+\boxed{4.13}\frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_N^{1/2}(\tau',u)\,\mathrm{d}\tau'\\
&\quad+C_*\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',u)\left(\mathbb{Q}_N^{(\mathrm{Partial})}\right)^{1/2}(\tau',u)\,\mathrm{d}\tau'\\
&\quad+C_*\frac{1}{|\tau|^{1/2}}\mathbb{Q}_N^{1/2}(\tau,u)\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\left(\mathbb{Q}_N^{(\mathrm{Partial})}\right)^{1/2}(\tau',u)\,\mathrm{d}\tau'\\
&\quad+\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u),
\end{aligned}\tag{29.54}$$

where $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ *satisfies* (29.2).

Moreover, for every $\Psi\in\vec{\Psi}_{(\mathrm{Partial})}=\{\mathcal{R}_{(-)},v^2,v^3,s\}$, we have the following less degenerate estimates:

$$\left|\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}(1+2\mu)(L\mathcal{Y}^N\Psi)(\breve{X}\Psi)\mathcal{Y}^N\mathrm{tr}_{\not g}\chi\,\mathrm{d}\varpi\right|\lesssim\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u),\tag{29.55}$$

where $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ *satisfies* (29.2).

Proof. We first prove (29.54). The operator $\mathcal{Y}^N$ on LHS (29.54) is of the form $\mathcal{Y}^N=Y_{(A)}\mathcal{Y}^{N-1}$ for some $A\in\{2,3\}$. We now use (19.7a) to decompose the factor $\mathcal{Y}^N\mathrm{tr}_{\not g}\chi$ on LHS (29.54) as follows: $\mathcal{Y}^N\mathrm{tr}_{\not g}\chi=Y_{(A)}\mathcal{Y}^{N-1}\mathrm{tr}_{\not g}\chi=Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}-Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}$. We insert this decomposition into LHS (29.54) and will handle each of the two integrals separately, starting with the one generated by the piece $Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}$, which is easier. Specifically, the pointwise estimate (22.5b), the estimate $|\breve{X}\mathcal{R}_{(+)}|\lesssim 1$ (see (17.12)), and Def. 22.1 imply that the product $(\breve{X}\mathcal{R}_{(+)})Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}$ is of type $\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N]}$. Hence, the estimate (29.6) implies that the corresponding integral $-\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)L\mathcal{Y}^N\mathcal{R}_{(+)}\right\}\left\{(\breve{X}\mathcal{R}_{(+)})Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathfrak{X}}\right\}\mathrm{d}\varpi$ is of type $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ as desired.

To complete the proof of (29.54), it remains for us to bound the following spacetime integral:

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}}\frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)L\mathcal{Y}^N\mathcal{R}_{(+)}\right\}\left\{(\breve{X}\mathcal{R}_{(+)})Y_{(A)}{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\right\}\mathrm{d}\varpi.\tag{29.56}$$

To bound (29.56), we first integrate by parts using the identity (20.15) with $\mathcal{R}_{(+)}$ in the role of φ, ${}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}$ in the role of η, and $\mathcal{Y}^N$ in the role of $\mathcal{P}^N$. The first two integrals on RHS (20.15), namely

$$\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,u]}} (1+2\mu)(\check{X}\mathcal{R}_{(+)})(Y_{(A)}\mathcal{Y}^N\mathcal{R}_{(+)})^{(\mathfrak{n})}\widetilde{L}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\varpi,$$
$$-\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}} (1+2\mu)(\check{X}\mathcal{R}_{(+)})(Y_{(A)}\mathcal{Y}^N\mathcal{R}_{(+)})^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\underline{\varpi},$$

are the main ones, and in (29.43)–(29.44), we showed that they are bounded in magnitude by $\leq$ RHS (29.54) as desired. The third and fifth integrals on RHS (20.15), namely $\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,u]}}(Y_{(A)}{}^{(\mathfrak{n})}\tau)(1+2\mu)(\check{X}\mathcal{R}_{(+)})({}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^N\mathcal{R}_{(+)})^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\underline{\varpi}$ and $-\int_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,u]}}(Y_{(A)}{}^{(\mathfrak{n})}\tau)(1+2\mu)(\check{X}\mathcal{R}_{(+)})({}^{(\mathfrak{n})}\widetilde{L}\mathcal{P}^N\mathcal{R}_{(+)})^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}}\,\mathrm{d}\underline{\varpi}$, were shown to be of type $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ in Lemma 29.15. The last integral on RHS (20.15), whose integrand is $\mathrm{Error}[\mathcal{R}_{(+)};{}^{(\mathcal{Y}^{N-1})}\widetilde{\mathscr{X}};\mathcal{Y}^N;Y_{(A)}]$, was shown to be of type $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ in Lemma 29.14.

The estimate (29.55) can be proved using nearly identical arguments. The difference compared to (29.54) is that the integrand factor $\check{X}\Psi$ on LHS (29.55) satisfies the pointwise bound $|\check{X}\Psi|\lesssim\varepsilon$ (see (17.11)); this provides a smallness factor that allows us to relegate all the terms to the error term $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ on RHS (29.55).

□

29.6. **Estimates for error integrals involving a loss of one derivative.** Prop. 24.1 states that the wave energies become less singular by a factor of $|\tau|^2$ at every level of descent below top-order, until one reaches a level where the energies are bounded. Our proof of this "descent scheme" for below-top-order estimates relies on bounding the difficult error integrals generated by the $\mathcal{Y}^N\mathrm{tr}_{\not{g}}\chi$-involving products on RHSs (22.3a)–(22.3b) in a different way. That is, below top-order, we control the $\mathcal{Y}^N\mathrm{tr}_{\not{g}}\chi$ factors via transport equation estimates that *lose one derivative*; this is very different compared to, for example, the top-order error integrals we bounded in Lemmas 29.11 and 29.16, where we could not afford any derivative loss. While losing one derivative is permissible below top-order, this approach couples the below top-order energy estimates to the top-order ones. The main merit of this approach is that it leads to estimates that are less singular with respect to powers of $|\tau|$, which ultimately allows us to implement the energy estimate descent scheme. In the next lemma, we prove the main estimates for below top-order error integrals that lose one derivative.

Lemma 29.17 (Estimates for wave equation error integrals involving a loss of one derivative)**.** *Assume that* $2\leq N\leq N_{\mathrm{top}}$ *and* $\Psi\in\vec{\Psi}=\{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^2,s\}$, *and recall that* $\check{T}$ *is the multiplier vectorfield defined in* (20.22). *Then the following estimates hold for* $(\tau,u)\in[\tau_0,\tau_{\mathrm{Boot}})\times[-U_1,U_2]$*:*

$$\begin{aligned}
&\int_{{}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left|\check{T}\mathcal{P}^{N-1}\Psi\right|\left|\begin{pmatrix}(\check{X}\Psi)\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi\\ (\not{d}^{\#}\Psi)\cdot\mu\not{d}\mathcal{P}^{N-2}\mathrm{tr}_{\not{g}}\chi\\ c^{-2}X^A(\not{d}^{\#}\Psi)\cdot\mu\not{d}\mathcal{P}^{N-2}\mathrm{tr}_{\not{g}}\chi\end{pmatrix}\right|_{\not{g}}\mathrm{d}\varpi\\
&\lesssim\int_{\tau'=\tau_0}^{\tau}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau',u)\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{Q}_N^{1/2}(\tau'',u)\,\mathrm{d}\tau''\mathrm{d}\tau'\\
&\quad+\mathrm{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u),
\end{aligned}\tag{29.57}$$

where $\mathrm{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u)$ *satisfies* (29.5) *with* $N-1$ *in the role of* M.

Proof. We prove (29.57) only for the term generated by the first entry $(\check{X}\Psi)\mathcal{P}^{N-1}\mathrm{tr}_{\not{g}}\chi$ on LHS (29.57); the remaining terms on the LHS are easier to estimate because they enjoy an additional power of μ, and we omit the details. Since $\check{T}\mathcal{P}^{N-1}\Psi=(1+2\mu)L\mathcal{P}^{N-1}\Psi+2\check{X}\mathcal{P}^{N-1}\Psi$, we can use (18.9b), (20.53) (25.1b), the bootstrap assumptions, Young's inequality, and the fact that the $\mathbb{Q}_M(\tau,u)$ are increasing in their arguments to deduce:

$$
\begin{aligned}
&\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left|\breve{T}\mathcal{P}^{N-1}\Psi\right| \left|(\breve{X}\Psi)\mathcal{P}^{N-1}\mathrm{tr}_{\not g}\chi\right| \mathrm{d}\varpi \\
&\lesssim \int_{\tau'=\tau_0}^{\tau} \left\{ \left\|L\mathcal{P}^{N-1}\Psi\right\|_{L^2\left(^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} + \left\|\breve{X}\mathcal{P}^{N-1}\Psi\right\|_{L^2\left(^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} \right\} \left\|\mathcal{P}^{N-1}\mathrm{tr}_{\not g}\chi\right\|_{L^2\left(^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,u]}\right)} \mathrm{d}\tau' \\
&\lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N-1]}^{1/2}(\tau',u) \left\{ \mathring{\epsilon} + \int_{\tau''=\tau_0}^{\tau'} \frac{\mathbb{Q}_{[1,N]}^{1/2}(\tau'',u)}{|\tau''|^{1/2}} \mathrm{d}\tau'' \right\} \mathrm{d}\tau' \\
&\lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N-1]}^{1/2}(\tau',u) \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \mathbb{Q}_N^{1/2}(\tau'',u)\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
&\quad + \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N-1]}(\tau',u)\,\mathrm{d}\tau' + \mathring{\epsilon}^2 \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}}\,\mathrm{d}\tau'.
\end{aligned}
\tag{29.58}
$$

Noting that the last term on RHS (29.58) is $\lesssim \mathring{\epsilon}^2$, we conclude that RHS (29.58) $\lesssim$ RHS (29.57) as desired.

□

29.7. **Proof of Prop. 29.1.** We now prove the main integral inequalities for the wave-variables, i.e., Prop. 29.1.

Proof of (29.1)**:** We first prove the top-order estimate (29.1). Let $N = N_{\mathrm{top}}$ and $\Psi \in \{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$, and let $\mathfrak{G}$ denote μ times the inhomogeneous term in the geometric wave equation (2.22) satisfied by Ψ, i.e. $\mu\square_{\mathbf{g}}\Psi = \mathfrak{G}$. Fix any $\mathcal{P}^N \in \mathfrak{P}^{(N)}$, where $\mathfrak{P}^{(N)}$ is defined in Def. 8.10, and let $(\tau,u) \in [\tau_0,\tau_{\mathrm{Boot}}) \times [-U_1,U_2]$. The starting point of the proof is the fundamental energy–null-flux identity (20.26) with $f \overset{\mathrm{def}}{=} \mathcal{P}^N\Psi$:

$$
\begin{aligned}
&\mathbb{E}_{(\mathrm{Wave})}[\mathcal{P}^N\Psi](\tau,u) + \mathbb{F}_{(\mathrm{Wave})}[\mathcal{P}^N\Psi](\tau,u) + \mathbb{K}[\mathcal{P}^N\Psi](\tau,u) \\
&= \mathbb{E}_{(\mathrm{Wave})}[\mathcal{P}^N\Psi](\tau_0,u) + \mathbb{F}_{(\mathrm{Wave})}[\mathcal{P}^N\Psi](\tau,-U_1) \\
&\quad + \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} {}^{(\breve{T})}\mathfrak{B}[\mathcal{P}^N\Psi]\,\mathrm{d}\varpi \\
&\quad - \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left\{(1+2\mu)L\mathcal{P}^N\Psi + 2\breve{X}\mathcal{P}^N\Psi\right\} \mu\square_{\mathbf{g}}\mathcal{P}^N\Psi\,\mathrm{d}\varpi.
\end{aligned}
\tag{29.59}
$$

We will show that RHS (29.59) $\leq$ RHS (29.1), which is the difficult step. After that, we can take the supremum of the resulting estimate over the relevant values of τ and u, then take the maximum over $\Psi \in \{\mathcal{R}_{(+)},\mathcal{R}_{(-)},v^2,v^3,s\}$ and over all $\mathscr{P}^N \in \mathfrak{P}^{(N)}$, and appeal to definitions (20.43a)–(20.43c), finally concluding the desired top-order estimate (29.1).

To complete the proof of (29.1), it remains for us to show that RHS (29.59) $\leq$ RHS (29.1). In the rest of the proof, $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ denotes a term of type $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ on RHS (29.1), i.e., any term that satisfies (29.2). First, using (24.7), we see that the initial data energy and null-flux terms on RHS (29.59) satisfy $\mathbb{E}_{(\mathrm{Wave})}[\mathcal{P}^N\Psi](\tau_0,u)+\mathbb{F}_{(\mathrm{Wave})}[\mathcal{P}^N\Psi](\tau,-U_1) \lesssim \mathring{\epsilon}^2 = \mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$ as desired.

Next, using Lemma 29.6, we see that the ${}^{(\breve{T})}\mathfrak{B}[\mathcal{P}^N\Psi]$-involving integral on RHS (29.59) is type $\mathrm{Error}_N^{(\mathrm{Sub\text{-}critical})}(\tau,u)$ (and hence of type $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$) as desired.

It remains for us to bound the last integral $-\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left\{(1+2\mu)L\mathcal{P}^N\Psi + 2\breve{X}\mathcal{P}^N\Psi\right\} \mu\square_{\mathbf{g}}\mathcal{P}^N\Psi\,\mathrm{d}\varpi$ on RHS (29.59). We split the argument into Steps 1, 2, 3A, and 3B. We stress that *none of the error integrals we have treated thus far and none of the error integrals that we treat in Steps 1, 2, or 3B generate the boxed-constant-multiplied integrals or C_*-multiplied integrals on RHS* (29.1)*; they are generated only in Step 3A.*

Step 1: the case $\mathcal{P}^N \notin \left\{\mathcal{Y}^{N-1}L, \mathcal{Y}^N\right\}$. If $\mathcal{P}^N$ is any string of $\mathcal{P}_u$-tangential commutation vectorfields other than $\mathcal{Y}^{N-1}L$ or $\mathcal{Y}^N$, then by substituting (22.3c) for $\mu\square_{\mathbf{g}}\mathcal{P}^N\Psi$ on RHS (29.59) and using Lemma 29.4 to handle the $\mathrm{Harmless}_{(\mathrm{Wave})}^{[1,N_{\mathrm{top}}]}$ terms and Lemma 29.5 to handle the terms generated by $\mathfrak{G}$, we find that the corresponding error integrals are of type $\mathrm{Error}_N^{(\mathrm{Top})}(\tau,u)$. In total, we have shown that, except for the cases of $\mathcal{P}^N = \mathcal{Y}^{N-1}L$ or $\mathcal{P}^N = \mathcal{Y}^N$, all error integrals on

RHS (29.59) are of type $\text{Error}_N^{(\text{Top})}(\tau, u)$. In particular, *none of the integrals handled thus far generate the boxed-constant-multiplied integrals or C_*-multiplied integrals on RHS* (29.1).

Step 2: the case $\mathcal{P}^N = \mathcal{Y}^{N-1}L$. We now consider the last integral on RHS (29.59) in the case $\mathcal{P}^N = \mathcal{Y}^{N-1}L$. We use (22.3a) to substitute for $\mu\Box_{\mathbf{g}}\mathcal{P}^N\Psi$ on RHS (29.59). All error integrals except for the one generated by the first product on RHS (22.3a) can be handled using the same arguments given in Step 1. The error integral generated by the first product on RHS (22.3a) is:

$$\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)L\mathcal{Y}^N\Psi + 2\breve{X}\mathcal{Y}^N\Psi\right\}(\not{d}^{\#}\Psi)\cdot\mu\not{d}\mathcal{Y}^{N-1}\mathrm{tr}_{\not{g}}\chi\,\mathrm{d}\varpi, \tag{29.60}$$

and the estimate (29.20) implies that the integral is also of type $\text{Error}_N^{(\text{Top})}(\tau, u)$.

Step 3A: the case $\mathcal{P}^N = \mathcal{Y}^N$ and $\Psi = \mathcal{R}_{(+)}$. We now consider the most difficult case, $\mathcal{P}^N = \mathcal{Y}^N$ and $\Psi = \mathcal{R}_{(+)}$. We substitute RHS (22.3b) for $\mu\Box_{\mathbf{g}}\mathcal{Y}^N\Psi$ on RHS (29.59). All error integrals except for the ones generated by the first two products on RHS (22.3b) can be handled using the same arguments given in Step 1. The error integral generated by the second product on RHS (22.3a) is:

$$\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)L\mathcal{Y}^N\mathcal{R}_{(+)} + 2\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right\}(c^{-2}X^A)(\not{d}^{\#}\mathcal{R}_{(+)})\cdot\mu\not{d}\mathcal{Y}^{N-1}\mathrm{tr}_{\not{g}}\chi\,\mathrm{d}\varpi, \tag{29.61}$$

and the estimate (29.20) implies that the integral is also of type $\text{Error}_N^{(\text{Top})}(\tau, u)$.

The first product on RHS (22.3a) generates the following two difficult error integrals:

$$\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)L\mathcal{Y}^N\mathcal{R}_{(+)}\right\}(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^{N-1}Y_{(A)}\mathrm{tr}_{\not{g}}\chi\,\mathrm{d}\varpi, \tag{29.62}$$

$$2\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left\{\breve{X}\mathcal{Y}^N\mathcal{R}_{(+)}\right\}(\breve{X}\mathcal{R}_{(+)})\mathcal{Y}^{N-1}Y_{(A)}\mathrm{tr}_{\not{g}}\chi\,\mathrm{d}\varpi, \tag{29.63}$$

and in (29.54) and (29.25) respectively, we showed that the integrals (29.62)–(29.63) are bounded in magnitude by $\leq$ RHS (29.1) as desired. *It is precisely this step that generates all the boxed-constant-multiplied integrals and C_*-multiplied integrals on RHS* (29.1).

Step 3B: the case $\mathcal{P}^N = \mathcal{Y}^N$ and $\Psi \in \{\mathcal{R}_{(-)}, v^2, v^3, s\}$. This case can be handled as in Step 3A, but the analogs of the error integrals (29.62)–(29.63), specifically the following integrals:

$$\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left\{(1+2\mu)L\mathcal{Y}^N\Psi\right\}(\breve{X}\Psi)\mathcal{Y}^{N-1}Y_{(A)}\mathrm{tr}_{\not{g}}\chi\,\mathrm{d}\varpi, \tag{29.64}$$

$$2\int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau}\left\{\breve{X}\mathcal{Y}^N\Psi\right\}(\breve{X}\Psi)\mathcal{Y}^{N-1}Y_{(A)}\mathrm{tr}_{\not{g}}\chi\,\mathrm{d}\varpi, \tag{29.65}$$

where $\Psi \in \{\mathcal{R}_{(-)}, v^2, v^3, s\}$, can be bounded in magnitude via the less degenerate estimates (29.55) and (29.26). In particular, all error integrals that we encounter in Step 3B are of type $\text{Error}_N^{(\text{Top})}(\tau, u)$.

We have therefore proved (29.1).

Proof of (29.3)**:** We now prove the top-order estimate (29.3). The proof mirrors the proof of (29.1), except that, in view of definitions (20.44a)–(20.44c), we do not have to derive energy estimates for $\mathcal{R}_{(+)}$. Consequently, the proof of (29.3) does not involve the difficult error integrals (29.62)–(29.63) from Step 3A, which are the only error integrals that generate the difficult boxed-constant-involving terms. This explains why there are no such boxed-constant-involving terms on RHS (29.3). In particular, all error integrals that we encounter in the proof of (29.3) are of type $\text{Error}_N^{(\text{Top})}(\tau, u)$.

Proof of (29.4)**:** We now prove the below-top-order estimates (29.4). We fix any $\Psi \in \{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$, and we consider the integral identity (29.59) with N' in the role of N, i.e.,

$$\begin{aligned}
&\mathbb{E}[\mathcal{P}^{N'}\Psi]_{(\mathrm{Wave})}(\tau,u) + \mathbb{F}[\mathcal{P}^{N'}\Psi]_{(\mathrm{Wave})}(\tau,u) + \mathbb{K}[\mathcal{P}^{N'}\Psi](\tau,u) \\
&= \mathbb{E}[\mathcal{P}^{N'}\Psi]_{(\mathrm{Wave})}(\tau_0,u) + \mathbb{F}[\mathcal{P}^{N'}\Psi]_{(\mathrm{Wave})}(\tau,-U_1) \\
&\quad + \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} {}^{(\check{T})}\mathfrak{B}[\mathcal{P}^{N'}\Psi]\,d\varpi \\
&\quad - \int_{^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau],[-U_1,u]}} \frac{1}{L^{(\mathfrak{n})}\tau} \left\{(1+2\mu)L\mathcal{P}^{N'}\Psi + 2\check{X}\mathcal{P}^{N'}\Psi\right\} \mu\square_{\mathbf{g}}\mathcal{P}^{N'}\Psi\,d\varpi.
\end{aligned} \tag{29.66}$$

We will show that if $2 \leq N \leq N_{\mathrm{top}}$ and $1 \leq N' \leq N-1$, then RHS (29.66) $\leq$ RHS (29.4). Then for the same reasons given just below (29.59), we see, taking into account the Def. 20.12 of $\mathbb{W}_{[1,N-1]}(\tau,u)$, that this implies the desired estimate (29.4). The analogs of all the estimates through Step 1 above can be carried out just as before; the arguments we have given show that since $1 \leq N' \leq N-1$, all the corresponding error integrals are of type $\mathrm{Error}_{N-1}^{(\text{Sub-critical})}$, i.e., that they are $\lesssim$ RHS (29.5) with $N-1$ in the role of M in (29.5). The big difference occurs in Steps 2 and 3, where we now use a different method to control the error integrals generated by the terms on RHSs (22.3a)–(22.3b) that explicitly depend on $\mathrm{tr}_{g\!\!/}\chi$. More precisely, we control them all using the derivative-losing estimate (29.57), which shows that they are bounded in magnitude by:

$$\lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N']}^{1/2}(\tau',u) \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \mathbb{Q}_{N'+1}^{1/2}(\tau'',u)\,d\tau''\,d\tau' + \mathrm{Error}_{N'}^{(\text{Sub-critical})}(\tau,u). \tag{29.67}$$

Since $1 \leq N' \leq N-1$, we have $\mathrm{Error}_{N'}^{(\text{Sub-critical})} \lesssim \mathrm{Error}_{N-1}^{(\text{Sub-critical})}$, which is $\leq$ RHS (29.4) as desired. We handle the remaining double integral term in (29.67) by splitting it into two cases, the first being $N' = N-1$. Then the double integral is bounded by the first term on RHS (29.4). In the second case, which is $1 \leq N' \leq N-2$, using the fact that the $\mathbb{Q}_M(\tau,u)$ are increasing in their arguments, we see that the double time integral in (29.67) is $\lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N-2]}^{1/2}(\tau',u) \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \mathbb{Q}_{[1,N-1]}^{1/2}(\tau'',u)\,d\tau''\,d\tau' \lesssim \int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{1/2}} \mathbb{Q}_{[1,N-1]}(\tau',u)\,d\tau'$, which in turn is of type $\mathrm{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u)$, i.e., it is $\lesssim$ RHS (29.5) with $N-1$ in the role of M in (29.5), as desired. We have therefore proved (29.4), which finishes the proof of the proposition.

□

29.7.1. *The proof of Prop. 24.1.* In this section, we prove Prop. 24.1, which provides the main a priori estimates for the wave-variables. Throughout the proof, we will refer to the L^2-controlling quantities defined in Defs. 20.10 and 20.12. Our proof relies on the a priori estimates for the transport-variables from Prop. 24.2, which we already proved in Sects. 26 and 27. To help the reader follow the global structure of the paper, we recall that our proof of Prop. 24.2 relied on the wave energy bootstrap assumptions (24.12a)–(24.12b), and that the conclusions of Prop. 24.1 yield strict improvements of those bootstrap assumptions.

Estimates for $\mathbb{W}_{N_{\mathrm{top}}}$, $\mathbb{W}_{N_{\mathrm{top}}}^{(\mathrm{Partial})}$, and $\mathbb{W}_{[1,N_{\mathrm{top}}-1]}$.

The setup: The proof of the a priori estimates for $\mathbb{W}_{N_{\mathrm{top}}}$ is coupled to the ones for $\mathbb{W}_{N_{\mathrm{top}}}^{(\mathrm{Partial})}$ and $\mathbb{W}_{[1,N_{\mathrm{top}}-1]}$. Hence, we prove the a priori estimates for all three energies simultaneously via a coupled Grönwall argument. To start, we set:

$$\mathscr{F}(\tau,u) \overset{\mathrm{def}}{=} \sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]} \iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u}) \mathbb{W}_{N_{\mathrm{top}}}(\hat{\tau},\hat{u}) \tag{29.68a}$$

$$\mathscr{G}(\tau,u) \overset{\mathrm{def}}{=} \sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]} \iota_{\mathscr{G}}^{-1}(\hat{\tau},\hat{u}) \mathbb{W}_{N_{\mathrm{top}}}^{(\mathrm{Partial})}(\hat{\tau},\hat{u}), \tag{29.68b}$$

$$\mathscr{H}(\tau,u) \overset{\mathrm{def}}{=} \sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]} \iota_{\mathscr{H}}^{-1}(\hat{\tau},\hat{u}) \mathbb{W}_{[1,N_{\mathrm{top}}-1]}(\hat{\tau},\hat{u}), \tag{29.68c}$$

where:

$$\mathfrak{l}(\tau) \stackrel{\text{def}}{=} \exp\left(\int_{\tau'=\tau_0}^{\tau} \frac{1}{|\tau'|^{9/10}}\,\mathrm{d}\tau'\right), \tag{29.69a}$$

$$\iota_{\mathscr{F}}(\tau,u) = \iota_{\mathscr{G}}(\tau,u) \stackrel{\text{def}}{=} |\tau|^{-15.6}\mathfrak{l}^{\mathfrak{c}}(\tau)e^{\mathfrak{c}(u+U_1)}, \tag{29.69b}$$

$$\iota_{\mathscr{H}}(\tau,u) \stackrel{\text{def}}{=} |\tau|^{-13.6}\mathfrak{l}^{\mathfrak{c}}(\tau)e^{\mathfrak{c}(u+U_1)}, \tag{29.69c}$$

and $\mathfrak{c}$ is a sufficiently large positive constant that we choose below. For future use, we note that when $\mathfrak{c} \geq 1$ is fixed, the functions $\mathfrak{l}^{\mathfrak{c}}(\tau)$ and $e^{\mathfrak{c}(u+U_1)}$ are uniformly bounded from above by a positive, $\mathfrak{c}$-dependent constant for $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$. Moreover, we will silently use that for all $\mathfrak{c} \geq 1$ and $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$, $\mathfrak{l}^{\mathfrak{c}}(\tau)$ and $e^{\mathfrak{c}(u+U_1)}$ are uniformly bounded from below by 1. Moreover, in view of our assumption (10.7) and the fact that $\tau_{\text{Boot}} \leq 0$, we see that for $\tau \in [\tau_0,\tau_{\text{Boot}})$, $|\tau|$ is uniformly bounded from above by 1. We will also silently use the basic facts that the functions $\tau \to \mathfrak{l}^{\mathfrak{c}}(\tau)$ and $u \to e^{\mathfrak{c}(u+U_1)}$ are increasing. Finally, we will also use the following estimates, whose straightforward proofs we omit:

$$\int_{\tau'=\tau_0}^{\tau} \frac{\mathfrak{l}^{\mathfrak{c}}(\tau')}{|\tau'|^{9/10}}\,\mathrm{d}\tau' \leq \frac{1}{\mathfrak{c}}\mathfrak{l}^{\mathfrak{c}}(\tau), \qquad \int_{u'=-U_1}^{u} e^{\mathfrak{c}(u'+U_1)}\,\mathrm{d}u' \leq \frac{1}{\mathfrak{c}}e^{\mathfrak{c}(u+U_1)}. \tag{29.70}$$

From the above discussion, it follows that the top-order estimate (i.e., (24.1a) with $K \stackrel{\text{def}}{=} 0$) and the just-below-top-order estimate (i.e., (24.1a) with $K \stackrel{\text{def}}{=} 1$) follow once we prove that there is a uniform $C > 0$, **independent** of all $\mathfrak{c} \geq 1$ and all sufficiently small $\varsigma \in (0,1]$, and a $\mathfrak{c} \gg 1$ such that the following estimates hold for $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$:

$$\mathscr{F}(\tau,u) \leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2, \qquad \mathscr{G}(\tau,u) \leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2, \qquad \mathscr{H}(\tau,u) \leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2. \tag{29.71}$$

We clarify that even though the constants C in (29.71) are independent of ς and $\mathfrak{c}$, the constants on RHS (24.1a) can depend on ς and, in view of definitions (29.68a)–(29.68c) and (29.70), on $\mathfrak{c}$ as well. We further clarify that our final choice of ς and $\mathfrak{c}$ will not be made until the very end of the proof. The reason is that later on, we will use a downward induction scheme to obtain the lower order estimates, and that scheme could in principle require choosing ς to be smaller and $\mathfrak{c}$ to be larger at each step. Moreover, for convenience, in the proof, we will set $\mathfrak{c} \stackrel{\text{def}}{=} \varsigma^{-2}$, so that our final choice of $\mathfrak{c}$ will in fact be determined by choosing and fixing $\varsigma \in (0,1]$ to be sufficiently small, where the final choice of ς will not be made until the very end of the proof.

To prove (29.71), we will show that there is a uniform $C > 0$ such that for every $\varsigma \in (0,1]$, $\mathfrak{c} \geq 1$, sufficiently small $\varepsilon \geq 0$, and $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$, the following estimates hold:

$$\begin{aligned}
\mathscr{F}(\tau,u) &\leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2 \\
&\quad + \left\{\frac{\frac{4\times 1.01}{1.99}+4.13}{15.6} + \frac{\frac{8(1.01)^2}{1.99}}{15.6\times 7.8} + \frac{4.13}{7.3} + C\varepsilon + C\varsigma + \frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\mathscr{F}(\tau,u) \\
&\quad + C\left\{\varepsilon+\varsigma+\frac{1}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\mathscr{H}(\tau,u) + C\mathscr{F}^{1/2}(\tau,u)\mathscr{G}^{1/2}(\tau,u),
\end{aligned} \tag{29.72}$$

$$\begin{aligned}
\mathscr{G}(\tau,u) &\leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2 + C\left\{\varepsilon+\varsigma+\frac{1}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\mathscr{F}(\tau,u) \\
&\quad + C\left\{\varepsilon+\varsigma+\frac{1}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\mathscr{H}(\tau,u),
\end{aligned} \tag{29.73}$$

$$\mathscr{H}(\tau,u) \leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2 + C\mathscr{F}(\tau,u) + \left\{\frac{1}{2}+C\varsigma+\frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\mathscr{H}(\tau,u). \tag{29.74}$$

Before proving (29.72)–(29.74), we first show that these estimates imply (29.71). To see this, we set (for convenience) $\mathfrak{c} \stackrel{\text{def}}{=} \varsigma^{-2}$. Also noting that $\frac{\frac{4\times 1.01}{1.99}+4.13}{15.6} + \frac{\frac{8(1.01)^2}{1.99}}{15.6\times 7.8} + \frac{4.13}{7.3} < .995 < 1$, we see that for all sufficiently small $\varsigma > 0$ and $\varepsilon \geq 0$, we can soak the second product on RHS (29.72) back into LHS (29.72) and soak the last factor on RHS (29.74) back into

LHS (29.74), thereby deducing:

$$\mathscr{F}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2 + C\left\{\epsilon + \varsigma + \underbrace{\frac{1}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)}_{\varsigma^2+\varsigma}\right\}\mathscr{H}(\tau,u) + C\mathscr{F}^{1/2}(\tau,u)\mathscr{G}^{1/2}(\tau,u) \tag{29.75}$$

and:

$$\mathscr{H}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2 + C\mathscr{F}(\tau,u). \tag{29.76}$$

Inserting (29.76) estimate into RHS (29.73), we find that for all sufficiently small $\varsigma > 0$ and $\epsilon \ge 0$, we have:

$$\mathscr{G}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2 + C\left\{\epsilon+\varsigma\right\}\mathscr{F}(\tau,u). \tag{29.77}$$

Next, using Young's inequality, we bound the last product on RHS (29.75) as follows: $C\mathscr{F}^{1/2}(\tau,u)\mathscr{G}^{1/2}(\tau,u) \le \frac{1}{2}\mathscr{F}(\tau,u) + C\mathscr{G}(\tau,u)$. The term $\frac{1}{2}\mathscr{F}(\tau,u)$ can be absorbed back into LHS (29.75), which yields:

$$\mathscr{F}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2 + C\left\{\epsilon + \varsigma + \underbrace{\frac{1}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)}_{\varsigma^2+\varsigma}\right\}\mathscr{H}(\tau,u) + C\mathscr{G}(\tau,u). \tag{29.78}$$

Inserting (29.76)–(29.77) into (29.78), we find that:

$$\mathscr{F}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2 + C\left\{\epsilon+\varsigma\right\}\mathscr{F}(\tau,u). \tag{29.79}$$

Hence, if $\varsigma > 0$ is sufficiently small, then for all sufficiently small $\epsilon \ge 0$, we can absorb the last product on RHS (29.79) back into the LHS. This implies the desired bound $\mathscr{F}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2$. Inserting this bound into RHSs (29.76)–(29.77), we find that $\mathscr{G}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2$ and $\mathscr{H}(\tau,u) \le C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2$. We have therefore proved (29.71).

Proof of (29.72)–(29.74): It remains for us to prove (29.72)–(29.74). We set $N \stackrel{\text{def}}{=} N_{\text{top}}$. We fix any $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}] \times [-U_1,U_2]$, and we let $(\hat{\tau},\hat{u}) \in [\tau_0,\tau] \times [-U_1,u]$. We evaluate the top-order integral inequality (29.1) at $(\hat{\tau},\hat{u})$ and multiply it by $\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})$. Similarly, we evaluate (29.3)–(29.4) at $(\hat{\tau},\hat{u})$ and respectively multiply by $\iota_{\mathscr{G}}^{-1}(\hat{\tau},\hat{u})$ and $\iota_{\mathscr{H}}^{-1}(\hat{\tau},\hat{u})$. We then obtain suitable bounds from the resulting products and then take $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$. The left-hand sides of the resulting inequalities are, by definition, equal to the left-hand sides of (29.72)–(29.74), while our "suitable bounds," which we derive below, will yield the right-hand sides of (29.72)–(29.74).

We now prove (29.72). We will explain how to handle several representative terms generated by the top-order integral inequality (29.1), including the most difficult terms. The remaining terms can be handled using similar or simpler arguments, and we omit the details. As our first example, we consider the term $C\int_{\tau'=\tau_0}^{\hat{\tau}} \frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\tau_0}^{\tau'}\left(\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right)(\tau'',u)\,d\tau''\right\}^2 d\tau'$ generated by the 3rd line of RHS (29.2). Using the already proved a priori estimates (24.3a) for $\mathbb{C}_{N_{\text{top}}}$ and $\mathbb{D}_{N_{\text{top}}}$, we bound this term as follows:

$$\begin{aligned}
&C\int_{\tau'=\hat{\tau}_0}^{\hat{\tau}} \frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\hat{\tau}_0}^{\tau'}\left(\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right)(\tau'',\hat{u})\,d\tau''\right\}^2 d\tau' \\
&\lesssim \mathring{\epsilon}^2\int_{\tau'=\tau_0}^{\hat{\tau}} \frac{1}{|\tau'|^{4/3}}\left\{\int_{\tau''=\hat{\tau}_0}^{\tau'}|\tau''|^{-8.55}\,d\tau''\right\}^2 d\tau' \\
&\lesssim \mathring{\epsilon}^2\int_{\tau'=\tau_0}^{\hat{\tau}} \frac{1}{|\tau'|^{4/3}}|\tau'|^{-15.1}\,d\tau' \\
&\lesssim \mathring{\epsilon}^2|\hat{\tau}|^{-(15+13/30)} \lesssim \mathring{\epsilon}^2|\hat{\tau}|^{-15.6}.
\end{aligned} \tag{29.80}$$

Multiplying (29.80) by $\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})$ and taking $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$, we obtain:

$$C \sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]} \iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u}) \int_{\tau'=\hat{\tau}_0}^{\hat{\tau}} \frac{1}{|\tau'|^{4/3}} \left\{ \int_{\tau''=\hat{\tau}_0}^{\tau'} \left(\mathbb{C}_N^{1/2}+\mathbb{D}_N^{1/2}\right)(\tau'',\hat{u})\,\mathrm{d}\tau'' \right\}^2 \mathrm{d}\tau' \leq C\mathring{\epsilon}^2, \tag{29.81}$$

which is $\leq$ RHS (29.72) as desired. The term $\int_{\tau'=\tau_0}^{\hat{\tau}} \frac{1}{|\tau'|^{4/3}} \left\{ \int_{\tau''=\tau_0}^{\tau'} \frac{1}{|\tau''|^{1/2}} \left(\mathbb{C}_{\leq N-1}^{1/2}+\mathbb{D}_{\leq N-1}^{1/2}\right)(\tau'',u)\,\mathrm{d}\tau'' \right\}^2 \mathrm{d}\tau'$ generated by the 4^{th} line of RHS (29.2) can be handled using similar arguments, this time with the help of the already proven estimate (24.3b)–(24.3c). Similarly, the following estimate holds for the term generated by the first term on RHS (29.2):

$$C \sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]} \iota_{\mathscr{F}}^{-1} \left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2 \frac{1}{|\hat{\tau}|^{3/2}} \leq C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2, \tag{29.82}$$

which is $\leq$ RHS (29.72) as desired.

We now handle the first term on RHS (29.1), i.e., the one multiplied by the boxed constant $\boxed{\left\{\frac{4\times 1.01}{1.99}+4.13\right\}}$. Evaluating the term at $(\hat{\tau},\hat{u})$, multiplying it by $\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})$, multiplying and dividing the integrand by $|\tau'|^{15.6}$, taking $\sup_{\tau'\in[\tau_0,\hat{\tau}]}|\tau'|^{15.6}\mathbb{Q}_N(\tau',\hat{u})$, and pulling the sup-ed out quantity outside of the integral, we deduce:

$$\begin{aligned}
&\left\{\frac{4\times 1.01}{1.99}+4.13\right\}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|}\mathbb{Q}_N(\tau',\hat{u})\,\mathrm{d}\tau' \\
&\leq \left\{\frac{4\times 1.01}{1.99}+4.13\right\}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\times\sup_{\tau''\in[\tau_0,\hat{\tau}]}\left\{|\tau''|^{15.6}\mathbb{Q}_N(\tau'',\hat{u})\right\}\times\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{16.6}}\,\mathrm{d}\tau' \\
&\leq \frac{1}{15.6}\left\{\frac{4\times 1.01}{1.99}+4.13\right\}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\times\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''|^{15.6}\mathbb{Q}_N(\tau'',u'')\right\}\times|\hat{\tau}|^{-15.6} \\
&\leq \frac{1}{15.6}\left\{\frac{4\times 1.01}{1.99}+4.13\right\}\mathscr{F}(\hat{\tau},\hat{u}) \\
&\leq \frac{1}{15.6}\left\{\frac{4\times 1.01}{1.99}+4.13\right\}\mathscr{F}(\tau,u).
\end{aligned} \tag{29.83}$$

Taking $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$ of (29.83), we find that $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$ LHS (29.83) $\leq$ RHS (29.83). We incorporate the terms on RHS (29.83) as part of the "main terms" located on the second line of RHS (29.72). We clarify that to obtain the third inequality in (29.83), we multiplied by $1=\mathfrak{l}^{\mathfrak{c}}(\tau'')\mathfrak{l}^{-\mathfrak{c}}(\tau'')e^{\mathfrak{c}(u''+U_1)}e^{-\mathfrak{c}(u''+U_1)}$ under the sup, used the monotonicity properties $\mathfrak{l}^{\mathfrak{c}}(\tau'')\leq\mathfrak{l}^{\mathfrak{c}}(\hat{\tau})$ and $e^{\mathfrak{c}(u''+U_1)}\leq e^{\mathfrak{c}(\hat{u}+U_1)}$, pulled $\mathfrak{l}^{\mathfrak{c}}(\hat{\tau})$ and $e^{\mathfrak{c}(\hat{u}+U_1)}$ out of the sup (so the remaining sup-ed quantity is $\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}],[-U_1,\hat{u}]}\left\{|\tau''|^{15.6}\mathfrak{l}^{-\mathfrak{c}}(\tau'')e^{-\mathfrak{c}(u''+U_1)}\mathbb{Q}_{N_{\text{top}}}(\tau'',u'')\right\}\leq\mathscr{F}(\hat{\tau},\hat{u})$), and then noted that the factors $\mathfrak{l}^{\mathfrak{c}}(\hat{\tau})$, $e^{\mathfrak{c}(\hat{u}+U_1)}$, and $|\hat{\tau}|^{-15.6}$ (outside of the sup) multiply together to exactly cancel $\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})$.

We can handle the second term on RHS (29.1) (which is multiplied by $\boxed{\frac{8\times(1.01)^2}{1.99}}$) using similar arguments, but we have to integrate twice in time. We find that the corresponding term is:

$$\begin{aligned}
&\leq \boxed{\frac{8\times(1.01)^2}{1.99}}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',\hat{u})\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\mathbb{Q}_N^{1/2}(\tau'',\hat{u})\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
&\leq \frac{1}{15.6\times 7.8}\left\{\frac{8\times(1.01)^2}{1.99}\right\}\mathscr{F}(\hat{\tau},\hat{u})\leq\frac{1}{15.6\times 7.8}\left\{\frac{8\times(1.01)^2}{1.99}\right\}\mathscr{F}(\tau,u).
\end{aligned} \tag{29.84}$$

The terms on RHS (29.84) are also part of the "main terms" located on the second line of RHS (29.72).

We can handle the third term on RHS (29.1) (which is multiplied by $\boxed{4.13}$) using similar arguments, involving only one integration in time, to deduce:

$$\begin{aligned}
&\boxed{4.13}\,\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\frac{1}{|\hat{\tau}|^{1/2}}\mathbb{Q}_N^{1/2}(\hat{\tau},\hat{u})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_N^{1/2}(\tau',\hat{u})\,\mathrm{d}\tau' \\
&\leq\left\{\frac{4.13}{7.3}\right\}\mathscr{F}(\hat{\tau},\hat{u})\leq\left\{\frac{4.13}{7.3}\right\}\mathscr{F}(\tau,u).
\end{aligned} \tag{29.85}$$

The terms on RHS (29.85) provide the last contribution to the "main terms" located on the second line of RHS (29.72).

Using the same arguments we used to prove (29.80)–(29.85), we can bound the contribution of the three C_*-multiplied terms on RHS (29.1) by $\leq C\mathscr{F}^{1/2}(\tau,u)\mathscr{G}^{1/2}(\tau,u)$, which in turn is bounded by the last term on RHS (29.72).

The remaining terms on RHS (29.72), which are all generated by the terms on the right-hand side of the inequality (29.2) for $|\mathrm{Error}_N^{(\mathrm{Top})}|$, are less dangerous than the three main terms we treated in (29.83)–(29.85) because they are either **I)** critical with respect to the energy blow-up rates but feature a small factor of $C\varepsilon$, **II)** sub-critical[62] with respect to the blow-up rates, or **III)** controlled by the already proven a priori estimates for the transport-variables from Prop. 24.2 (it turns out that all these terms are sub-critical too). The type **I** terms can be handled as in (29.83)–(29.85), but they are much less delicate because we do not have to be careful about the size of the constants; such terms contribute to the $C\varepsilon$-multiplied terms on the second line of RHS (29.72). The type **III** terms have already been adequately controlled in Prop. 24.2 and contribute only to the term $C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2$ on RHS (29.72), as in (29.81)–(29.82). The type **II** terms can be handled using arguments that rely only on the multiplicative factors $\mathfrak{l}^{\mathfrak{c}}(\tau)$ and $e^{\mathfrak{c}(u+U_1)}$ in (29.69b)–(29.69c). We will handle three representative type **II** terms: the term on the next-to-last line of RHS (29.2) featuring a triple integral, the u'-integral term on the fifth-from-last line of RHS (29.2), and the τ' integral featuring the lower order term $\mathbb{Q}_{[1,N-1]}$ on the last line of RHS (29.2). The remaining terms on RHS (29.2) can be handled using similar arguments, where we use the estimates of Prop. 24.2 and argue as in (29.80)–(29.81) to handle error terms that involve the (already bounded) quantities $\mathbb{V}_M$, $\mathbb{V}_M$, $\mathbb{C}_M$, and $\mathbb{D}_M$; we omit the details.

We will show that the contribution of the first representative term can be bounded as follows:

$$
\begin{aligned}
& C\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',\hat{u})\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\int_{\tau'''=\tau_0}^{\tau''}\frac{1}{|\tau'''|^{1/2}}\mathbb{Q}_N^{1/2}(\tau''',\hat{u})\,\mathrm{d}\tau'''\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
&\leq \frac{C}{\mathfrak{c}}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\mathfrak{l}^{\frac{\mathfrak{c}}{2}}(\hat{\tau})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|}\mathbb{Q}_N^{1/2}(\tau',\hat{u})\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|}\sup_{(\tau''',u''')\in[\tau_0,\tau'']\times[-U_1,\hat{u}]}\left\{\mathfrak{l}^{-\frac{\mathfrak{c}}{2}}(\tau''')\mathbb{Q}_N^{1/2}(\tau''',u''')\right\}\mathrm{d}\tau''\,\mathrm{d}\tau' \\
&\leq \frac{C}{\mathfrak{c}}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\mathfrak{l}^{\frac{\mathfrak{c}}{2}}(\hat{\tau}) \\
&\quad\times\sup_{(\tau''',u''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'''|^{7.8}\mathfrak{l}^{-\frac{\mathfrak{c}}{2}}(\tau''')\mathbb{Q}_N^{1/2}(\tau''',u''')\right\}\times\sup_{(\tau'''',u'''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''''|^{7.8}\mathbb{Q}_N^{1/2}(\tau'''',u'''')\right\} \\
&\quad\times\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{8.8}}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{8.8}}\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
&\leq \frac{C}{\mathfrak{c}}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\mathfrak{l}^{\mathfrak{c}}(\hat{\tau})\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''|^{15.6}\mathfrak{l}^{-\mathfrak{c}}(\tau'')\mathbb{Q}_N(\tau'',u'')\right\}\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{16.6}}\,\mathrm{d}\tau' \\
&\leq \frac{C}{\mathfrak{c}}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})|\hat{\tau}|^{-15.6}\mathfrak{l}^{\mathfrak{c}}(\hat{\tau})e^{\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''|^{15.6}\mathfrak{l}^{-\mathfrak{c}}(\tau'')e^{-\mathfrak{c}(u''+U_1)}\mathbb{Q}_N(\tau'',u'')\right\} \\
&\leq \frac{C}{\mathfrak{c}}\mathscr{F}(\hat{\tau},\hat{u})\leq\frac{C}{\mathfrak{c}}\mathscr{F}(\tau,u).
\end{aligned}
\tag{29.86}
$$

Taking $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$ of (29.86), we find that $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$ LHS (29.86) $\leq$ RHS (29.86), which is $\leq$ RHS (29.72) as desired. We now justify the sequence of inequalities in (29.86). The first inequality follows from multiplying and dividing the $\mathrm{d}\tau'''$ integrand by $\mathfrak{l}^{\frac{\mathfrak{c}}{2}}(\tau''')$, pulling out $\sup_{(\tau''',u''')\in[\tau_0,\tau'']\times[-U_1,\hat{u}]}\left\{\mathfrak{l}^{-\frac{\mathfrak{c}}{2}}(\tau''')\mathbb{Q}_N^{1/2}(\tau''',u''')\right\}$ from the $\mathrm{d}\tau'''$ integral, and using (29.70) to bound the remaining inner-most time integral $\int_{\tau'''=\tau_0}^{\tau''}\frac{\mathfrak{l}^{\frac{\mathfrak{c}}{2}}(\tau''')}{|\tau'''|^{1/2}}\,\mathrm{d}\tau'''$ by $\leq\frac{C}{\mathfrak{c}}\mathfrak{l}^{\frac{\mathfrak{c}}{2}}(\hat{\tau})$. The second inequality follows from multiplying and dividing the $\mathrm{d}\tau''$ integrand by $|\tau''|^{7.8}$ and pulling out a sup-ed quantity from the integral, and from multiplying and dividing the $\mathrm{d}\tau'$ integrand by $|\tau'|^{7.8}$ and pulling out a sup-ed quantity from the integral. The remaining inequalities follow from straightforward integration, the definitions of the quantities involved, and the monotonicity of various factors.

[62] By a "critical term," we mean a term whose blowup-rate (in terms of powers of $|\tau|^{-1}$) is exactly compatible with the energy blowup-rates stated in Prop. 24.1. By "sub-critical term," we mean one whose blowup-rate is less singular (in terms of powers of $|\tau|^{-1}$) than a critical term, i.e., a term that is not among the most singular.

Next, using (29.70) and arguments similar to but simpler than the ones we used to prove (29.86), we bound the term generated by the u'-integral on the fifth-from-last line of (29.2) as follows:

$$
\begin{aligned}
& C\left(1+\varsigma^{-1}\right)\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\int_{u'=-U_1}^{\hat{u}}\mathbb{Q}_N(\hat{\tau},u')\,\mathrm{d}u' \\
& \leq \left(1+\varsigma^{-1}\right)\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{e^{-\mathfrak{c}(u''+U_1)}\mathbb{Q}_N(\tau'',u'')\right\}\int_{u'=-U_1}^{\hat{u}}e^{\mathfrak{c}(u'+U_1)}\,\mathrm{d}u' \\
& \leq \frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})e^{\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{e^{-\mathfrak{c}(u''+U_1)}\mathbb{Q}_N(\tau'',u'')\right\} \\
& \leq \frac{C}{\mathfrak{c}}\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})|\hat{\tau}|^{-15.6}\mathfrak{l}^{\mathfrak{c}}(\hat{\tau})e^{\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau',u')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'|^{15.6}\mathfrak{l}^{-\mathfrak{c}}(\tau')e^{-\mathfrak{c}(u'+U_1)}\mathbb{Q}_N(\tau',u')\right\} \\
& \leq \frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\mathscr{F}(\hat{\tau},\hat{u}) \leq \frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\mathscr{F}(\tau,u),
\end{aligned}
\tag{29.87}
$$

which is $\leq$ RHS (29.72) as desired.

Similarly, we bound the term generated by the $\mathbb{Q}_{[1,N-1]}$ integral on the last line of RHS (29.2) as follows:

$$
\begin{aligned}
& C\iota_{\mathscr{F}}^{-1}(\hat{\tau},\hat{u})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{5/2}}\mathbb{Q}_{[1,N-1]}(\tau',\hat{u})\,\mathrm{d}\tau' \\
& = C|\hat{\tau}|^{15.6}\mathfrak{l}^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)}\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{5/2}}\mathbb{Q}_{[1,N-1]}(\tau',\hat{u})\mathfrak{l}^{\mathfrak{c}}(\tau')\mathfrak{l}^{-\mathfrak{c}}(\tau')\,\mathrm{d}\tau' \\
& \leq C|\hat{\tau}|^{13.6}\mathfrak{l}^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{\mathfrak{l}^{-\mathfrak{c}}(\tau'')\mathbb{Q}_{[1,N-1]}(\tau'',u'')\right\}\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{1/2}}\mathfrak{l}^{\mathfrak{c}}(\tau')\,\mathrm{d}\tau' \\
& \leq \frac{C}{\mathfrak{c}}|\hat{\tau}|^{13.6}e^{-\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{\mathfrak{l}^{-\mathfrak{c}}(\tau'')\mathbb{Q}_{[1,N-1]}(\tau'',u'')\right\} \\
& \leq \frac{C}{\mathfrak{c}}\sup_{(\tau'',u'')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''|^{13.6}\mathfrak{l}^{-\mathfrak{c}}(\tau'')e^{-\mathfrak{c}(u''+U_1)}\mathbb{Q}_{[1,N-1]}(\tau'',u'')\right\} \\
& \leq \frac{C}{\mathfrak{c}}\mathscr{H}(\hat{\tau},\hat{u}) \leq \frac{C}{\mathfrak{c}}\mathscr{H}(\tau,u),
\end{aligned}
\tag{29.88}
$$

which is $\leq$ RHS (29.72) as desired. This completes our proof of (29.72).

The estimate (29.73) can be proved via similar arguments that start with evaluating both sides of inequality (29.3) at $(\hat{\tau},\hat{u})$, multiplying the inequality by $\iota_{\mathscr{G}}^{-1}(\hat{\tau},\hat{u})$, and then taking $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$. The key difference between the estimates (29.72) and (29.73) is that the critical boxed-constant-multiplied terms and C_*-multiplied terms appearing on RHS (29.1) are *absent* from RHS (29.3). Consequently, using arguments similar to the ones given above, we find that the terms $\left\{\frac{\frac{4\times 1.01}{1.99}+4.13}{15.6}+\frac{\frac{8(1.01)^2}{1.99}}{15.6\times 7.8}+\frac{4.13}{7.3}\right\}\mathscr{F}(\tau,u)$ and $C\mathscr{F}^{1/2}(\tau,u)\mathscr{G}^{1/2}(\tau,u)$ from RHS (29.72) are *absent* from RHS (29.73).

The estimate (29.74) can be proved via similar arguments that start with evaluating both sides of inequality (29.4) at $(\hat{\tau},\hat{u})$, multiplying the inequality by $\iota_{\mathscr{H}}^{-1}(\hat{\tau},\hat{u})$, and then taking the supremum $\sup_{(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]}$. Using arguments similar to the ones given above, we find that the terms $\mathrm{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u)$ on RHS (29.4) generate terms that can be bounded by $\leq C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2+\left\{C\varsigma+\frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\mathscr{H}(\tau,u)$, which is $\leq$ RHS (29.74) as desired. To handle the term generated by the remaining term on RHS (29.4) (i.e., the double time-integral involving $\mathbb{Q}_N^{1/2}$), we can bound it using straightforward arguments based on multiplying and dividing by $|\tau'|^{6.8}$ and $|\tau''|^{7.8}$ in the two time integrals:

$$\begin{aligned}
& C\iota_{\mathscr{H}}^{-1}(\hat{\tau},\hat{u})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau',\hat{u})\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{Q}_{N}^{1/2}(\tau'',\hat{u})\,d\tau''\,d\tau' \\
&= C|\hat{\tau}|^{13.6}\mathfrak{l}^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)}\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau',\hat{u})\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{Q}_{N}^{1/2}(\tau'',\hat{u})\,d\tau''\,d\tau' \\
&\leq C|\hat{\tau}|^{13.6}\mathfrak{l}^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau''',u''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'''|^{6.8}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau''',u''')\right\} \\
&\quad\times\sup_{(\tau'''',u'''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''''|^{7.8}\mathbb{Q}_{N}^{1/2}(\tau'''',u'''')\right\} \\
&\quad\times\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{7.3}}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{8.3}}\,d\tau''\,d\tau' \\
&\leq C\mathfrak{l}^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau''',u''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'''|^{6.8}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau''',u''')\right\} \\
&\quad\times\sup_{(\tau'''',u'''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''''|^{7.8}\mathbb{Q}_{N}^{1/2}(\tau'''',u'''')\right\} \\
&\leq C\sup_{(\tau''',u''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'''|^{6.8}\mathfrak{l}^{-\frac{\mathfrak{c}}{2}}(\tau''')e^{-\frac{\mathfrak{c}}{2}(u'''+U_1)}\mathbb{Q}_{[1,N-1]}^{1/2}(\tau''',u''')\right\} \\
&\quad\times\sup_{(\tau'''',u'''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''''|^{7.8}\mathfrak{l}^{-\frac{\mathfrak{c}}{2}}(\tau'')e^{-\frac{\mathfrak{c}}{2}(u''''+U_1)}\mathbb{Q}_{N}^{1/2}(\tau'''',u'''')\right\} \\
&\leq C\mathscr{H}^{1/2}(\tau,u)\mathscr{F}^{1/2}(\tau,u)\leq\frac{1}{2}\mathscr{H}(\tau,u)+C\mathscr{F}(\tau,u),
\end{aligned} \tag{29.89}$$

where to obtain the last inequality, we used Young's inequality. We finally observe that RHS (29.89) ≤ RHS (29.74), and we note that the inequality (29.89) is the only one that contributes the term $\frac{1}{2}\mathscr{H}(\tau,u)$ to RHS (29.74). We have therefore proved (29.89), which completes our proof of (29.72)–(29.74). This also yields the desired bounds for $\mathbb{W}_{N_{\text{top}}}$ and $\mathbb{W}_{N_{\text{top}}-1}$ stated in (24.1a), aside from the issue that, as we noted below (29.71), we will not make the final choice of $\mathfrak{c}$ until the end of the proof.

Estimates for $\mathbb{W}_{[1,N_{\text{top}}-2]}$, $\mathbb{W}_{[1,N_{\text{top}}-3]}$, $\cdots$, $\mathbb{W}_1$. We now explain how to derive the a priori estimates (24.1a)–(24.1b) for $\mathbb{W}_{[1,N_{\text{top}}-2]}$, $\mathbb{W}_{[1,N_{\text{top}}-3]}$, $\cdots$, $\mathbb{W}_1$ via downward induction, starting with $\mathbb{W}_{[1,N_{\text{top}}-2]}$.

Unlike our analysis of the strongly coupled triple $\mathbb{W}_{N_{\text{top}}}$, $\mathbb{W}_{N_{\text{top}}}^{(\text{Partial})}$, and $\mathbb{W}_{[1,N_{\text{top}}-1]}$, we can derive the estimate for $\mathbb{W}_{[1,N_{\text{top}}-2]}$ using only the integral inequality (29.4), the already proven vorticity and entropy estimates provided by Prop. 24.2, and our already proven bounds for $\mathbb{W}_{[1,N_{\text{top}}-1]}$ (more precisely, the already proven bound (29.71) for $\mathscr{H}$). To begin, we define an analog of (29.69c): $\iota_{\widetilde{\mathscr{H}}}(\tau,u)\overset{\text{def}}{=}|\tau|^{-11.6}\mathfrak{l}^{\mathfrak{c}}(\tau)e^{\mathfrak{c}(u+U_1)}$, as well as an analog of (29.68c): $\widetilde{\mathscr{H}}(\tau,u)\overset{\text{def}}{=}\sup_{(\tau',u')\in[\tau_0,\tau]\times[-U_1,u]}\left\{\iota_{\widetilde{H}}^{-1}(\tau',u')\mathbb{W}_{[1,N_{\text{top}}-2]}(\tau',u')\right\}$. Note that compared to our definition (29.69c) of $\iota_{\mathscr{H}}$, we have reduced the power of $|\tau|^{-1}$ by two in our definition of $\iota_{\widetilde{\mathscr{H}}}$. As before, we will prove (24.1a) for $K=2$ by showing that there is a uniform $C>0$ such that for all sufficiently small $\varsigma\in(0,1]$ and $\varepsilon\geq 0$, we have:

$$\widetilde{\mathscr{H}}(\tau,u)\leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2. \tag{29.90}$$

To prove (29.90), we will show that:

$$\widetilde{\mathscr{H}}(\tau,u)\leq C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2+\left\{\frac{1}{2}+C\varsigma+\frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\widetilde{\mathscr{H}}(\tau,u), \tag{29.91}$$

where as before, we set $\mathfrak{c}\overset{\text{def}}{=}\varsigma^{-2}$. Once we have proved (29.91), then if $\varsigma\in(0,1]$ and $\varepsilon\geq 0$ are sufficiently small, we can absorb all terms on RHS (29.91) except for $C\left(1+\varsigma^{-1}\right)\mathring{\varepsilon}^2$ back into the left, thereby arriving at the desired bound (29.90).

It remains for us to prove (29.91). To proceed, we set $N=N_{\text{top}}-1$, multiply both sides of (29.4) by $\iota_{\widetilde{\mathscr{H}}}^{-1}$, and evaluate the resulting expression at $(\hat{\tau},\hat{u})\in[\tau_0,\tau]\times[-U_1,u]$. Using same arguments we used to prove (29.74) (in particular using the already proven estimates of Prop. 24.2), we find that the terms $\text{Error}_{N-1}^{(\text{Sub-critical})}(\tau,u)$ on RHS (29.4) generate terms that we can bound by $\leq C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2+\left\{C\varsigma+\frac{C}{\mathfrak{c}}\left(1+\varsigma^{-1}\right)\right\}\widetilde{\mathscr{H}}(\tau,u)$. We now handle the remaining term, i.e., the term generated by the first term on RHS (29.4), which is a double time-integral involving the above-present-order factor $\mathbb{Q}_{N_{\text{top}}-1}^{1/2}$. Multiplying and dividing by $|\tau'|^{5.8}$ and $|\tau''|^{6.8}$ in the two time integrals, arguing as in the proof of (29.89), and using the already proven bound (29.71) for $\mathscr{H}$, we bound this term as follows:

$$
\begin{aligned}
& C\iota_{\widetilde{\mathscr{H}}}^{-1}(\hat{\tau},\hat{u})\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N_{\text{top}}-2]}^{1/2}(\tau',\hat{u})\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{Q}_{N_{\text{top}}-1}^{1/2}(\tau'',\hat{u})\,d\tau''\,d\tau' \\
&= C|\hat{\tau}|^{11.6}\iota^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)}\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{1/2}}\mathbb{Q}_{[1,N_{\text{top}}-2]}^{1/2}(\tau',\hat{u})\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{1/2}}\mathbb{Q}_{N_{\text{top}}-1}^{1/2}(\tau'',\hat{u})\,d\tau''\,d\tau' \\
&\leq C|\hat{\tau}|^{11.6}\iota^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)} \\
&\quad\times\sup_{(\tau''',u''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'''|^{5.8}\mathbb{Q}_{[1,N_{\text{top}}-2]}^{1/2}(\tau''',u''')\right\} \\
&\quad\times\sup_{(\tau'''',u'''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''''|^{6.8}\mathbb{Q}_{N_{\text{top}}-1}^{1/2}(\tau'''',u'''')\right\} \\
&\quad\times\int_{\tau'=\tau_0}^{\hat{\tau}}\frac{1}{|\tau'|^{6.3}}\int_{\tau''=\tau_0}^{\tau'}\frac{1}{|\tau''|^{7.3}}\,d\tau''\,d\tau' \\
&\leq C\iota^{-\mathfrak{c}}(\hat{\tau})e^{-\mathfrak{c}(\hat{u}+U_1)}\sup_{(\tau''',u''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'''|^{5.8}\mathbb{Q}_{[1,N_{\text{top}}-2]}^{1/2}(\tau''',u''')\right\} \\
&\quad\times\sup_{(\tau'''',u'''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''''|^{6.8}\mathbb{Q}_{N_{\text{top}}-1}^{1/2}(\tau'''',u'''')\right\} \\
&\leq C\sup_{(\tau''',u''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau'''|^{5.8}\iota^{-\frac{\mathfrak{c}}{2}}(\tau''')e^{-\frac{\mathfrak{c}}{2}(u'''+U_1)}\mathbb{Q}_{[1,N_{\text{top}}-2]}^{1/2}(\tau''',u''')\right\} \\
&\quad\times\sup_{(\tau'''',u'''')\in[\tau_0,\hat{\tau}]\times[-U_1,\hat{u}]}\left\{|\tau''''|^{6.8}\iota^{-\frac{\mathfrak{c}}{2}}(\tau'''')e^{-\frac{\mathfrak{c}}{2}(u''''+U_1)}\mathbb{Q}_{N_{\text{top}}-1}^{1/2}(\tau'''',u'''')\right\} \\
&\leq C\widetilde{\mathscr{H}}^{1/2}(\hat{\tau},\hat{u})\mathscr{H}^{1/2}(\hat{\tau},\hat{u})\leq\frac{1}{2}\widetilde{\mathscr{H}}(\tau,u)+C\mathscr{H}(\tau,u) \\
&\leq\frac{1}{2}\widetilde{\mathscr{H}}(\tau,u)+C\left(1+\varsigma^{-1}\right)\mathring{\epsilon}^2,
\end{aligned}
\tag{29.92}
$$

which is $\leq$ RHS (29.91) as desired. We have therefore proved (29.90), which in particular implies (24.1a) for $K=2$.

The desired bounds (24.1a)–(24.1b) for $\mathbb{W}_{[1,N_{\text{top}}-3]}$, $\cdots$, $\mathbb{W}_1$ can be derived by downward induction based on an argument that is very similar to the one we used to prove the bound (29.90) for $\mathbb{W}_{[1,N_{\text{top}}-2]}$. The only difference is that we define an analogous multiplicative factor $\iota_{\widetilde{\mathscr{H}},P}(\tau,u)\overset{\text{def}}{=}|\tau|^{-P}\iota^{\mathfrak{c}}(\tau)e^{\mathfrak{c}(u+U_1)}$, where $P=9.6$ for $\mathbb{W}_{[1,N_{\text{top}}-3]}$, $P=7.6$ for $\mathbb{W}_{[1,N_{\text{top}}-4]}$, $P=5.6$ for $\mathbb{W}_{[1,N_{\text{top}}-5]}$, $P=3.6$ for $\mathbb{W}_{[1,N_{\text{top}}-6]}$, $P=1.6$ for $\mathbb{W}_{[1,N_{\text{top}}-7]}$, and $P=0$ for $\mathbb{W}_{[1,N_{\text{top}}-8]}$. We stress that these latter estimates (i.e., (24.1b)) *do not involve any singular factor of* $|\tau|^{-1}$.

□

29.8. **Proof of Prop. 24.4.** In this short section, we prove Prop. 24.4, which yields our energy estimates for the acoustic geometry on the rough foliations, thereby completing our proof of the energy estimates.

To proceed, we note that (18.1) implies $\|f\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}\right)}\leq|\tau|^{-1}\|\mu f\|_{L^2\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}\right)}$. Hence, the desired bound (24.6a) follows from inserting the already proven estimates of Props. 24.1 and 24.2 for $\mathbb{Q}_M$, $\mathbb{C}_M$, $\mathbb{D}_M$, $\mathbb{V}_M$, and $\mathbb{S}_M$ (for the relevant values of M) into RHS (29.14) and integrating in τ. Similarly, (24.6b)–(24.6c) follow from inserting the estimates (24.1a)–(24.1b) into RHS (25.1b).

□

30. Improvements of the fundamental quantitative L^∞ bootstrap assumptions

We continue to work under the assumptions of Sect. 13.2. In this short section, we derive L^∞ estimates that yield an improvement of the fundamental quantitative bootstrap assumptions stated in Sect. 12.3.1. The results follow easily from the Sobolev embedding estimate (20.6b) and the estimates for the non-singular energies that we have already derived in Prop. 24.1.

Proposition 30.1 (Improvement of the fundamental quantitative bootstrap assumptions)**.** *Under the parameter-size assumptions of Sect. 10.2, the initial data assumptions of Sects. 11.2.1–11.2.3, and the bootstrap assumptions of Sects. 12 and 24.3, there exists a constant $C > 0$ such that the following estimates hold for $(\tau, u) \in [\tau_0, \tau_{\text{Boot}}) \times [-U_1, U_2]$:*

$$\left\| \mathcal{P}^{[1,N_{\text{top}}-10]} \vec{\Psi} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}, \left\| \mathcal{P}^{\leq N_{\text{top}}-11} (\Omega, S) \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}, \left\| \mathcal{P}^{\leq N_{\text{top}}-12} (\mathcal{C}, \mathcal{D}) \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \leq C \mathring{\varepsilon}. \tag{30.1}$$

In particular, if $\mathring{\varepsilon}$ is small enough such that $C\mathring{\varepsilon} < \varepsilon$, then (30.1) *yields a strict improvement of the fundamental quantitative bootstrap assumptions* (**BA** L^∞ **FUND**).

Proof. The estimate (30.1) for $\left\| \mathcal{P}^{\leq N_{\text{top}}-11} (\Omega, S) \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}$ follows from the Sobolev embedding estimate (20.6a), the definitions (20.46a)–(20.46b) of $\mathbb{V}_N^{(\text{Rough Tori})}(\tau, u)$ and $\mathbb{S}_N^{(\text{Rough Tori})}(\tau, u)$, and the rough tori energy estimates (24.4c). Similarly, the estimate (30.1) for $\left\| \mathcal{P}^{\leq N_{\text{top}}-12} (\mathcal{C}, \mathcal{D}) \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}$ follows from the Sobolev embedding estimate (20.6a), the definitions (20.48a)–(20.48b) of $\mathbb{C}_N^{(\text{Rough Tori})}(\tau, u)$ and $\mathbb{D}_N^{(\text{Rough Tori})}(\tau, u)$, and the rough tori energy estimates (24.5c).

To prove the estimate (30.1) for $\left\| \mathcal{P}^{[1,N_{\text{top}}-10]} \vec{\Psi} \right\|_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)}$, we use the estimate (20.6b), the data-assumption (11.13a), Lemma 20.14, and the estimate (24.1b) to conclude that $\left\| \mathcal{P}^{[1,N_{\text{top}}-10]} \vec{\Psi} \right\|^2_{L^\infty\left({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}\right)} \lesssim \mathring{\varepsilon}^2 + \mathbb{W}_{[1,N_{\text{top}}-8]}(\tau, u) \lesssim \mathring{\varepsilon}^2$ as desired. □

31. Existence up to the singular boundary at fixed $\mathfrak{n}$ via continuation criteria

In this section, we prove our first main theorem, Theorem 31.1, which shows that at fixed $\mathfrak{n} \in [0, \mathfrak{n}_0]$, the solution exists on the domain ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, which in particular contains the μ-adapted torus $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$, a subset of the singular boundary (see Sect. 32.2). To prove the theorem, we rely mainly on results that we have already established, though we also rely on standard continuation criteria, which we prove independently in Prop. 31.2. Roughly, given the results we have already proved, the continuation criteria allow us to continue the solution classically as long as μ has not vanished. We highlight that our setup guarantees (see in particular (18.1)) that within ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, the vanishing of μ happens only along the μ-adapted torus $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$, which is contained in the top boundary ${}^{(\mathfrak{n})}\widetilde{\Sigma}_0^{[-U_1,U_2]}$, where ${}^{(\mathfrak{n})}\tau = 0$.

31.1. Existence up to the singular boundary torus $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$.

In this section, we state and prove our first main theorem, which concerns fixed $\mathfrak{n} \in [0, \mathfrak{n}_0]$.

Theorem 31.1 (Existence in a region containing the singular boundary torus $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$)**.** *Fix any of the compactly supported admissible simple isentropic plane symmetric "background" solutions $\mathcal{R}_{(+)}^{\text{PS}}$ from Def. A.7 (recall that $\mathcal{R}_{(-)}$, v^2, v^3, s, Ω, S, $\mathcal{C}$, and $\mathcal{D}$ vanish for these background solutions).*

Let $(\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)|_{\Sigma_0} \stackrel{def}{=} \left(\mathring{\mathcal{R}}_{(+)}, \mathring{\mathcal{R}}_{(-)}, \mathring{v}^2, \mathring{v}^3, \mathring{s}\right)$ be perturbed fluid data on the flat Cartesian hypersurface Σ_0, as in (11.1), *and let $u|_{\Sigma_0} = -x^1$ be the initial condition of the eikonal function, as in* (3.1) *and* (A.7b)*. Let $\left(\mathring{\Omega}^i, \mathring{S}^i, \mathring{\mathcal{C}}^i, \mathring{\mathcal{D}}\right)_{i=1,2,3}$ denote the initial data on Σ_0 of $\left(\Omega^i, S^i, \mathcal{C}^i, \mathcal{D}\right)_{i=1,2,3}$. Note that these data are determined by $\left(\mathring{\mathcal{R}}_{(+)}, \mathring{\mathcal{R}}_{(-)}, \mathring{v}^2, \mathring{v}^3, \mathring{s}\right)$, the compressible Euler equations* (2.6a)–(2.6c), *definition* (2.7), *and Def. 2.7. Also recall that these data and the data of the eikonal function determine the data of all the acoustic geometry on Σ_0; see Remark 11.1.*

Assume the following:

- ***A1)*** $N_{\text{top}} \geq 24$.
- ***A2)*** *The quantity $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ defined in* (11.4) *is **sufficiently small**, where $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ is a Sobolev norm of the perturbation of the fluid data away from the background solution, and $U_0 > 0$ and $U_2 > 0$ are parameters from Sect. 10.1.*
- ***A3)*** *Recall that in Appendix B (see in particular Prop. B.2), we showed that the smallness of $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ implies that the parameter-size assumptions of Sect. 10.2 hold and that the fluid variable and acoustic geometry data induced on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$, ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, and $\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}$ satisfy the assumptions stated in Sects. 11.2.1–11.2.3, where $U_1 > 0$ and $\mathring{\delta}_* > 0$ are parameters from Sect. 10.1.*

A4) *In particular, Prop. B.2 implies that the parameter* $\mathring{\epsilon}$ *from Sect. 10.1 can be chosen to satisfy* $\mathring{\epsilon} = \mathcal{O}\left(\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)$, *where the implicit constants in "$\mathcal{O}(\cdot)$" depend on the background solution.*

A5) $\mathfrak{n} \in [0, \mathfrak{n}_0]$, *where* $\mathfrak{n}_0$ *is the parameter from Sect. 10.2.*

Then the following conclusions hold.

The rough time function and classical existence relative to the geometric coordinates.

- *There exists a rough time function* ${}^{(\mathfrak{n})}\tau = {}^{(\mathfrak{n})}\tau(t,u,x^2,x^3)$ *(constructed in Sect. 4) with range* $[\tau_0, 0] = [-\mathfrak{m}_0, 0]$ *(see Def. 4.8), and we denote its level-set portions, viewed as subsets of geometric coordinate space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$, *as follows:* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[u_1,u_2]} = \{(t,u,x^2,x^3) \mid {}^{(\mathfrak{n})}\tau(t,u,x^2,x^3) = \tau,\, u_1 \le u \le u_2,\, (x^2,x^3) \in \mathbb{T}^2\}$. *More precisely,* ${}^{(\mathfrak{n})}\tau$ *is defined on the portion* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]} = \bigcup_{\tau \in [\tau_0,0]} {}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}$ *of the maximal classical development of the data with respect to the differential structure of the geometric coordinates* (t,u,x^2,x^3).
- *The change of variables map* ${}^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\mathfrak{n})}\tau,u,x^2,x^3)$ *is a diffeomorphism from* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ *onto its image* $[-\mathfrak{m}_0,0] \times [-U_1,U_2] \times \mathbb{T}^2$ *satisfying* $\|{}^{(\mathfrak{n})}\mathscr{T}\|_{C^{2,1}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]})} \le C$. *Moreover,* $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau \approx 1$ *on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$.
- *The fluid variables* $\vec{\Psi}$, Ω^i, S^i, $\mathcal{C}^i$, *and* $\mathcal{D}$, *the eikonal function* u, μ, L^i, *the Cartesian coordinate functions* (t,x^1,x^2,x^3) *and all of the auxiliary quantities constructed out of these quantities exist classically with respect to the geometric coordinates* (t,u,x^2,x^3) *on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$. *In particular, with respect to the geometric coordinates, the compressible Euler equations* (2.6a)–(2.6c) *are satisfied, and the equations of Theorem 2.15 are also satisfied.*
- *The Hölder estimates of Lemma 15.6, the* L^∞ *estimates of Prop. 17.1 with* ε *replaced by* $C\mathring{\epsilon}$, *and the energy estimates of Props. 24.1, 24.2, 24.3, and 24.4 hold with* $\tau_{\mathrm{Boot}} = 0$, *i.e., they hold on* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$.

The behavior of μ ***and properties of*** Υ.

- *For* $\tau \in [-\mathfrak{m}_0, 0]$, *we have:*

$$\min_{{}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}} \mu = -\tau. \tag{31.1}$$

Moreover, within ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}$, *the minimum value* $-\tau$ *in* (31.1) *is achieved by* μ *precisely on the set* $\check{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ *from definition 4.3c, which is a* $C^{1,1}$*-embedded torus. In particular, in* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, μ *vanishes precisely along the* μ*-adapted torus* $\check{\mathbf{T}}_{0,-\mathfrak{n}}$, *which is a subset of* ${}^{(\mathfrak{n})}\widetilde{\Sigma}_0^{[-U_1,U_2]}$ *that is contained in the singular boundary (see Sect. 32.2).*
- *On* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, *the change of variables map* $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ *is an injection onto its image in Cartesian coordinate space satisfying* $\|\Upsilon\|_{C^{3,1}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]})} \le C$. *In particular,* Υ *is a homeomorphism from the compact set* ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ *onto its image.*
- *With* $d_{\mathrm{geo}}\Upsilon$ *denoting the Jacobian matrix of* Υ, *we have:*

$$\det d_{\mathrm{geo}}\Upsilon \approx -\mu. \tag{31.2}$$

Hence, on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]} \backslash \check{\mathbf{T}}_{0,-\mathfrak{n}}$, Υ *is a diffeomorphism.*
- *For* $\mathfrak{m} \in [0,\mathfrak{m}_0]$, $\Upsilon\left(\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\right)$ *is an embedded two-dimensional* $C^{1,1}$ *torus in Cartesian coordinate space. In particular, the restriction of* Υ *to* $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *is a diffeomorphism from* $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *onto its image* $\Upsilon\left(\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\right)$.

A description of the solution's singular and regular behavior with respect to Cartesian coordinates.

- *(Region without singularities). On the subset* $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]} \backslash \check{\mathbf{T}}_{0,-\mathfrak{n}}\right)$ *of Cartesian coordinate space, the solution exists classically with respect to the Cartesian coordinates.*
- *(The fluid singularity). The following lower bound holds in* $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_{☢},U_{☢}]}\right)$:

$$|X\mathcal{R}_{(+)}| \ge \frac{\mathring{\delta}_*}{\mu|\bar{c}_{;\rho} + 1|}, \tag{31.3}$$

where $\mathring{\delta}_* > 0$ *is the data-parameter from* (11.6), $\bar{c}_{;\rho} \overset{def}{=} c_{;\rho}(\rho = 0, s = 0)$ *is* $c_{;\rho}$ *evaluated at the trivial solution,* $\bar{c}_{;\rho} + 1$ *is a* <u>non-zero</u> *constant by assumption, and the* Σ_t*-tangent vectorfield* X *has Euclidean length satisfying*

$\sqrt{\sum_{a=1}^{3}(X^a)^2} = 1 + \mathcal{O}(\mathring{\alpha})$, *where* $\mathring{\alpha}$ *is the small parameter from Sect. 10.2. In particular, if* $q \in \Upsilon\left(\breve{\mathbf{T}}_{0,-\mathfrak{n}}\right)$, *then since* $\breve{\mathbf{T}}_{0,-\mathfrak{n}} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-\frac{1}{2}U_{\clubsuit},\frac{1}{2}U_{\clubsuit}]}$ *by* (18.3b), *and since* $\mu = 0$ *along* $\Upsilon\left(\breve{\mathbf{T}}_{0,-\mathfrak{n}}\right)$, *it follows that* $|X\mathcal{R}_{(+)}|(q') \to \infty$ *as* $q' \to q$ *in* $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}\backslash\breve{\mathbf{T}}_{0,-\mathfrak{n}}\right)$. *Similarly, the following lower bounds hold in* $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_{\clubsuit},U_{\clubsuit}]}\right)$, *where* ρ *is the logarithmic density (see* (2.3)*):*

$$|X\rho| \geq \frac{\mathring{\delta}_*}{4\mu|\bar{c}_{;\rho}+1|}, \qquad |Xv^1| \geq \frac{\mathring{\delta}_*}{4\mu|\bar{c}_{;\rho}+1|}. \tag{31.4}$$

- *(Regular behavior along the characteristics). The derivatives of* $\vec{\Psi}$, Ω^i, S^i *up to order* $N_{\text{top}} - 11$ *with respect to the vectorfields in the* $\mathcal{P}_u$*-tangent commutation set* $\mathscr{P}$ *defined in* (3.16) *and the derivatives of* $\mathcal{C}^i$ *and* $\mathcal{D}$ *up to order* $N_{\text{top}} - 12$ *with respect to the elements of* $\mathscr{P}$ *are* L^∞*-bounded on* $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}\right)$ *by* $\leq C$. *Finally, for* $\alpha = 0,1,2,3$ *and* $A = 2,3$, *the derivatives of* $\mathbf{g}_{ab}Y^a_{(A)}\partial_\alpha v^b$ *up to order* $N_{\text{top}} - 11$ *with respect to the elements of* $\mathscr{P}$ *are* L^∞*-bounded on* $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}\right)$ *by* $\leq C$.

Proof. The standard local well-posedness results and Cauchy stability provided by Prop. B.2 imply that there exists a $\tau_{\text{Local}} \in (\frac{3}{4}\tau_0, 0)$ such that the solution variables $\vec{\Psi}$, Ω^i, S^i, $\mathcal{C}^i$, $\mathcal{D}$, u, and ${}^{(\mathfrak{n})}\tau$ are classical solutions on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Local}}),[-U_1,U_2]}$ and such that all of the bootstrap assumptions from Sect. 12 hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Local}}),[-U_1,U_2]}$. Let τ_{Max} be the supremum over all such τ_{Local}. Then the solution exists classically and satisfies all the bootstrap assumptions from Sect. 12 on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Max}}),[-U_1,U_2]}$. If it were true that $\tau_{\text{Max}} < 0$, then Prop. 31.2 (which we prove independently in the next section) would imply that there is a $\Delta > 0$ with $\tau_{\text{Max}} + \Delta < 0$ such that the solution exists classically and satisfies the bootstrap assumptions from Sect. 12 on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Max}}+\Delta),[-U_1,U_2]}$, which is impossible in view of the definition of τ_{Max}. Hence, $\tau_{\text{Max}} = 0$.

Aside from (31.4) and the results concerning the boundedness of the quantities $\mathbf{g}_{ab}Y^a_{(A)}\partial_\alpha v^b$, the remaining conclusions of the theorem now follow from (18.1) and the statements just below it, (18.11), Lemma 15.6, and Props. 17.1, 18.4, 24.1, 24.2, 24.3, 24.4, and 30.1, with 0 in the role of τ_{Boot} in all these results.

The lower bounds stated in (31.4) follow from (31.3) and the estimates $|\mu X\rho| = \frac{1}{2}\{1 + \mathcal{O}(\mathring{\alpha})\}|\mu X\mathcal{R}_{(+)}| + \mathcal{O}(\mathring{\epsilon})$ and $|\mu Xv^1| = \frac{1}{2}\{1 + \mathcal{O}(\mathring{\alpha})\}|\mu X\mathcal{R}_{(+)}| + \mathcal{O}(\mathring{\epsilon})$, which follow from (2.5), (2.7) (see also Remark 2.6), (9.3e) for $c - 1$, and the estimates of Prop. 17.1 and Cor. 17.2.

Finally, we show that the derivatives of $\mathbf{g}_{ab}Y^a_{(A)}\partial_\alpha v^b$ up to order $N_{\text{top}} - 11$ with respect to the elements of $\mathscr{P}$ are L^∞-bounded on $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}\right)$. We first consider the case $\alpha = 0$. We start by using (5.9a) to deduce that $\mathbf{g}_{ab}Y^a_{(A)}\partial_t v^b = -\frac{L^1X^1+L^2X^2+L^3X^3}{c^2}\mathbf{g}_{ab}Y^a_{(A)}Xv^b + \text{Error}$, where Error involves only $\mathcal{P}_u$-tangential derivatives of v. From Prop. 9.1 and Prop. 17.1, it follows that the derivatives of Error up to order $N_{\text{top}} - 11$ with respect to the elements of $\mathscr{P}$ are L^∞-bounded on $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}\right)$ by $\leq C$. Next, we use (2.8) and (2.15a)–(2.15b) to compute that relative to the Cartesian coordinates, we have (where ϵ_{ijk} is the fully antisymmetric symbol normalized by $\epsilon_{123} = 1$): $Xv^b = X^d\partial_d v^b = X^d\partial_b v^d + \epsilon_{dbe}X^d(\text{curl}\, v)^e = c^2X_d\partial_b v^d + \exp(\rho)\epsilon_{dbe}X^d\Omega^e$. Again using (2.15a), we find that:

$$\begin{aligned} -\frac{L^1X^1+L^2X^2+L^3X^3}{c^2}\mathbf{g}_{ab}Y^a_{(A)}Xv^b = & -\frac{L^1X^1+L^2X^2+L^3X^3}{c^2}X_dY_{(A)}v^d \\ & - \exp(\rho)\frac{L^1X^1+L^2X^2+L^3X^3}{c^2}\mathbf{g}_{ab}Y^a_{(A)}\epsilon_{dbe}X^d\Omega^e. \end{aligned} \tag{31.5}$$

Finally, using Prop. 9.1 and Prop. 17.1, we see that the derivatives of the products on RHS (31.5) up to order $N_{\text{top}} - 11$ with respect to the elements of $\mathscr{P}$ are L^∞-bounded on $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}\right)$ by $\leq C$. We have therefore proved the desired result in the case $\alpha = 0$. To handle the cases $\alpha = 1,2,3$, we use a similar argument that relies on the identities (5.9b)–(5.9d) in place of (5.9a).

□

31.2. **Continuation criteria.** In the next proposition, we provide the continuation criteria that are needed for the proof of Theorem 31.1. Roughly, the proposition shows that in the solution regime under study, if $\tau_{\text{Boot}} < 0$, then the solution can be continued beyond ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ as a classical solution with respect to the Cartesian coordinates *and* with respect to the geometric coordinates. Central to the proof is the estimate (18.1), which in particular shows that μ is strictly positive on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ whenever $\tau_{\text{Boot}} < 0$, i.e., no shocks are present in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$. Given the results we have already established, the proposition is rather standard.

Proposition 31.2 (Continuation criteria). *Assume the following:*

- *The assumptions of Theorem 31.1 hold.*
- $\tau_{\text{Boot}} < 0$.
- *The rough time function ${}^{(\mathfrak{n})}\tau$, the solution variables $\vec{\Psi}$, Ω^i, S^i, $\mathcal{C}^i$, $\mathcal{D}$, u, etc. are classical solutions on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$.*
- *The bootstrap assumptions of Sect. 12 and 24.3 hold (e.g., the energy bootstrap assumptions* (24.12a)–(24.12b) *hold for $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}) \times [-U_1,U_2]$).*

Then there exists a $\Delta \in (0,|\tau_{\text{Boot}}|)$ such that the rough time function ${}^{(\mathfrak{n})}\tau$, the solution variables $\vec{\Psi}$, Ω^i, S^i, $\mathcal{C}^i$, $\mathcal{D}$, u, and all of the other geometric quantities defined throughout the article can be uniquely extended (where the solution variables are classical solutions) to a strictly larger region of the form ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}+\Delta),[-U_1,U_2]}$ on which all of the bootstrap assumptions of Sects. 12 and 24.3 hold.

Proof. Throughout this proof, we allow the small positive numbers $m > 0$ and $\Delta > 0$ to vary from line to line, sometimes silently shrinking them as necessary.

Step 1: Extension to ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$. Since $\tau_{\text{Boot}} = -\mathfrak{m}_{\text{Boot}} < 0$ by assumption, Lemmas 15.5 and 15.6, Prop. 18.4, and the bootstrap assumptions imply that the quantities $\vec{\Psi}$, Ω^i, S^i, $\mathcal{C}^i$, $\mathcal{D}$, Υ, t, x^1, x^2, x^3, u, and all the other geometric quantities defined throughout the article extend from ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$ to the compact set ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ as classical solutions relative to the geometric coordinates (t,u,x^2,x^3) such that Υ is a diffeomorphism on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$. The same results yields that these quantities extend to the compact set $[\tau_0,\tau_{\text{Boot}}] \times [-U_1,U_2] \times \mathbb{T}^2$ as classical solutions relative to the adapted rough coordinates $({}^{(\mathfrak{n})}\tau,u,x^2,x^3)$, and to the compact set $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}\right)$ as classical solutions relative to the Cartesian coordinates (t,x^1,x^2,x^3). Moreover, in view of the energy estimates of Sect. 24, it is a standard result that for all $(\tau,u) \in [\tau_0,\tau_{\text{Boot}}] \times [-U_1,U_2]$, the extended quantities enjoy the same Sobolev and Lebesgue regularity (i.e., the corresponding norms are all finite) with respect to the geometric coordinates on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, and ${}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau]}$ as the data on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$, and ${}^{(\mathfrak{n})}\mathcal{P}_{-U_1}^{[\tau_0,\tau_{\text{Boot}}]}$, and that relative to all of the corresponding Sobolev and Lebesgue function space topologies on these surfaces that we have used throughout the paper, the solution is continuous with respect to (τ,u) on $[\tau_0,\tau_{\text{Boot}}] \times [-U_1,U_2]$; we refer readers to [68, Section 2.7] for the main ideas behind the proof of these "propagation of regularity" and "continuity-in-norm" results in the context of the relativistic Euler equations coupled to Nördstrom's theory of gravity.

Furthermore, by (15.27)–(15.28), there is a τ_{Boot}-dependent function $\mathfrak{t}_{\tau_{\text{Boot}},\mathfrak{n}}$ on $[-U_1,U_2] \times \mathbb{T}^2$ such that relative to the geometric coordinates, we have:

$$
{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-U_1,U_2]} = \left\{ \left(\mathfrak{t}_{\tau_{\text{Boot}},\mathfrak{n}}(u,x^2,x^3),u,x^2,x^3\right) \mid (u,x^2,x^3) \in [-U_1,U_2] \times \mathbb{T}^2 \right\},
$$

and such that $\|\mathfrak{t}_{\tau_{\text{Boot}},\mathfrak{n}}\|_{C^{2,1}([-U_1,U_2]\times\mathbb{T}^2)} \leq C$. In addition, using (6.20a), (6.20b), (6.20c), (18.1), (18.8a), and (18.27), and our crucial assumption that $\tau_{\text{Boot}} < 0$, we see that $\mathbf{g}({}^{(\mathfrak{n})}\widetilde{N},{}^{(\mathfrak{n})}\widetilde{N}) < 0$ on ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-U_1,U_2]}$, i.e., that ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-U_1,U_2]}$ is $\mathbf{g}$-spacelike. Also using Prop. 18.4, we further deduce that the hypersurface $\Upsilon\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-U_1,U_2]}\right)$ in Cartesian coordinate space is $C^{2,1}$.

Step 2: Extending all quantities – except for the rough time function – beyond ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$. Let Q denote the quantities $\vec{\Psi}$, Ω^i, S^i, $\mathcal{C}^i$, $\mathcal{D}$, Υ, t, x^1, x^2, x^3, u, etc. from the beginning of Step 1. Note that the rough time function ${}^{(\mathfrak{n})}\tau$ is not among these quantities. To extend Q beyond ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$, we will use the Cauchy stability arguments given in Appendix B; while it is not essential for us to use the results of Appendix B here, it allows us to avoid treating the characteristic initial value problem for the compressible Euler equations, which would have involved inessential technical complications. Specifically, in Step 3 of the proof of Prop. B.2, we used Cauchy stability to show that Q exists classically in the (geometric coordinate) region $\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$ depicted in Fig. 16. Moreover, in the proof of Lemma 27.3, we showed that we can extend the rough time function ${}^{(\mathfrak{n})}\tau$ into a subset ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_*,-U_1]}$ of $\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$ (where $U_* > U_1 > 0$) so that it is defined on a region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_*,U_2]}$ containing ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}$ for all sufficiently small $\epsilon > 0$ (see Footnote 73) and such that, by (15.22) and (27.6), we have $\|{}^{(\mathfrak{n})}\tau\|_{C_{\text{geo}}^{2,1}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_*,U_2]})} \leq C$. In particular, combining these results with (11.17a), (B.2), and the results we derived in Step 1 of the present proof, we see that there is an $\epsilon > 0$ such that

${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-(U_1+\epsilon),U_2]} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]} \cup \mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$. Shrinking ϵ if necessary, and using a standard compactness argument, we can assume that Υ is a diffeomorphism on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-(U_1+\epsilon),U_2]}$. Hence, the quantities Q and the rough time function ${}^{(\mathfrak{n})}\tau$ extend to ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-(U_1+\epsilon),U_2]}$ as classical solutions with respect to the geometric coordinates, and similarly with respect to the Cartesian coordinates (on $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-(U_1+\epsilon),U_2]}\right)$). Moreover, by continuity and the results of Step 1 of the present proof, we see that ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}$ is a $\mathbf{g}$-spacelike hypersurface portion if $\epsilon > 0$ is small enough.

We now use standard local well-posedness (see [68, Section 2.7] for the main ideas behind the analysis in the context of the relativistic Euler equations coupled to Nördstrom's theory of gravity) for the compressible Euler equations and the eikonal equation (3.1) relative to the Cartesian coordinates. That is, starting from the data on the $\mathbf{g}$-spacelike hypersurface portion $\Upsilon\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}\right)$ in Cartesian coordinate space, we consider the corresponding local solution to (2.6a)–(2.6c) and the solution u to the eikonal equation (3.1). By Theorem 2.15, $\vec{\Psi}$, Ω^i, S^i, $\mathcal{C}^i$, and $\mathcal{D}$ are solutions to the equations of Theorem 2.15. This "extended" solution is classical and enjoys the same regularity as the solution in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-(U_1+\epsilon),U_2]}$; we will discuss this in more detail below. For the extended solution, we let $\mathscr{D}_\epsilon^+ = \mathscr{D}_\epsilon^+\left(\Upsilon\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}\right)\right)$ denote the future domain of dependence in Cartesian coordinate space $\mathbb{R}_t \times \mathbb{R}_{x^1} \times \mathbb{T}^2$ of the set $\Upsilon\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}\right)$ with respect to the acoustical metric $\mathbf{g}$. For each small $\delta > 0$, let $\mathscr{D}_{\epsilon;\delta}^+$ denote the subset of $\mathscr{D}_\epsilon^+$ consisting of the points in $\mathscr{D}_\epsilon^+$ that can be joined to $\Upsilon\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}\right)$ by a C^1 curve in $\mathscr{D}_\epsilon^+$ that has length $\leq \delta$ with respect to the standard Euclidean metric on $\mathbb{R}_t \times \mathbb{R}_{x^1} \times \mathbb{T}^2$; $\mathscr{D}_{\epsilon;\delta}^+$ is, in particular, a neighborhood of $\Upsilon\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}\right)$ in $\mathscr{D}_\epsilon^+$ such that the lateral boundaries of $\mathscr{D}_{\epsilon;\delta}^+$ contain $\mathbf{g}$-null hypersurface portions, one of which is a portion of $\mathcal{P}_{U_2}$; see Fig. 12 for a picture of the setup in geometric coordinate space, with the (x^2, x^3)-directions suppressed. Since Υ is a diffeomorphism on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-(U_1+\epsilon),U_2]}$, a standard compactness argument yields that if δ and $\epsilon > 0$ are small enough, then Υ^{-1} is a diffeomorphism from $\mathscr{D}_{\epsilon;\delta}^+$ onto its image in geometric coordinate space $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$. In particular, by using Υ^{-1} to change variables to geometric coordinates after extending in Cartesian coordinates, we see that the quantities Q – but not yet the rough time function – can be extended as classical solutions relative to the geometric coordinates to a larger region ${}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ of the following form for some sufficiently small $m > 0$:

$$
{}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m \stackrel{\text{def}}{=} {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]} \cup {}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m^{(\text{New region})}, \tag{31.6}
$$

$$
\begin{aligned}
{}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m^{(\text{New region})} &\stackrel{\text{def}}{=} \bigcup \left\{ \left(\mathrm{t}_{\tau_{\text{Boot}},\mathfrak{n}}(u,x^2,x^3) + m', u, x^2, x^3 \right) \mid (m',u,x^2,x^3) \in [0,m] \times [-U_1,U_2] \times \mathbb{T}^2 \right\} \\
&\subset \Upsilon^{-1}\left(\mathscr{D}_{\epsilon;\delta}^+\right),
\end{aligned} \tag{31.7}
$$

such that each of the following hypersurfaces:

$$
\left\{ \left(\mathrm{t}_{\tau_{\text{Boot}},\mathfrak{n}}(u,x^2,x^3) + m', u, x^2, x^3 \right) \mid (u,x^2,x^3) \times [-U_1,U_2] \times \mathbb{T}^2 \right\} \tag{31.8}
$$

is $\mathbf{g}$-spacelike and such that Υ is a diffeomorphism from ${}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ onto its image in Cartesian coordinate space. We clarify that in carrying out this argument, we have taken into account that the $\mathbf{g}$-null boundary of ${}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ on which $u \equiv U_2$ lies in the future domain of dependence (with respect to the acoustical metric $\mathbf{g}$) of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-U_1,U_2]}$ (without needing to extend to $u \in [-(U_1+\epsilon), U_2]$); this is tied to the fact that by construction, the level-sets of u are "right-moving" in Cartesian coordinate space, as is shown in Fig. 1B. In contrast, the $\mathbf{g}$-null boundary of ${}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ on which $u \equiv -U_1$ does not lie in the future domain of dependence of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-U_1,U_2]}$. It does, however, lie in the future domain of dependence of ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}$ whenever $\epsilon > 0$ and m is small compared to ϵ (this is the reason we are working with the extended hypersurface ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\text{Boot}}}^{[-(U_1+\epsilon),U_2]}$). Moreover, thanks to the bound $\|\mathrm{t}_{\tau_{\text{Boot}},\mathfrak{n}}\|_{C^{2,1}([-U_1,U_2]\times\mathbb{T}^2)} \leq C$ from Step 1, arguments similar to the ones we used in the proof of Lemma 15.5 can be used to show that ${}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ is quasi-convex in the sense of Item 6 in the statement of that lemma.

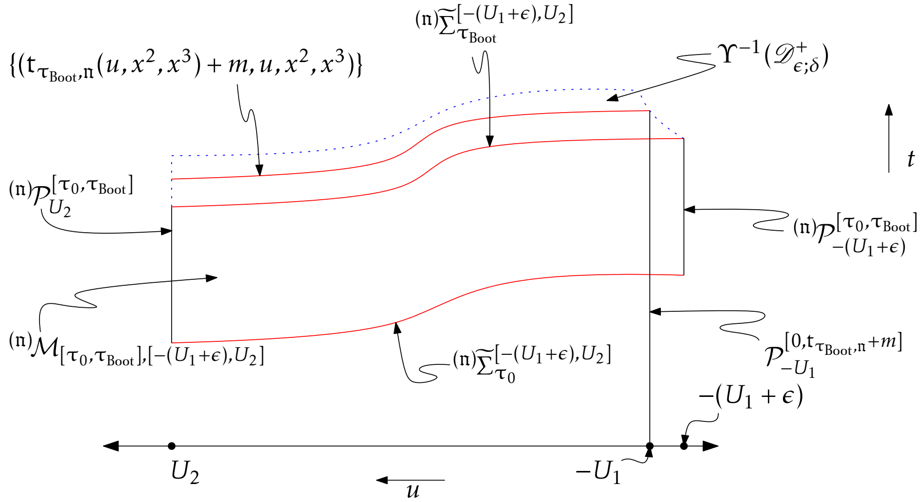


Figure 12. The extension to $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m^{(\text{New region})}$ in geometric coordinates

Given the energy estimate framework we established in the bulk of the paper (which yields L^2estimates up to top-order under commutations with the elements of the $\mathcal{P}_u$-tangent commutation set $\mathscr{P}$), it is also standard (see [68, Section 2.7] for the main ideas) that, as in Step 1, the solution enjoys the same regularity with respect to the geometric coordinates in the extended region $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ as it does in $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}$, where in $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m^{(\text{New region})}$, regularity is measured on the spacelike hypersurfaces (31.8) in geometric coordinate space as well as constant-u $\mathbf{g}$-null hypersurface portions contained in $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$. Similarly, the solution enjoys the same regularity with respect to the Cartesian coordinates in $\Upsilon\left({}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m\right)$ as it does in $\Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}\right)$.

Step 3: Extending the rough time function beyond $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}$. We first note that Lemmas 15.5, 15.6, and 15.7 and the relation (5.5) imply that the map $\mathscr{M}(t,u,x^2,x^3)=(\mu,\breve{X}\mu,x^2,x^3)$ defined in (5.3a) is a $C^{1,1}_{\mathrm{geo}}$ diffeomorphism from $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_{☢},U_{☢}]}$ onto a set containing $[\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]\times\{-\mathfrak{n}\}\times\mathbb{T}^2$ such that $\{\mathfrak{m}_{\mathrm{Boot}}\}\times\{-\mathfrak{n}\}\times\mathbb{T}^2\subset\mathscr{M}\left({}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\mathrm{Boot}}}^{(-U_{☢},U_{☢})}\right)$. By exploiting the compactness of $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$, we can, recalling that $\frac{\partial}{\partial t}\mu<0$ when $|u|\leq U_{☢}$ by (18.8b) and shrinking m if necessary, assume that $\mathscr{M}$ is a $C^{1,1}_{\mathrm{geo}}$ diffeomorphism from $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m\cap\{|u|\leq U_{☢}\}$ onto a set containing $\{\mathfrak{m}_{\mathrm{Boot}}\}\times\{-\mathfrak{n}\}\times\mathbb{T}^2$ in its interior, and that there is a $\Delta\in(0,\mathfrak{m}_{\mathrm{Boot}})$ such that $[\mathfrak{m}_{\mathrm{Boot}}-\Delta,\mathfrak{m}_{\mathrm{Boot}}+\Delta]\times[-\mathfrak{n}-\Delta,-\mathfrak{n}+\Delta]\times\mathbb{T}^2$ is contained in the interior of $\mathscr{M}\left({}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m\cap\{|u|\leq U_{☢}\}\right)$. In particular, for $\mathfrak{m}\in[\mathfrak{m}_{\mathrm{Boot}}-\Delta,\mathfrak{m}_{\mathrm{Boot}}+\Delta]$, the μ-adapted tori $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ defined in (4.3c) are two-dimensional $C^{1,1}_{\mathrm{geo}}$ sub-manifolds contained in the interior of $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m\cap\{|u|\leq U_{☢}\}$, and the hypersurface portion $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[-\mathfrak{m}_{\mathrm{Boot}}-\Delta,-\mathfrak{m}_{\mathrm{Boot}}+\Delta]}\overset{\text{def}}{=}\breve{\mathbb{X}}_{-\mathfrak{n}}\cap\left\{(t,u,x^2,x^3)\in\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2\mid\mathfrak{m}_{\mathrm{Boot}}-\Delta\leq\mu(t,u,x^2,x^3)\leq\mathfrak{m}_{\mathrm{Boot}}+\Delta\right\}$ is a three-dimensional $C^{1,1}_{\mathrm{geo}}$ sub-manifold contained in the interior of $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m\cap\{|u|\leq U_{☢}\}$ (see (4.3c) for the definition of $\breve{\mathbb{X}}_{-\mathfrak{n}}$, and compare with the alternate definition (4.7b) of $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[-\mathfrak{m}_{\mathrm{Boot}}-\Delta,-\mathfrak{m}_{\mathrm{Boot}}+\Delta]}$, which will eventually agree with the definition given above). Moreover, the estimate (**BA** μ cnvx), which by continuity holds in $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m\cap\{|u|\leq U_{☢}\}$ with different constants, implies that the vectorfield $^{(\mathfrak{n})}\breve{W}$ is transversal to $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[-\mathfrak{m}_{\mathrm{Boot}}-\Delta,-\mathfrak{m}_{\mathrm{Boot}}+\Delta]}$. Considering also that $^{(\mathfrak{n})}\breve{W}$ is tangent to the lower boundary $^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_{\mathrm{Boot}}}^{[-U_1,U_2]}$ of $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m^{(\text{New region})}$, we see that these results are sufficient to allow us to extend the construction of the rough time function $^{(\mathfrak{n})}\tau$ (see Sect. 4.1, Lemma 14.2, and Lemma 15.1) from $^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}$ to all of $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$, i.e., so that $^{(\mathfrak{n})}\tau$ is defined on $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ (where we have perhaps shrunk m if necessary) and satisfies $\|{}^{(\mathfrak{n})}\tau\|_{W^{3,\infty}_{\mathrm{geo}}(\mathrm{int}({}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m))}\leq C$. From this bound, the quasi-convexity of $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ mentioned in Step 2, and the Sobolev embedding result (15.26) – which also holds for the domain $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ thanks to its quasi-convexity – we also find that $\|{}^{(\mathfrak{n})}\tau\|_{C^{2,1}_{\mathrm{geo}}({}^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m)}\leq C$. Considering also the estimate $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau\approx 1$, which holds in $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ by (15.12a) and continuity, we further deduce that the map $^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3)=({}^{(\mathfrak{n})}\tau,u,x^2,x^3)$ is a diffeomorphism from $^{(\mathfrak{n})}\widetilde{\mathcal{M}}_m$ onto its image, and that there is a $\Delta > 0$ (perhaps smaller than before) such that the image set contains $[\tau_0, \tau_{\text{Boot}} + \Delta] \times [-U_1, U_2] \times \mathbb{T}^2$. That is, $\|{}^{(\mathfrak{n})}\tau\|_{C^{2,1}_{\text{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}+\Delta],[-U_1,U_2]})} \lesssim 1$, and ${}^{(\mathfrak{n})}\mathscr{T}$ is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}+\Delta],[-U_1,U_2]}$ onto its image.

Step 4: Propagation of regularity on the foliation induced by the extended ${}^{(\mathfrak{n})}\tau$. We have constructed the solution on the extended region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}+\Delta],[-U_1,U_2]}$ as well as the foliation $\left\{{}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}\right\}_{\tau\in[\tau_0,\tau_{\text{Boot}}+\Delta]}$ of it. We can now argue as in Step 2 to deduce that the solution enjoys the same regularity with respect to the geometric coordinates in the extended region ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}+\Delta],[-U_1,U_2]}$ as it does in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$, where, as in the bulk of the paper, regularity is measured on the hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ (which, by continuity, are $\mathbf{g}$-spacelike for Δ sufficiently small), on constant-u $\mathbf{g}$-null hypersurface portions ${}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_0,\tau_0+\Delta]}$, and on the rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$.

Step 5: The bootstrap assumptions hold for the extended solution. The results we proved throughout the paper have yielded, on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}),[-U_1,U_2]}$, strict improvements of all the bootstrap assumptions of Sects. 12 and 24.3; see Sect. 12.5 for a description of the results that yield the improvements. Hence, by exploiting the continuity guaranteed by local well-posedness, we conclude that all the bootstrap assumptions of Sects. 12 and 24.3 also hold on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}+\Delta),[-U_1,U_2]}$. □

32. Developments of the data, the singular boundary and the crease, and a new time function

In Sect. 34, we will state and prove Theorem 34.1, which is our main theorem on the behavior of the solution up to the singular boundary. To prove Theorem 34.1, we will amalgamate some of our prior results that we derived at fixed $\mathfrak{n} \in [0, \mathfrak{n}_0]$. In this section, we carry out some of these tasks by constructing a new region $\mathcal{M}_{\text{Interesting}}$ that contains the portion of the singular boundary featured in Theorem 34.1. We also construct a corresponding time function ${}^{(\text{Interesting})}\tau$ that foliates $\mathcal{M}_{\text{Interesting}}$, and our construction is such that the singular boundary portion of interest (including the crease) is contained in the level-set $\{{}^{(\text{Interesting})}\tau = 0\}$, which forms the top boundary of $\mathcal{M}_{\text{Interesting}}$. Finally, we derive fundamental properties of $\mathcal{M}_{\text{Interesting}}$ and ${}^{(\text{Interesting})}\tau$.

Actually, as part of our construction in this section (see Sect. 32.2), we rigorously *define* the singular boundary and crease. In Theorem 34.1, we reveal the behavior of the solution up to these sets, and the results of the theorem will justify our definitions.

32.1. Definitions of developments.

For each $\mathfrak{n} \in [0, \mathfrak{n}_0]$, Theorem 31.1 yields the development ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ of the data, which contains the μ-adapted torus $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$, a subset of the singular boundary. Using the ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ as building blocks, we now define other developments, including $\mathcal{M}_{\text{Interesting}}$, on which we will derive refined estimates at the low derivative levels, revealing the detailed structure of the singular boundary. In particular, in Prop. 32.3, we exhibit various key properties of $\mathcal{M}_{\text{Interesting}}$.

Definition 32.1 (The development $\mathcal{M}_{\text{Interesting}}$ and constituent subsets)**.** We define the following subsets of geometric coordinate space (see Fig. 13), where the subset ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$ is defined in (4.6d), the hypersurface portion $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]}$ is defined in (4.7b), and to obtain the second equality in (32.1b), we used (15.44):

$$\mathcal{M}_{\text{Left}} \stackrel{\text{def}}{=} {}^{(0)}\mathcal{M}_{[\tau_0,0],[U_{\clubsuit},U_2]} \cup \left({}^{(0)}\mathcal{M}_{[\tau_0,0],[-U_{\clubsuit},U_{\clubsuit}]} \cap \{\breve{X}\mu > 0\}\right), \tag{32.1a}$$

$$\mathcal{M}_{\text{Singular}} \stackrel{\text{def}}{=} \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]} = \bigcup_{(\mathfrak{m},\mathfrak{n})\in[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}, \tag{32.1b}$$

$$\mathcal{M}_{\text{Right}} \stackrel{\text{def}}{=} {}^{(\mathfrak{n}_0)}\mathcal{M}_{[\tau_0,0],[-U_1,-U_{\clubsuit}]} \cup \left({}^{(\mathfrak{n}_0)}\mathcal{M}_{[\tau_0,0],[-U_{\clubsuit},U_{\clubsuit}]} \cap \{\breve{X}\mu < -\mathfrak{n}_0\}\right), \tag{32.1c}$$

$$\mathcal{M}_{\text{Interesting}} \stackrel{\text{def}}{=} \mathcal{M}_{\text{Left}} \cup \mathcal{M}_{\text{Singular}} \cup \mathcal{M}_{\text{Right}}. \tag{32.1d}$$

Remark 32.2 ($|u| \leq \frac{1}{2}U_{\clubsuit}$ in $\mathcal{M}_{\text{Singular}}$)**.** Note that by (18.3a), we have $|u| \leq \frac{1}{2}U_{\clubsuit}$ in $\mathcal{M}_{\text{Singular}}$. In our subsequent analysis, we often silently use this fact.

Remark 32.3 (The constituent pieces of $\mathcal{M}_{\text{Left}}$ join smoothly, and similarly for $\mathcal{M}_{\text{Right}}$)**.** The two subsets ${}^{(0)}\mathcal{M}_{[\tau_0,0],[U_{\clubsuit},U_2]}$ and ${}^{(0)}\mathcal{M}_{[\tau_0,0],[-U_{\clubsuit},U_{\clubsuit}]} \cap \{\breve{X}\mu > 0\}$, whose (non-disjoint) union defines $\mathcal{M}_{\text{Left}}$, join smoothly. The reason is that by (15.35a),

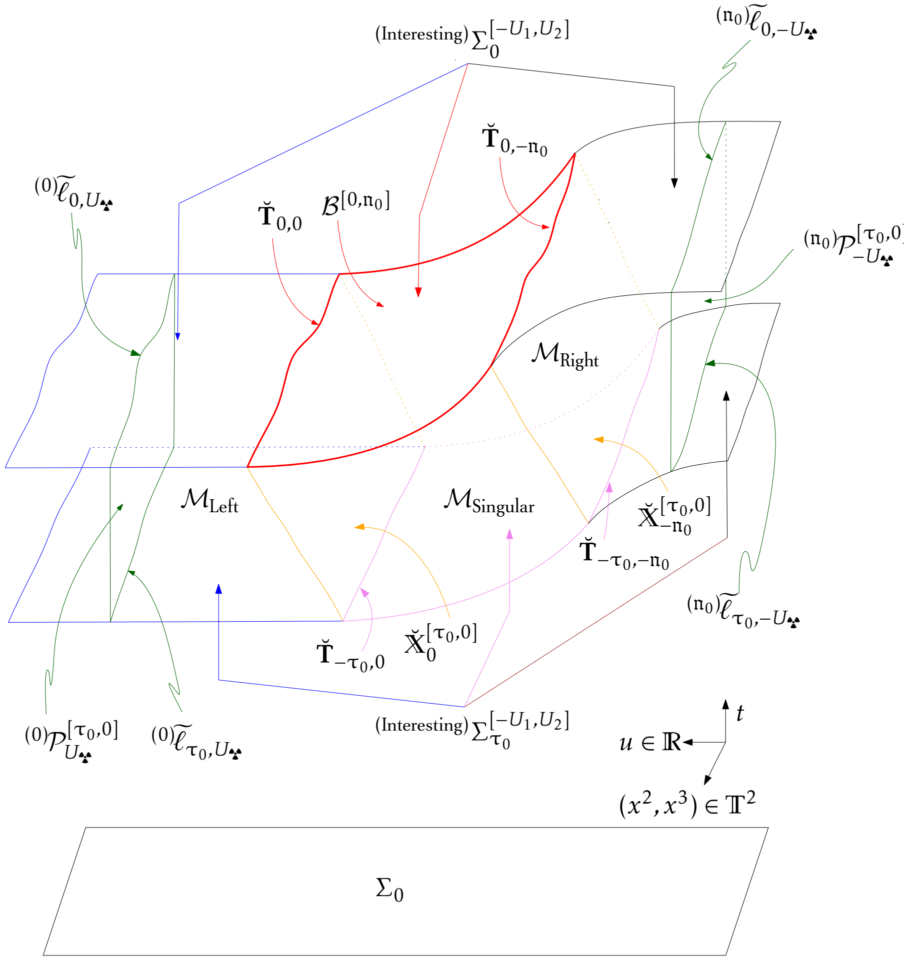


Figure 13. The region $\mathcal{M}_{\text{Interesting}}$ featured in Theorems 1.4 and 34.1

$\breve{X}\mu$ is strictly positive along $^{(0)}\mathcal{P}_{U_{☢}}^{[\tau_0,0]}$. Analogous remarks apply for $\mathcal{M}_{\text{Right}}$, in view of the estimate (15.35b), which shows that $\breve{X}\mu$ is strictly less than $-\mathfrak{n}_0$ along $^{(\mathfrak{n}_0)}\mathcal{P}_{-U_{☢}}^{[\tau_0,0]}$.

32.2. **The singular boundary and the crease.** We are now ready to define the singular boundary and the crease. In Prop. 32.3, we derive some crucial properties that these sets enjoy.

Definition 32.4 ($\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$, the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]}$, and the crease $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$)**.** For each fixed $\mathfrak{m} \in [0,\mathfrak{m}_0]$, we define the set $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ as follows, where $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ is the μ-adapted torus defined in (4.3c):

$$\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]} \stackrel{\text{def}}{=} \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}. \tag{32.2}$$

We define the **singular boundary** portion $\mathcal{B}^{[0,\mathfrak{n}_0]}$ as follows:

$$\mathcal{B}^{[0,\mathfrak{n}_0]} \stackrel{\text{def}}{=} \breve{\mathbb{M}}_0^{[0,\mathfrak{n}_0]} = \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{0,-\mathfrak{n}}. \tag{32.3}$$

We define the **crease** $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$ as follows:

$$\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]} \stackrel{\text{def}}{=} \breve{\mathbf{T}}_{0,0}. \tag{32.4}$$

From (32.1b) and definitions (32.3)–(32.4), it follows that $\mathcal{B}^{[0,\mathfrak{n}_0]} \subset \mathcal{M}_{\text{Singular}}$ and $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]} \subset \mathcal{M}_{\text{Singular}}$. We also note that from Def. 4.2, (32.1b), and (32.2), it follows that:

$$\check{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]} = \{(t,u,x^2,x^3) \mid \upmu(t,u,x^2,x^3) = \mathfrak{m}\} \cap \mathcal{M}_{\text{Singular}} \tag{32.5}$$

and that:

$$\mathcal{M}_{\text{Singular}} = \bigcup_{\mathfrak{m}\in[0,\mathfrak{m}_0]} \check{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}. \tag{32.6}$$

32.3. **The structure of $\mathcal{M}_{\text{Left}}$, $\mathcal{M}_{\text{Right}}$, $\mathcal{M}_{\text{Singular}}$, $\mathcal{B}^{[0,\mathfrak{n}_0]}$, and $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}$.** In the next proposition, we derive key properties of the sets that we defined in Sects. 32.1 and 32.2.

Proposition 32.5 (The structure of $\mathcal{M}_{\text{Left}}$, $\mathcal{M}_{\text{Right}}$, $\mathcal{M}_{\text{Singular}}$, $\mathcal{B}^{[0,\mathfrak{n}_0]}$, and $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}$)**.** *Assume the hypotheses and conclusions of Theorem 31.1 for* $\mathfrak{n} \in [0,\mathfrak{n}_0]$. *Let* ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$, ${}^{(\mathfrak{n})}\mathcal{P}_u^I$, $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$, *and* $\check{\mathbb{X}}_{-\mathfrak{n}}^I$ *be the sets defined in* (4.6b), (4.6c), (4.7b), *and* (4.3c) *respectively. Then the following conclusions hold (see Figs. 6 and 13).*

Differential-topological structure of $\mathcal{M}_{\text{Left}}$ ***in geometric coordinate space.***

- *The left lateral boundary of the set* $\mathcal{M}_{\text{Left}}$ *defined in* (32.1a) *is* ${}^{(0)}\mathcal{P}_{U_2}^{[\tau_0,0]}$, *which is a smooth hypersurface with boundary components* ${}^{(0)}\widetilde{\ell}_{0,U_2}$ *and* ${}^{(0)}\widetilde{\ell}_{\tau_0,U_2}$, *each of which are* $C^{2,1}$ *graphs over* $\mathbb{T}^2$.
- *The right lateral boundary of* $\mathcal{M}_{\text{Left}}$ *is* $\check{\mathbb{X}}_0^{[\tau_0,0]}$, *which is a* $C^{1,1}$ *hypersurface with boundary components* $\check{\mathbf{T}}_{0,0}$ *and* $\check{\mathbf{T}}_{-\tau_0,0}$, *each of which are* $C^{1,1}$ *graphs over* $\mathbb{T}^2$. *Moreover,* $\check{\mathbb{X}}_0^{[\tau_0,0]} = \{(t,u,x^2,x^3) \mid (t,x^2,x^3) \in {}^{(0)}\mathscr{H}_{[0,\mathfrak{m}_0]}, u = {}^{(0)}h(t,x^2,x^3)\}$, *where* ${}^{(0)}h$ *is the* $C^{1,1}$ *function on* ${}^{(0)}\mathscr{H}_{[0,\mathfrak{m}_0]} = \{(t,x^2,x^3) \in \mathbb{R}\times\mathbb{T}^2 \mid \mathfrak{T}_{\mathfrak{m}_0,0}(x^2,x^3) \leq t \leq \mathfrak{T}_{0,0}(x^2,x^3)\}$ *from Cor. 15.8.*
- *The top boundary of* $\mathcal{M}_{\text{Left}}$ *is the* $C^{2,1}$ *hypersurface* $\{(t,u,x^2,x^3) \mid (x^2,x^3) \in \mathbb{T}^2, \mathfrak{U}_{0,0}(x^2,x^3) \leq u \leq U_2$, *and* $t = \mathfrak{t}_{0,0}(u,x^2,x^3)\}$, *where* $\mathfrak{U}_{0,0}$ *is the* $C^{1,1}$ *function from Lemma 15.7 and* $\mathfrak{t}_{0,0}$ *is the* $C^{2,1}$ *function from Lemma 15.5. The two boundary components of this hypersurface are* ${}^{(0)}\widetilde{\ell}_{0,U_2}$, *which is a* $C^{2,1}$ *graph over* $\mathbb{T}^2$, *and the crease* $\check{\mathbf{T}}_{0,0}$, *which is a* $C^{1,1}$ *graph over* $\mathbb{T}^2$.
- *The bottom boundary of* $\mathcal{M}_{\text{Left}}$ *has an analogous structure: it is the* $C^{2,1}$ *hypersurface* $\{(t,u,x^2,x^3) \mid (x^2,x^3) \in \mathbb{T}^2, \mathfrak{U}_{\tau_0,0}(x^2,x^3) \leq u \leq U_2$, *and* $t = \mathfrak{t}_{\tau_0,0}(u,x^2,x^3)\}$, *which has the* $C^{2,1}$ *boundary component* ${}^{(0)}\widetilde{\ell}_{\tau_0,U_2}$ *and the* $C^{1,1}$ *boundary component* $\check{\mathbf{T}}_{\tau_0,0}$, *each of which are graphs over* $\mathbb{T}^2$.
- $\mathcal{M}_{\text{Left}}$ *is not closed, but it contains all of its limit points, except for the points in its right lateral boundary* $\check{\mathbb{X}}_0^{[\tau_0,0]}$.

Differential-topological structure of $\mathcal{M}_{\text{Right}}$ ***in geometric coordinate space.***

- *The left lateral boundary of the set* $\mathcal{M}_{\text{Right}}$ *defined in* (32.1c) *is equal to* $\check{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$, *which is a* $C^{1,1}$ *hypersurface with boundary components* $\check{\mathbf{T}}_{0,-\mathfrak{n}_0}$ *and* $\check{\mathbf{T}}_{-\tau_0,-\mathfrak{n}_0}$, *each of which are* $C^{1,1}$ *graphs over* $\mathbb{T}^2$. *Moreover,* $\check{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]} = \{(t,u,x^2,x^3) \mid (t,x^2,x^3) \in \mathscr{H}_{\mathfrak{n}_0}^{[0,\mathfrak{m}_0]}, u = {}^{(\mathfrak{n}_0)}h(t,x^2,x^3)\}$, *where* ${}^{(\mathfrak{n}_0)}h$ *is the* $C^{1,1}$ *function on* ${}^{(\mathfrak{n}_0)}\mathscr{H}_{[0,\mathfrak{m}_0]} = \{(t,x^2,x^3) \in \mathbb{R}\times\mathbb{T}^2 \mid \mathfrak{T}_{\mathfrak{m}_0,-\mathfrak{n}_0}(x^2,x^3) \leq t \leq \mathfrak{T}_{0,-\mathfrak{n}_0}(x^2,x^3)\}$ *from Cor. 15.8.*
- *The right lateral boundary of* $\mathcal{M}_{\text{Right}}$ *is* ${}^{(\mathfrak{n}_0)}\mathcal{P}_{-U_1}^{[\tau_0,0]}$, *which is a smooth hypersurface with boundary components* ${}^{(\mathfrak{n}_0)}\widetilde{\ell}_{0,-U_1}$ *and* ${}^{(\mathfrak{n}_0)}\widetilde{\ell}_{\tau_0,-U_1}$, *each of which are* $C^{2,1}$ *graphs over* $\mathbb{T}^2$.
- *The top boundary of* $\mathcal{M}_{\text{Right}}$ *is the* $C^{2,1}$ *hypersurface* $\{(t,u,x^2,x^3) \mid (x^2,x^3) \in \mathbb{T}^2, -U_1 \leq u \leq \mathfrak{U}_{0,\mathfrak{n}_0}(x^2,x^3)$, *and* $t = \mathfrak{t}_{0,\mathfrak{n}_0}(u,x^2,x^3)\}$, *where* $\mathfrak{U}_{0,\mathfrak{n}_0}$ *is the* $C^{1,1}$ *function from Lemma 15.7 and* $\mathfrak{t}_{0,\mathfrak{n}_0}$ *is the* $C^{2,1}$ *function from Lemma 15.5. The two boundary components are* ${}^{(\mathfrak{n}_0)}\widetilde{\ell}_{0,-U_1}$, *which is a* $C^{2,1}$ *graph over* $\mathbb{T}^2$, *and* $\check{\mathbf{T}}_{0,-\mathfrak{n}_0}$, *which is a* $C^{1,1}$ *graph over* $\mathbb{T}^2$.
- *The bottom boundary of* $\mathcal{M}_{\text{Right}}$ *has an analogous structure: it is the* $C^{2,1}$ *hypersurface* $\{(t,u,x^2,x^3) \mid (x^2,x^3) \in \mathbb{T}^2, \mathfrak{U}_{\tau_0,\mathfrak{n}_0}(x^2,x^3) \leq u \leq U_2$, *and* $t = \mathfrak{t}_{\tau_0,\mathfrak{n}_0,}(u,x^2,x^3)\}$, *which has the* $C^{1,1}$ *boundary component* $\check{\mathbf{T}}_{\tau_0,\mathfrak{n}_0}$ *and the* $C^{2,1}$ *boundary component* ${}^{(\mathfrak{n}_0)}\widetilde{\ell}_{\tau_0,-U_1}$, *each of which are graphs over* $\mathbb{T}^2$.
- $\mathcal{M}_{\text{Right}}$ *is not closed, but it contains all of its limit points except for the points in its left lateral boundary* $\check{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$.

A diffeomorphism onto $\mathcal{M}_{\text{Singular}}$. *Let* $\mathfrak{E}$ *be the map from* $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$ *into geometric coordinate space* $\mathbb{R}_t\times\mathbb{R}_u\times\mathbb{T}^2$ *defined by:*

$$\mathfrak{E}(\mathfrak{m},\mathfrak{n},x^2,x^3)\stackrel{def}{=}{}^{(\mathfrak{n})}E(\mathfrak{m},x^2,x^3)=\Big(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\Big), \tag{32.7}$$

where ${}^{(\mathfrak{n})}E$, $\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}$, *and* $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}$ *are the functions from* (12.4) *and Cor. 15.8. Then the following results hold.*

- *For* $(\mathfrak{m},\mathfrak{n})\in[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]$, *we have:*
$$\mathfrak{E}(\{\mathfrak{m}\}\times\{\mathfrak{n}\}\times\mathbb{T}^2)=\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}. \tag{32.8}$$
- $\mathfrak{E}$ *is a diffeomorphism from* $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$ *onto the set* $\mathcal{M}_{\text{Singular}}$ *(which we view to be a subset of geometric coordinate space) defined in* (32.1b) *such that the following estimate holds:*
$$\|\mathfrak{E}\|_{C^{1,1}([0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2)}\leq C. \tag{32.9}$$
- *In particular, the map* $\mathfrak{S}$ *defined by:*
$$\mathfrak{S}(\mathfrak{n},x^2,x^3)\stackrel{def}{=}\mathfrak{E}(0,\mathfrak{n},x^2,x^3) \tag{32.10}$$
is a diffeomorphism from $[0,\mathfrak{n}_0]\times\mathbb{T}^2$ *onto the singular boundary portion* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *defined in* (32.3) *such that:*
$$\mathfrak{S}(\{\mathfrak{n}\}\times\mathbb{T}^2)=\check{\mathbf{T}}_{0,-\mathfrak{n}} \tag{32.11}$$
and:
$$\|\mathfrak{S}\|_{C^{1,1}([0,\mathfrak{n}_0]\times\mathbb{T}^2)}\leq C. \tag{32.12}$$
- *For* $(\mathfrak{m},\mathfrak{n})\in[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]$ *the torus* $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *is a* $C^{1,1}$ *graph over* $\mathbb{T}^2$ *that is* $\mathbf{g}$*-spacelike.*
- *There exists a constant* $C>1$ *such that for* $(\mathfrak{m},\mathfrak{n})\in[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]$, *we have:*
$$-C\leq\min_{(x^2,x^3)\in\mathbb{T}^2}\frac{\partial}{\partial\mathfrak{m}}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)\leq\max_{(x^2,x^3)\in\mathbb{T}^2}\frac{\partial}{\partial\mathfrak{m}}\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)\leq-\frac{1}{C}, \tag{32.13a}$$
$$-C\leq\min_{(x^2,x^3)\in\mathbb{T}^2}\frac{\partial}{\partial\mathfrak{n}}\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)\leq\max_{(x^2,x^3)\in\mathbb{T}^2}\frac{\partial}{\partial\mathfrak{n}}\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3)\leq-\frac{1}{C}. \tag{32.13b}$$
- *For each fixed* $\mathfrak{m}\in[0,\mathfrak{m}_0]$, *the map* ${}^{(\mathfrak{m})}J$ *defined by:*
$${}^{(\mathfrak{m})}J(\mathfrak{n},x^2,x^3)\stackrel{def}{=}\Big(\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\Big) \tag{32.14}$$
satisfies the following estimates, where $d_{(\mathfrak{n},x^2,x^3)}{}^{(\mathfrak{m})}J$ *denotes the differential of* ${}^{(\mathfrak{m})}J$ *with respect to* $(\mathfrak{n},x^2,x^3)$*:*
$$\|{}^{(\mathfrak{m})}J\|_{C^{1,1}([0,\mathfrak{n}_0]\times\mathbb{T}^2)}\leq C, \tag{32.15a}$$
$$\det d_{(\mathfrak{n},x^2,x^3)}{}^{(\mathfrak{m})}J\approx-1, \qquad \text{on } [0,\mathfrak{n}_0]\times\mathbb{T}^2. \tag{32.15b}$$
Moreover, ${}^{(\mathfrak{m})}J$ *is a diffeomorphism from* $[0,\mathfrak{n}_0]\times\mathbb{T}^2$ *onto the image set* ${}^{(\mathfrak{m})}J([0,\mathfrak{n}_0]\times\mathbb{T}^2)$, *which is:*
$$\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}\stackrel{def}{=}\Big\{(u,x^2,x^3)\mid(x^2,x^3)\in\mathbb{T}^2,\,\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}_0}(x^2,x^3)\leq u\leq\mathfrak{U}_{\mathfrak{m},0}(x^2,x^3)\Big\}. \tag{32.16}$$
Furthermore, $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *is a quasi-convex subset of* $\mathbb{R}_u\times\mathbb{T}^2$ *in the following sense: every pair of points* $r_1,r_2\in\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *is connected by a* C^1 *curve in* $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *whose length with respect to the standard Euclidean metric on* $\mathbb{R}_u\times\mathbb{T}^2$ *is* $\lesssim\text{dist}_{\text{flat}}(r_1,r_2)$, *where* $\text{dist}_{\text{flat}}(r_1,r_2)$ *is the standard Euclidean distance between* r_1 *and* r_2 *in the flat space* $\mathbb{R}_u\times\mathbb{T}^2$.
Finally, the map ${}^{(\mathfrak{m})}J^{-1}$ *satisfies the following estimate:*
$$\|{}^{(\mathfrak{m})}J^{-1}\|_{C^{1,1}\left(\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}\right)}\leq C. \tag{32.17}$$

t ***is a function of*** (u,x^2,x^3) ***along the level-sets of*** μ ***in*** $\mathcal{M}_{\text{Singular}}$. *Let* $\check{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *be the set defined in* (32.2), *and recall that* $\check{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *is the portion of the level-set* $\{(t,u,x^2,x^3)\mid\mu(t,u,x^2,x^3)=\mathfrak{m}\}$ *in* $\mathcal{M}_{\text{Singular}}$. *Then the following results hold.*

- *Recall that the set* $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *is defined in* (32.16). *Then for each* $\mathfrak{m} \in [0, \mathfrak{m}_0]$, *there exists a function* $\breve{T}_{\mathfrak{m}} : \mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]} \to \mathbb{R}$ *such that relative to the geometric coordinates, we have:*

$$\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]} = \left\{ \left(\breve{T}_{\mathfrak{m}}(u, x^2, x^3), u, x^2, x^3 \right) \mid (u, x^2, x^3) \in \mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]} \right\}. \tag{32.18}$$

- *There exists a* $C > 0$ *such that following estimate holds for* $\mathfrak{m} \in [0, \mathfrak{m}_0]$:

$$\|\breve{T}_{\mathfrak{m}}\|_{C^{2,1}\left(\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}\right)} \leq C. \tag{32.19}$$

In particular, $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *is a* 3*-dimensional* $C^{2,1}$ *sub-manifold-with-boundary in geometric coordinate space.*

- *The boundary of* $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *in geometric coordinate space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$ *satisfies* $\partial \breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]} = \breve{\mathbf{T}}_{\mathfrak{m},0} \cup \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}_0}$, *where the* μ*-adapted tori* $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *are* $C^{1,1}$ *graphs over* $\mathbb{T}^2$ *(as is indicated in* (15.43)*).*

Differential-topological structure of $\mathcal{M}_{\text{Singular}}$.

- (***Quasi-convexity***) $\mathcal{M}_{\text{Singular}}$ *is quasi-convex. That is, there is a constant* $C > 0$ *such that every pair of points* $p_1, p_2 \in \mathcal{M}_{\text{Singular}}$ *is connected by a* C^1_{geo} *curve in* $\mathcal{M}_{\text{Singular}}$ *whose length with respect to the standard flat Euclidean metric on geometric coordinate space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$ *is* $\leq C\text{dist}_{\text{flat}}(p_1, p_2)$, *where* $\text{dist}_{\text{flat}}(p_1, p_2)$ *is the standard Euclidean distance between* p_1 *and* p_2 *in the flat space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$.
- (***Sobolev embedding***). *There is a constant* $C > 0$ *such that the following Sobolev embedding result holds for scalar functions* f *on* $\text{int}(\mathcal{M}_{\text{Singular}})$:

$$\|f\|_{C^{0,1}_{\text{geo}}(\mathcal{M}_{\text{Singular}})} \leq C\|f\|_{W^{1,\infty}_{\text{geo}}(\text{int}(\mathcal{M}_{\text{Singular}}))}. \tag{32.20}$$

- *Let* $\mathfrak{L}^{-1}$ *denote the inverse function of the map* $\mathfrak{L}$ *from* (32.7), *i.e.,* $\mathfrak{L}^{-1}(t, u, x^2, x^3) = \left(\mu, -\breve{X}\mu, x^2, x^3\right)$. *Then* $\mathfrak{L}^{-1}$ *is a* $C^{1,1}_{\text{geo}}$ *diffeomorphism from* $\mathcal{M}_{\text{Singular}}$ *onto* $[0, \mathfrak{m}_0] \times [0, \mathfrak{n}_0] \times \mathbb{T}^2$.
- $\mathfrak{L}^{-1}(\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}) = \{\mathfrak{m}\} \times \{-\mathfrak{n}\} \times \mathbb{T}^2$.
- *The following estimates hold:*

$$\left\|\mathfrak{L}^{-1}\right\|_{C^{1,1}_{\text{geo}}(\mathcal{M}_{\text{Singular}})} \leq C, \tag{32.21a}$$

$$\|\mu\|_{C^{2,1}_{\text{geo}}(\mathcal{M}_{\text{Singular}})} \leq C. \tag{32.21b}$$

- *The following estimates hold on* $\mathcal{M}_{\text{Singular}}$, *where* $d_{\text{geo}}\mathfrak{L}^{-1}$ *denotes the differential of* $\mathfrak{L}^{-1}$ *with respect to the geometric coordinates:*

$$\det d_{\text{geo}}\mathfrak{L}^{-1} \approx 1 \qquad \text{on } \mathcal{M}_{\text{Singular}}, \tag{32.22}$$

$$-\frac{9}{8}\mathring{\delta}_* \leq \min_{\mathcal{M}_{\text{Singular}}} \frac{\partial}{\partial t}\mu \leq \max_{\mathcal{M}_{\text{Singular}}} \frac{\partial}{\partial t}\mu \leq -\frac{7}{8}\mathring{\delta}_* \tag{32.23}$$

$$\frac{M_2}{2} \leq \min_{\mathcal{M}_{\text{Singular}}} \frac{\partial}{\partial u}\breve{X}\mu \leq \max_{\mathcal{M}_{\text{Singular}}} \frac{\partial}{\partial u}\breve{X}\mu \leq \frac{2}{M_2}. \tag{32.24}$$

- *The two lateral boundaries of* $\mathcal{M}_{\text{Singular}}$ *are the* $C^{1,1}$ *embedded hypersurfaces* $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ *and* $\breve{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$ *mentioned above, which have* $C^{1,1}$ *boundaries. In particular,* $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ *is the left lateral boundary of* $\mathcal{M}_{\text{Singular}}$, *while* $\breve{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$ *is its right lateral boundary.*
- *The top boundary of* $\mathcal{M}_{\text{Singular}}$ *is* $\breve{\mathbb{M}}_0^{[0,\mathfrak{n}_0]}$, *which is equal to the singular boundary portion* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *defined in* (32.3). *It is a* $C^{2,1}$ *embedded hypersurface with the boundary components* $\breve{\mathbf{T}}_{0,0}$ *and* $\breve{\mathbf{T}}_{0,-\mathfrak{n}_0}$, *which are* $C^{1,1}$ *graphs over* $\mathbb{T}^2$.
- *The bottom boundary of* $\mathcal{M}_{\text{Singular}}$ *is* $\breve{\mathbb{M}}_{\mathfrak{m}_0}^{[0,\mathfrak{n}_0]}$, *and it is a* $C^{2,1}$ *embedded hypersurface with the boundary components* $\breve{\mathbf{T}}_{\mathfrak{m}_0,0}$ *and* $\breve{\mathbf{T}}_{\mathfrak{m}_0,-\mathfrak{n}_0}$, *which are* $C^{1,1}$ *graphs over* $\mathbb{T}^2$.

Proof. Throughout the proof, we will silently use Theorem 31.1, which shows that the bootstrap assumptions and results proved prior to Theorem 31.1 hold with $\tau_{\text{Boot}} = 0$ and $\mathfrak{m}_{\text{Boot}} = 0$.

Proof of the properties $\mathcal{M}_{\mathrm{Left}}$ and $\mathcal{M}_{\mathrm{Right}}$: We give the proof only for $\mathcal{M}_{\mathrm{Left}}$ since the properties of $\mathcal{M}_{\mathrm{Right}}$ can be derived using similar arguments. The fact that the left lateral boundary of $\mathcal{M}_{\mathrm{Left}}$ is ${}^{(0)}\mathcal{P}_{U_2}^{[\tau_0,0]}$ follows from Def. 4.11 and definition (32.1a). These same definitions also imply that ${}^{(0)}\mathcal{P}_{U_2}^{[\tau_0,0]}$ is contained in $\mathcal{M}_{\mathrm{Left}}$. The regularity and structure of the boundary components of ${}^{(0)}\mathcal{P}_{U_2}^{[\tau_0,0]}$, namely ${}^{(0)}\widetilde{\ell}_{0,U_2}$ and ${}^{(0)}\widetilde{\ell}_{\tau_0,U_2}$, follow from the estimate (15.18) and the fact that ${}^{(0)}\widetilde{\ell}_{0,U_2} = {}^{(\mathfrak{n})}\mathscr{T}^{-1}\left(\{0\}\times\{U_2\}\times\mathbb{T}^2\right)$ and ${}^{(0)}\widetilde{\ell}_{\tau_0,U_2} = {}^{(\mathfrak{n})}\mathscr{T}^{-1}\left(\{\tau_0\}\times\{U_2\}\times\mathbb{T}^2\right)$.

The fact that the right lateral boundary of $\mathcal{M}_{\mathrm{Left}}$ is $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ follows from definition (32.1a), (18.3a), and Lemmas 15.5 Lemma 15.7 with $\mathfrak{n}=0$, which in particular imply that $\breve{\mathbb{X}}_0^{[\tau_0,0]} = {}^{(0)}\mathscr{T}^{-1}\circ{}^{(0)}\Phi^{-1}\left([0,\mathfrak{m}_0]\times\{0\}\times\mathbb{T}^2\right)\subset {}^{(0)}\mathcal{M}_{[\tau_0,0],[-\frac{3}{4}U_{☢},\frac{3}{4}U_{☢}]}$. We also see, in view of definition (32.1a) and the fact that $\breve{X}\mu=0$ along $\breve{\mathbb{X}}_0^{[\tau_0,0]}$, that $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ does not belong to $\mathcal{M}_{\mathrm{Left}}$. We established the regularity and structure of $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ and its boundary components $\breve{\mathbf{T}}_{0,0}$ and $\breve{\mathbf{T}}_{-\tau_0,0}$ in Cor. 15.8 (with $\mathfrak{n}=0$).

Next, we note that the arguments given in the previous paragraph, together with (15.32), (15.43), and (15.28), imply that the top boundary of $\mathcal{M}_{\mathrm{Left}}$ is the subset of ${}^{(0)}\widetilde{\Sigma}_0^{[-U_1,U_2]} = \{(t,u,x^2,x^3)\mid t=\mathfrak{t}_{0,0}(u,x^2,x^3),\,(u,x^2,x^3)\in[-U_1,U_2]\times\mathbb{T}^2\}$ in which $\mathfrak{U}_{0,0}(x^2,x^3)\leq u\leq U_2$. In particular, the two boundary components of the subset under consideration are ${}^{(0)}\widetilde{\Sigma}_0^{[-U_1,U_2]}\cap\{u=U_2\} = {}^{(0)}\widetilde{\ell}_{0,U_2}$ and ${}^{(0)}\widetilde{\Sigma}_0^{[-U_1,U_2]}\cap\{(t,u,x^2,x^3)\mid u=\mathfrak{U}_{0,0}(x^2,x^3)\} = \breve{\mathbf{T}}_{0,0}$, where to obtain the last identity, we used (15.37) with $\mathfrak{n}=0$. We derived the $C^{2,1}$-regularity of $\mathfrak{t}_{0,0}$ in (15.27). We derived the $C^{2,1}$-regularity of ${}^{(0)}\widetilde{\ell}_{0,U_2}$ and the $C^{1,1}$-regularity of $\breve{\mathbf{T}}_{0,0}$ earlier in the proof. We have therefore established the claimed properties of $\mathcal{M}_{\mathrm{Left}}$.

Proof of the properties of $\mathfrak{E}$ and $\mathfrak{S}$: To establish these results, we will study the following map:

$$\widetilde{\breve{\mathscr{M}}}(t,u,x^2,x^3) \stackrel{\text{def}}{=} \left(\mu,-\breve{X}\mu,x^2,x^3\right) \tag{32.25}$$

on the domain $\mathcal{M}_{\mathrm{Singular}}$ in geometric coordinate space. Note the sign difference of $-\breve{X}\mu$ on RHS (32.25) compared to the definition (5.3a) of $\breve{\mathscr{M}}$; we inserted the minus sign on RHS (32.25) to facilitate the discussion in parts of this proof. We will show that $\widetilde{\breve{\mathscr{M}}}$ is a diffeomorphism from $\mathcal{M}_{\mathrm{Singular}}$ onto $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$, and our proof will show that the desired map $\mathfrak{E}$ is equal to $\widetilde{\breve{\mathscr{M}}}^{-1}$. To proceed, we first compose the maps ${}^{(\mathfrak{n})}\mathscr{T}$ and ${}^{(\mathfrak{n})}\Phi$ (recall (5.5), which states that ${}^{(\mathfrak{n})}\Phi\circ{}^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3) = \breve{\mathscr{M}}(t,u,x^2,x^3) = (\mu,\breve{X}\mu,x^2,x^3)$) and use Lemmas 15.5 and 15.7 and (15.44) to deduce that for each fixed $\mathfrak{n}\in[0,\mathfrak{n}_0]$, $\breve{\mathscr{M}}$ is a $C^{1,1}_{\mathrm{geo}}$ diffeomorphism on a subset of ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ containing $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]} = \bigcup_{\mathfrak{m}\in[0,\mathfrak{m}_0]}\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$. Hence, using (15.33), definition (32.1b), Lemma 15.9, and the estimates of Prop. 17.1, we see (accounting for the minus sign difference between $\breve{\mathscr{M}}$ and $\widetilde{\breve{\mathscr{M}}}$) that $\widetilde{\breve{\mathscr{M}}}$ is a diffeomorphism from $\mathcal{M}_{\mathrm{Singular}}$ onto $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$ whose Jacobian determinant satisfies the bound $\det d_{\mathrm{geo}}\widetilde{\breve{\mathscr{M}}}\approx 1$ on $\mathcal{M}_{\mathrm{Singular}}$ and such that for $(\mathfrak{m},\mathfrak{n})\in[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]$, we have $\widetilde{\breve{\mathscr{M}}}(\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}) = \{\mu\}\times\{\mathfrak{n}\}\times\mathbb{T}^2$. We clarify that the estimates (15.46)–(15.47) (which hold on $\mathcal{M}_{\mathrm{Singular}}$) and the convexity of the image set $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$ guarantee the global injectivity of $\widetilde{\breve{\mathscr{M}}}$ on $\mathcal{M}_{\mathrm{Singular}}$. Also using the Hölder estimates provided by Lemma 15.6 and Rademacher's theorem, we see that $\|\widetilde{\breve{\mathscr{M}}}\|_{W^{2,\infty}_{\mathrm{geo}}(\mathrm{int}(\mathcal{M}_{\mathrm{Singular}}))}\lesssim 1$. From (**BA μ – TORI STRUCTURE**), Cor. 15.8, and (15.32), it follows that the inverse map $\widetilde{\breve{\mathscr{M}}}^{-1}$ is precisely the map $\mathfrak{E}$ defined in (32.7). Thus, denoting $\widetilde{\breve{\mathscr{M}}}^{-1}$ by $\mathfrak{E}$, we can use these estimates and differentiate the identity $\mathfrak{E}\circ\widetilde{\breve{\mathscr{M}}}(t,u,x^2,x^3) = (t,u,x^2,x^3)$ up to two times to deduce that $\|\mathfrak{E}\|_{W^{2,\infty}((0,\mathfrak{m}_0)\times(0,\mathfrak{n}_0)\times\mathbb{T}^2)}\lesssim 1$. Since $(0,\mathfrak{m}_0)\times(0,\mathfrak{n}_0)\times\mathbb{T}^2$ is convex, we further deduce from Sobolev embedding (as in the proof of (15.18)) that $\|\mathfrak{E}\|_{C^{1,1}([0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2)}\lesssim\|\mathfrak{E}\|_{W^{2,\infty}((0,\mathfrak{m}_0)\times(0,\mathfrak{n}_0)\times\mathbb{T}^2)}\lesssim 1$, which yields (32.9). The properties of the map $\mathfrak{S}$ defined in (32.10), including (32.11), follow from the above arguments.

Proof of the quasi-convexity of $\mathcal{M}_{\mathrm{Singular}}$ and (32.20): First, we note that Lemma 15.9 (in particular the Jacobian estimate (15.47)), the estimates of Prop. 17.1, and the convexity of $[0,\mathfrak{m}_0]\times[-\mathfrak{n}_0,0]\times\mathbb{T}^2$ imply that for every pair of points $q_1,q_2\in[0,\mathfrak{m}_0]\times[-\mathfrak{n}_0,0]\times\mathbb{T}^2$, we have the following estimates: $\mathrm{dist}_{\mathrm{flat}}(q_1,q_2)\approx\mathrm{dist}_{\mathrm{flat}}\left(\breve{\mathscr{M}}^{-1}(q_1),\breve{\mathscr{M}}^{-1}(q_2)\right)$, where $\mathrm{dist}_{\mathrm{flat}}(q_1,q_2)$ is the standard Euclidean distance between q_1 and q_2 in the flat space $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$, and $\mathrm{dist}_{\mathrm{flat}}\left(\breve{\mathscr{M}}^{-1}(q_1),\breve{\mathscr{M}}^{-1}(q_2)\right)$ is the standard Euclidean distance between $\breve{\mathscr{M}}^{-1}(q_1)$ and $\breve{\mathscr{M}}^{-1}(q_2)$ in the flat space

$\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$ (note that $\breve{\mathscr{M}}^{-1}(q_1), \breve{\mathscr{M}}^{-1}(q_2) \in \mathcal{M}_{\mathrm{Singular}}$). From this bound, the convexity of $[0,\mathfrak{m}_0] \times [-\mathfrak{n}_0, 0] \times \mathbb{T}^2$, and the estimates of Prop. 17.1, we conclude that $\mathcal{M}_{\mathrm{Singular}} = \breve{\mathscr{M}}^{-1}\left([0,\mathfrak{m}_0] \times [-\mathfrak{n}_0, 0] \times \mathbb{T}^2\right)$ is quasi-convex in the sense stated in the proposition. From this quasi-convexity, it is a standard result (see, for example, [40, Theorem 7]), that the Sobolev embedding result (32.20) holds on $\mathcal{M}_{\mathrm{Singular}}$, where the constant C on RHS (32.20) depends on the constant (a different one, also called C) in the definition of quasi-convexity.

Proof of (32.22)–(32.24): Since $\mathfrak{L}^{-1} = \widetilde{\breve{\mathscr{M}}}$, the estimate (32.22) follows from the fact that on $\mathcal{M}_{\mathrm{Singular}}$, we have $\det d_{\mathrm{geo}} \widetilde{\breve{\mathscr{M}}} \approx 1$, as we showed above.

(32.23) follows from the estimate (18.8b) (which holds on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-\frac{1}{2}U_{\clubsuit},\frac{1}{2}U_{\clubsuit}]}$ for every $\mathfrak{n} \in [0,\mathfrak{n}_0]$), the definition $\mathcal{M}_{\mathrm{Singular}}$ of (32.1b), and the fact that $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-\frac{1}{2}U_{\clubsuit},\frac{1}{2}U_{\clubsuit}]}$ by (18.3a). From similar reasoning, based on the estimate (18.5), we conclude (32.24).

Proof of (32.13a)–(32.13b): We define the vectorfields J and K as follows:

$$J \overset{\mathrm{def}}{=} \frac{1}{\frac{\partial}{\partial t}\mu - \frac{(\frac{\partial}{\partial t}\breve{X}\mu)\frac{\partial}{\partial u}\mu}{\frac{\partial}{\partial u}\breve{X}\mu}} \left\{ \frac{\partial}{\partial t} - \frac{\frac{\partial}{\partial t}\breve{X}\mu}{\frac{\partial}{\partial u}\breve{X}\mu}\frac{\partial}{\partial u} \right\}, \tag{32.26}$$

$$K \overset{\mathrm{def}}{=} \frac{-1}{\frac{\partial}{\partial u}\breve{X}\mu - \frac{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu}{\frac{\partial}{\partial t}\mu}} \left\{ \frac{\partial}{\partial u} - \frac{\frac{\partial}{\partial u}\mu}{\frac{\partial}{\partial t}\mu}\frac{\partial}{\partial t} \right\}. \tag{32.27}$$

From definitions (32.26)–(32.27) and straightforward computations, we find that:

$$J\mu = 1, \qquad -J\breve{X}\mu = Jx^2 = Jx^3 = 0. \tag{32.28}$$

Hence, J is the partial derivative with respect to μ in the coordinate system $(\mu, -\breve{X}\mu, x^2, x^3)$ (corresponding to RHS (32.25)) on the region $[0,\mathfrak{m}_0] \times [0,\mathfrak{n}_0] \times \mathbb{T}^2$. Similarly, we compute that:

$$-K\breve{X}\mu = 1, \qquad K\mu = Kx^2 = Kx^3 = 0, \tag{32.29}$$

and thus K is the partial derivative with respect to $-\breve{X}\mu$ in the coordinate system $(\mu, -\breve{X}\mu, x^2, x^3)$. From (32.25) and the inverse function theorem, we compute that the 2×2 upper left-hand block of the matrix $[d_{\mathrm{geo}}\widetilde{\breve{\mathscr{M}}}]^{-1}$ is equal to $\begin{pmatrix} Jt & Kt \\ Ju & Ku \end{pmatrix}$. Using (32.26)–(32.27), (18.5), and (18.8b), and the fact that $\widetilde{\breve{\mathscr{M}}}$ is a diffeomorphism from $\mathcal{M}_{\mathrm{Singular}}$ (see also Remark 32.2) onto $[0,\mathfrak{m}_0] \times [0,\mathfrak{n}_0] \times \mathbb{T}^2$, we deduce that the diagonal entries of the 2×2 upper left-hand block of the matrix $[d_{\mathrm{geo}}\widetilde{\breve{\mathscr{M}}}]^{-1}$ satisfy the following estimates on $[0,\mathfrak{m}_0] \times [0,\mathfrak{n}_0] \times \mathbb{T}^2$, where $C > 1$:

$$-C \leq Jt \leq -\frac{1}{C}, \qquad -C \leq Ku \leq -\frac{1}{C}. \tag{32.30}$$

Since $\mathfrak{L} = \widetilde{\breve{\mathscr{M}}}^{-1}$, we see that the estimates in (32.30) are precisely (32.13a)–(32.13b).

Proof of the diffeomorphism property of ${}^{(\mathfrak{m})}J$, the quasiconvexity of $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$, and the estimates (32.15a)–(32.15b) **and** (32.17): The estimates (32.15a)–(32.15b) and the fact that ${}^{(\mathfrak{m})}J$ is a diffeomorphism on $[0,\mathfrak{n}_0] \times \mathbb{T}^2$ follow from the estimate (32.9), the monotonicity estimate (32.13b), and the inverse function theorem.

The quasi-convexity of $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ follows by combining the estimate (32.15a) and the monotonicity estimate (32.13b) with arguments similar to the ones we used to prove (15.25) and the quasi-convexity of ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_1,U_2]}$; we omit the details.

The estimate (32.17) for ${}^{(\mathfrak{m})}J^{-1}$ follows from the estimates (32.15a)–(32.15b), the inverse function theorem, and the quasi-convexity of $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$.

Proof of (32.21a) **and** (32.21b): Lemma 5.5 and Prop. 17.1 imply $\|\mu\|_{W^{3,\infty}_{\mathrm{geo}}(\mathrm{int}(\mathcal{M}_{\mathrm{Singular}}))} \lesssim 1$ and $\|\breve{X}\mu\|_{W^{2,\infty}_{\mathrm{geo}}(\mathrm{int}(\mathcal{M}_{\mathrm{Singular}}))} \lesssim 1$. Also using (32.20), we deduce (32.21b) and the bound $\|\breve{X}\mu\|_{C^{1,1}_{\mathrm{geo}}(\mathcal{M}_{\mathrm{Singular}})} \lesssim 1$. From these estimates and (32.25), it follows that $\|\widetilde{\breve{\mathscr{M}}}\|_{C^{1,1}_{\mathrm{geo}}(\mathcal{M}_{\mathrm{Singular}})} \leq C$. Since $\mathfrak{L}^{-1} = \widetilde{\breve{\mathscr{M}}}$, we conclude (32.21a).

Proof of the existence of $\breve{T}_{\mathfrak{m}}$ and the properties of $\breve{T}_{\mathfrak{m}}$ and $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$. We define the function ${}^{(\mathfrak{m})}M(\mathfrak{n},x^2,x^3)$ to be the composition of the map $(\mathfrak{n},x^2,x^3) \to \mathfrak{L}(\mathfrak{m},\mathfrak{n},x^2,x^3)$ with ${}^{(\mathfrak{m})}J^{-1}$, where we clarify that ${}^{(\mathfrak{m})}M$ has domain equal to the set $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ from (32.16). In view of (32.2), (32.8), and the diffeomorphism properties of $\mathfrak{L}$ and ${}^{(\mathfrak{m})}J^{-1}$, we see that ${}^{(\mathfrak{m})}M$ is a diffeomorphism from $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ onto $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ (where we view $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ to be a subset of geometric coordinate space). Hence, the function $\breve{T}_{\mathfrak{m}}$ is equal to the first component of ${}^{(\mathfrak{m})}M$. We have therefore shown (32.18).

We now prove (32.19). First, recalling that $\breve{T}_{\mathfrak{m}}$ is the first component of ${}^{(\mathfrak{m})}M$, we use the bounds (32.9) and (32.17) to deduce that $\|\breve{T}_{\mathfrak{m}}\|_{C^{1,1}\left(\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}\right)} \leq C$. It remains for us to show that the second-order derivatives of the function $\breve{T}_{\mathfrak{m}}(\mathfrak{n},x^2,x^3)$ are bounded in the norm $\|\cdot\|_{C^{0,1}\left(\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}\right)}$ by $\leq C$. To this end, we differentiate the identity $\mu\left(\breve{T}_{\mathfrak{m}}(u,x^2,x^3),u,x^2,x^3\right) = \mathfrak{m}$ (which holds on $\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$) up to two times with elements of $\left\{\frac{\partial}{\partial u},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right\}$ and use the implicit function theorem, the chain rule, (18.8b), (32.21b), and the already proven bound $\|\breve{T}_{\mathfrak{m}}\|_{C^{1,1}\left(\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}\right)} \leq C$. We have therefore proved (32.19). From this bound, (32.16), and (32.18), we conclude that for each fixed $\mathfrak{m} \in [0,\mathfrak{m}_0]$, the set $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ is a $C^{2,1}$ embedded hypersurface in geometric coordinate space $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$. Also using the bounds $\sup_{(\mathfrak{m},\mathfrak{n})\in[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]} \|\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}\|_{C^{1,1}(\mathbb{T}^2)} \leq C$ and $\sup_{(\mathfrak{m},\mathfrak{n})\in[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]} \|\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}\|_{C^{1,1}(\mathbb{T}^2)} \leq C$ implied by (32.9), we conclude that the two boundary components of $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ are $C^{1,1}$. Finally, since diffeomorphisms map boundaries to boundaries, we conclude from (32.8) that the two boundary components of $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ are $\mathfrak{L}(\{\mathfrak{m}\}\times\{0\}\times\mathbb{T}^2) = \breve{\mathbf{T}}_{\mathfrak{m},0}$ and $\mathfrak{L}(\{\mathfrak{m}\}\times\{\mathfrak{n}_0\}\times\mathbb{T}^2) = \breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}_0}$.

Proof of the properties of the boundary of $\mathcal{M}_{\text{Singular}}$: Since $\mathfrak{L}$ is a diffeomorphism from $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$ onto $\mathcal{M}_{\text{Singular}}$, and since diffeomorphisms map boundaries to boundaries, it follows that the boundary of $\mathcal{M}_{\text{Singular}}$ is the union of four sets: $\mathfrak{L}([0,\mathfrak{m}_0]\times\{0\}\times\mathbb{T}^2)$, $\mathfrak{L}(\{0\}\times[0,\mathfrak{n}_0]\times\mathbb{T}^2)$, $\mathfrak{L}([0,\mathfrak{m}_0]\times\{\mathfrak{n}_0\}\times\mathbb{T}^2)$, and $\mathfrak{L}(\{\mathfrak{m}_0\}\times[0,\mathfrak{n}_0]\times\mathbb{T}^2)$. In view of the form (32.25) of the map $\widetilde{\breve{\mathscr{M}}}$ (which is equal to $\mathfrak{L}^{-1}$), definitions (4.3a)–(4.3c), (4.7b), and (32.1b), and (15.44), we conclude (recalling that $\tau_0 = -\mathfrak{m}_0$) that the boundary of $\mathcal{M}_{\text{Singular}}$ is the union of the four sets $\breve{\mathbb{X}}_0^{[\tau_0,0]}$, $\breve{\mathbb{M}}_0^{[0,\mathfrak{n}_0]} = \mathcal{B}^{[0,\mathfrak{n}_0]}$, $\breve{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$, and $\breve{\mathbb{M}}_{\mathfrak{m}_0}^{[0,\mathfrak{n}_0]}$, as is stated in the proposition. The regularity properties of these boundary portions was derived earlier in the proof. Finally, using the monotonicity guaranteed by (32.23)–(32.24), we find that the four sets mentioned above are respectively (see Fig. 13) the left lateral boundary of $\mathcal{M}_{\text{Singular}}$, the top boundary of $\mathcal{M}_{\text{Singular}}$, the right lateral boundary of $\mathcal{M}_{\text{Singular}}$, and the bottom boundary of $\mathcal{M}_{\text{Singular}}$.

This completes our proof of the proposition.

□

32.4. **The character of $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$, $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$, and $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}$.** In this section, we study the character of various sub-manifolds of geometric coordinate space, i.e., whether they are $\mathbf{g}$-timelike, null, or spacelike. The singular boundary is degenerate for reasons discussed in Remark 32.8. Hence, we postpone our investigation of the character of the singular boundary until Prop. 33.2, where we describe how it is embedded into the physical Cartesian coordinate space equipped with the acoustical metric $\mathbf{g}$.

We start with the following simple lemma, which provides various identities involving the gradient vectorfield of μ.

Lemma 32.6 (Identities involving $\mathbf{D}^{\#}\mu$). *Let $\mathbf{D}\mu$ denote the gradient one-form of μ, let $\mathbf{D}^{\#}\mu$ denote the $\mathbf{g}$-dual vectorfield of the gradient one-form, and let $\nabla\!\!\!/^{\#}\mu$ denote the $\ell_{t,u}$-tangent vectorfield equal to the $g\!\!\!/$-dual of $\nabla\!\!\!/\mu$. Then the following identities hold:*

$$\mu\mathbf{D}^{\#}\mu = -(\mu L\mu + \breve{X}\mu)L - (L\mu)\breve{X} + \mu\nabla\!\!\!/^{\#}\mu, \tag{32.31a}$$

$$\mu\mathbf{g}(\mathbf{D}^{\#}\mu,\mathbf{D}^{\#}\mu) = \mu(\mathbf{g}^{-1})^{\alpha\beta}(\partial_\alpha\mu)\partial_\beta\mu = -2(L\mu)\breve{X}\mu - \mu\left\{(L\mu)^2 - |\nabla\!\!\!/\mu|_{g\!\!\!/}^2\right\}. \tag{32.31b}$$

In particular, if $0 < \mu' \leq \mathfrak{m}_0$, $q \in \{(t,u,x^2,x^3) \mid \mu(t,u,x^2,x^3) = \mu'\} \cap \{|u| \leq U_{\clubsuit}\}$, and if $\breve{X}\mu|_q \leq 0$, then since (18.8a) *and* (28.31) *imply that RHS* (32.31b) < 0 *at q, it follows that $\{(t,u,x^2,x^3) \mid \mu(t,u,x^2,x^3) = \mu'\}$ is $\mathbf{g}$-spacelike (i.e., $\mathbf{g}(\mathbf{D}^{\#}\mu,\mathbf{D}^{\#}\mu) < 0$) at q.*

Proof. (32.31a)–(32.31b) follow from a straightforward computation based on the fact that $(\mathbf{D}^{\#})^\alpha \stackrel{\text{def}}{=} (\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta$ and the identity $(\mathbf{g}^{-1})^{\alpha\beta} = -L^\alpha L^\beta - X^\alpha L^\beta - L^\alpha X^\beta + (g\!\!\!/^{-1})^{\alpha\beta}$, which follows from (3.34b). □

In the next lemma, we exhibit the character of various sub-manifolds of geometric coordinate space.

Lemma 32.7 (The character of $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$, $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$, $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}$)**.** *Assume the hypotheses and conclusions of Theorem 31.1 for* $\mathfrak{n} \in [0,\mathfrak{n}_0]$. *Recall that for* $(\mathfrak{m},\mathfrak{n}) \in [0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]$, *the* μ*-adapted torus* $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *defined in* (4.3c) *is contained in* $\mathcal{M}_{\text{Singular}}$ *(see* (32.1b)*), and that for* $\mathfrak{m} \in [0,\mathfrak{m}_0]$, *the set* $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *defined in* (32.2) *is the* $\mathfrak{m}$*-level-set of* μ *in* $\mathcal{M}_{\text{Singular}}$. *Then the following results hold.*

The character of $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ ***and*** $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}$.

- *For* $(\mathfrak{m},\mathfrak{n}) \in [0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]$, *the torus* $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ *is a* 2*-dimensional,* $\mathbf{g}$*-spacelike sub-manifold.*
- *In particular, the crease* $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]} = \breve{\mathbf{T}}_{0,0}$ *is a* 2*-dimensional,* $\mathbf{g}$*-spacelike sub-manifold.*

The character of $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ ***and*** $\mathcal{B}^{[0,\mathfrak{n}_0]}$.

- *For* $0 < \mathfrak{m} \leq \mathfrak{m}_0$, $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ *is a* 3*-dimensional,* $\mathbf{g}$*-spacelike sub-manifold-with-boundary.*

Remark 32.8 (Acoustical metric degeneracies along the singular boundary)**.** Note that Lemma 32.7 does not address the causal structure of the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]} = \breve{\mathbb{M}}_0^{[0,\mathfrak{n}_0]}$ (see definition 32.3, and recall that μ vanishes along $\breve{\mathbb{M}}_0^{[0,\mathfrak{n}_0]}$), viewed as a subset of geometric coordinate space. We have avoided discussing the causal structure of $\mathcal{B}^{[0,\mathfrak{n}_0]}$ because relative to the geometric coordinates (t,u,x^2,x^3), some components of the acoustical metric $\mathbf{g}$ degenerate along $\breve{\mathbb{M}}_0^{[0,\mathfrak{n}_0]}$. In particular, using Lemma 3.9 and Lemma 5.5, one can check that along $\mathcal{B}^{[0,\mathfrak{n}_0]}$, all vectors $V \in \text{span}\left\{L, \frac{\partial}{\partial u}\right\}$ are $\mathbf{g}$-null, i.e, they satisfy $\mathbf{g}(V,V) = 0$. In contrast, even along $\mathcal{B}^{[0,\mathfrak{n}_0]}$, the Cartesian component matrix $\{\mathbf{g}_{\alpha\beta}\}_{\alpha,\beta=0,1,2,3}$ of the acoustical metric is a non-degenerate 4×4 Lorentzian matrix (in fact, by (2.16)–(2.17) and the estimates of Prop. 17.1, the matrix is close to the standard Minkowski matrix). The discrepancy between the properties of $\mathbf{g}$ in the two coordinate systems is caused by the fact that the change of variables map $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ has a non-injective Jacobian matrix along $\mathcal{B}^{[0,\mathfrak{n}_0]}$; see Prop. 33.2. We refer to Prop. 33.2 for a description of the structure of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, that is, the structure of the image of $\mathcal{B}^{[0,\mathfrak{n}_0]}$ in Cartesian coordinate space under the map Υ, which, in the proposition, is shown to be an injective map on all of $\mathcal{M}_{\text{Interesting}}$.

Proof of Lemma 32.7. We already showed in Prop. 32.5 that $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ is a 2-dimensional manifold and that $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ is a 3-dimensional, $\mathbf{g}$-spacelike sub-manifold-with-boundary.

Proof that $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ **is** $\mathbf{g}$**-spacelike**: Consider the map $\widetilde{\breve{\mathscr{M}}}(t,u,x^2,x^3) = \left(\mu, -\breve{X}\mu, x^2, x^3\right)$ from (32.25). In the proof of Prop. 32.5, we showed that $\widetilde{\breve{\mathscr{M}}}$ is a diffeomorphism from $\mathcal{M}_{\text{Singular}}$ onto $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$. Using Lemma 5.5, Prop. 9.1, and the L^∞ estimates of Prop. 17.1 (with ε replaced by $C\mathring{\epsilon}$, which Theorem 31.1 allows for) we compute that:

$$d_{\text{geo}}\widetilde{\breve{\mathscr{M}}} = \begin{pmatrix} \frac{\partial}{\partial t}\mu & \frac{\partial}{\partial u}\mu & * & * \\ -\frac{\partial}{\partial t}\breve{X}\mu & -\frac{\partial}{\partial u}\breve{X}\mu & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{32.32}$$

where $d_{\text{geo}}\widetilde{\breve{\mathscr{M}}}$ is the Jacobian matrix of $\widetilde{\breve{\mathscr{M}}}$ and "$*$" denotes quantities that are bounded in magnitude by $\mathcal{O}(\mathring{\epsilon})$. From (32.32), (18.5), and (18.8b), we deduce that $\det d_{\text{geo}}\widetilde{\breve{\mathscr{M}}} \approx 1$. It follows that:

$$[d_{\text{geo}}\widetilde{\breve{\mathscr{M}}}]^{-1} = \frac{1}{\det d_{\text{geo}}\widetilde{\breve{\mathscr{M}}}} \begin{pmatrix} -\frac{\partial}{\partial u}\breve{X}\mu & -\frac{\partial}{\partial u}\mu & * & * \\ \frac{\partial}{\partial t}\breve{X}\mu & \frac{\partial}{\partial t}\mu & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32.33}$$

Fix any $(\mathfrak{m},\mathfrak{n}) \in [0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]$. Since $\mu \equiv \mathfrak{m}$ and $\breve{X}\mu \equiv -\mathfrak{n}$ along $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$, the last two columns of (32.33) are the components of vectorfields $V_{(2)} = *\frac{\partial}{\partial t} + *\frac{\partial}{\partial u} + \frac{\partial}{\partial x^2}$ and $V_{(3)} = *\frac{\partial}{\partial t} + *\frac{\partial}{\partial u} + \frac{\partial}{\partial x^3}$ that, when restricted to $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$, span the tangent space of $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$. Using Lemma 3.9, Lemma 5.5, (3.31a), Prop. 9.1 and the L^∞ estimates of Prop. 17.1, we compute that $\mathbf{g}(V_{(A)}, V_{(B)}) = \delta_{AB} + \mathcal{O}(\mathring{\alpha})$, where δ_{AB} is the Kronecker delta. From this estimate, it easily follows that $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ is $\mathbf{g}$-spacelike, even in the case of the crease (in which $\mathfrak{m} = \mathfrak{n} = 0$).

Proof that $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ **is** $\mathbf{g}$**-spacelike when** $\mathfrak{m} > 0$: Fix any $\mathfrak{m} \in (0,\mathfrak{m}_0]$. Note that $\mu \equiv \mathfrak{m}$ along $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$, that $\breve{X}\mu \leq 0$ along $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ (by (32.2) and the fact that $\breve{X}\mu \equiv -\mathfrak{n}$ along $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$), and that $L\mu < 0$ along $\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ (by Remark 32.2 and

(18.8a)). Using these results, the estimate $|\not\nabla\mu|_{\not g} \lesssim \varepsilon$ (see (28.31)), and the identity (32.31b), we conclude that along $\check{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$, we have $\mathbf{g}(\mathbf{D}^{\#}\mu, \mathbf{D}^{\#}\mu) < 0$. This implies that the $\mathbf{g}$-normal to $\check{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$ is $\mathbf{g}$-timelike, which is the desired result.

□

32.5. **A new time function and related geometric objects.** In this section, on $\mathcal{M}_{\text{Interesting}}$, we define a time function ${}^{(\text{Interesting})}\tau$ as well as several related geometric objects. We also derive some geometric and analytic properties of these objects.

32.5.1. *Definitions.*

Definition 32.9 (${}^{(\text{Interesting})}\tau$, ${}^{(\text{Interesting})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\text{Interesting})}\tau, u, x^2, x^3)$, ${}^{(\text{Interesting})}\mathfrak{t}_{\tau}$, $\breve{H}$, and $\breve{G}$)**.** Let $\mathcal{M}_{\text{Interesting}} = \mathcal{M}_{\text{Left}} \cup \mathcal{M}_{\text{Singular}} \cup \mathcal{M}_{\text{Right}}$ be the set defined in (32.1d) and depicted in Figs. 6 and 13.

Definition of ${}^{(\text{Interesting})}\tau$. On $\mathcal{M}_{\text{Interesting}}$, we define the scalar function ${}^{(\text{Interesting})}\tau$ as follows:

$$
{}^{(\text{Interesting})}\tau(t,u,x^2,x^3) \overset{\text{def}}{=} \begin{cases} {}^{(0)}\tau(t,u,x^2,x^3), & \text{in } \mathcal{M}_{\text{Left}}, \\ -\mu(t,u,x^2,x^3), & \text{in } \mathcal{M}_{\text{Singular}}, \\ {}^{(\mathfrak{n}_0)}\tau(t,u,x^2,x^3), & \text{in } \mathcal{M}_{\text{Right}}. \end{cases} \tag{32.34}
$$

Definition of level-set portions of ${}^{(\text{Interesting})}\tau$. For $-U_1 \leq u_1 \leq u_2 \leq U_2$ and $\tau \in [\tau_0, 0]$, we define:

$$
{}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]} \overset{\text{def}}{=} \left\{(t,u,x^2,x^3) \mid (u,x^2,x^3) \in [u_1,u_2] \times \mathbb{T}^2, \ {}^{(\text{Interesting})}\tau(t,u,x^2,x^3) = \tau\right\}. \tag{32.35}
$$

Definition of ${}^{(\text{Interesting})}\mathscr{T}$ and ${}^{(\text{Interesting})}\mathfrak{t}_{\tau}$. We define the map ${}^{(\text{Interesting})}\mathscr{T} : \mathcal{M}_{\text{Interesting}} \to [\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$ as follows:

$$
{}^{(\text{Interesting})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\text{Interesting})}\tau, u, x^2, x^3). \tag{32.36}
$$

Next, for each $\tau \in [\tau_0, 0]$, we define the function ${}^{(\text{Interesting})}\mathfrak{t}_{\tau} : [-U_1, U_2] \times \mathbb{T}^2 \to \mathbb{R}$ as follows:

$$
{}^{(\text{Interesting})}\mathfrak{t}_{\tau}(u,x^2,x^3) \overset{\text{def}}{=} \begin{cases} \mathfrak{t}_{\tau,0}(u,x^2,x^3), & \text{if } \mathfrak{U}_{-\tau,0}(x^2,x^3) < u \leq U_2, \\ \breve{T}_{-\tau}(u,x^2,x^3), & \text{if } \mathfrak{U}_{-\tau,-\mathfrak{n}_0}(x^2,x^3) \leq u \leq \mathfrak{U}_{-\tau,0}(x^2,x^3), \\ \mathfrak{t}_{\tau,\mathfrak{n}_0}(u,x^2,x^3), & \text{if } -U_1 \leq u < \mathfrak{U}_{-\tau,-\mathfrak{n}_0}(x^2,x^3), \end{cases} \tag{32.37}
$$

where $\mathfrak{t}_{\tau,\mathfrak{n}}$ is the function from (15.28), $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}$ is the function on $\mathbb{T}^2$ from (15.37), and $\breve{T}_{\mathfrak{m}}$ is the function from (32.18).

Definition of $\breve{H}$ and $\breve{G}$. In $\mathcal{M}_{\text{Interesting}}$, we define the vectorfields $\breve{H}$ and $\breve{G}$ as follows, where ϕ is the cut-off function from Def. 4.1:

$$
\breve{H} \overset{\text{def}}{=} \begin{cases} \breve{X} - \breve{X}^A\left(\frac{\partial}{\partial x^A} - \frac{\frac{\partial}{\partial x^A}{}^{(0)}\tau}{\frac{\partial}{\partial t}{}^{(0)}\tau}\frac{\partial}{\partial t}\right), & \text{in } \mathcal{M}_{\text{Left}}, \\ \left(\breve{X} - \frac{\breve{X}\mu}{L\mu}L\right) - \left(\breve{X}^A - \frac{\breve{X}\mu}{L\mu}L^A\right)\left(\frac{\partial}{\partial x^A} - \frac{\frac{\partial}{\partial x^A}\mu}{\frac{\partial}{\partial t}\mu}\frac{\partial}{\partial t}\right), & \text{in } \mathcal{M}_{\text{Singular}}, \\ \left(\breve{X} + \phi\frac{\mathfrak{n}_0}{L\mu}L\right) - \left(\breve{X}^A + \phi\frac{\mathfrak{n}_0}{L\mu}L^A\right)\left(\frac{\partial}{\partial x^A} - \frac{\frac{\partial}{\partial x^A}{}^{(\mathfrak{n}_0)}\tau}{\frac{\partial}{\partial t}{}^{(\mathfrak{n}_0)}\tau}\frac{\partial}{\partial t}\right), & \text{in } \mathcal{M}_{\text{Right}}, \end{cases} \tag{32.38a}
$$

$$
\breve{G} \overset{\text{def}}{=} \begin{cases} \frac{1}{\frac{\partial}{\partial t}{}^{(0)}\tau}\frac{\partial}{\partial t}, & \text{in } \mathcal{M}_{\text{Left}}, \\ -\frac{1}{\frac{\partial}{\partial t}\mu}\frac{\partial}{\partial t}, & \text{in } \mathcal{M}_{\text{Singular}}, \\ \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n}_0)}\tau}\frac{\partial}{\partial t}, & \text{in } \mathcal{M}_{\text{Right}}. \end{cases} \tag{32.38b}
$$

Remark 32.10 (The regularity of ${}^{(\text{Interesting})}\tau$ and the connection to the causal structure of $\{\mu = 0\}$)**.** In Prop. 32.11, we show that ${}^{(\text{Interesting})}\tau \in C^{1,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})$. That regularity is optimal in the sense that generally, ${}^{(\text{Interesting})}\tau \notin C^2_{\text{geo}}(\mathcal{M}_{\text{Interesting}})$. The reason is that with respect to the geometric coordinates (t,u,x^2,x^3), τ and $-\mu$ generally agree only to first-order along $\breve{\mathbb{X}}_0^{[\tau_0,0]}$, which is the common boundary of the regions $\mathcal{M}_{\text{Left}}$ and $\mathcal{M}_{\text{Singular}}$ in the piecewise-defined definition (32.34) of ${}^{(\text{Interesting})}\tau$; the first-order agreement along $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ follows from definition (32.34), (4.4b), (15.44), and (15.10), while (4.4) implies that $\mathbf{D}^{(\mathfrak{n})}\breve{W}^{(\mathfrak{n})}\tau \equiv 0$ along $\breve{\mathbb{X}}_0^{[\tau_0,0]}$, an identity that does not hold for μ in the solution regime under study (see, for example, the transversal convexity condition (18.5)). It follows that the second-order partial derivatives of ${}^{(\text{Interesting})}\tau$ with respect to the geometric coordinates can jump across $\breve{\mathbb{X}}_0^{[\tau_0,0]}$.

Moreover, it is generally impossible to modify the construction of ${}^{(\text{Interesting})}\tau$ and $\mathcal{M}_{\text{Interesting}}$ to enforce C^2 agreement of ${}^{(\text{Interesting})}\tau$ with $-\mu$ across $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ in a manner such that the zero level-set of the new ${}^{(\text{Interesting})}\tau$, which we denote by $\tau'_{\text{Interesting}}$, still contains the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]}$. The reason is that if $\tau'_{\text{Interesting}}$ were C^2 with respect to the geometric coordinates and agreed with $-\mu$ up to second-order along $\breve{\mathbb{X}}_0^{[\tau_0,0]}$, then by Taylor expanding $\tau'_{\text{Interesting}}$ starting from the crease $\breve{\mathbf{T}}_{0,0}$ (i.e., the subset of $\breve{\mathbb{X}}_0^{[\tau_0,0]} \cap \{\tau'_{\text{Interesting}} = 0\}$ along which $\mu = \breve{X}\mu = 0$), and using the estimates (18.5), (18.8a), and (28.31) for μ (which by Taylor expansion imply corresponding estimates for $\tau'_{\text{Interesting}}$), one could prove the following result: on near-zero level-sets of $\tau'_{\text{Interesting}}$, in a region near the crease $\breve{\mathbf{T}}_{0,0}$ with $\mu > 0$, we would have: $\mu\mathbf{g}(\mathbf{D}^{\#}\tau'_{\text{Interesting}}, \mathbf{D}^{\#}\tau'_{\text{Interesting}}) > 0$. This would imply that the level-sets of $\tau'_{\text{Interesting}}$ are spacelike near the crease, i.e., $\tau'_{\text{Interesting}}$ could not be used as a time function. The main idea of the proof is that formally, along the level-set $\{\mu = 0\}$, RHS (32.31b) becomes positive as one passes through the crease in the direction of increasing u because $\breve{X}\mu$ becomes positive (thanks to (18.5)) while $L\mu$ is strictly negative everywhere near the crease (thanks to (18.8a)).

32.5.2. *Properties of* ${}^{(\text{Interesting})}\tau$ *and related quantities.* In the next proposition, we derive some fundamental properties of the quantities from Def. 32.9 as well as implications of these properties for the structure of $\mathcal{M}_{\text{Interesting}}$ and the behavior of μ on $\mathcal{M}_{\text{Interesting}}$.

Proposition 32.11 (Properties of ${}^{(\text{Interesting})}\tau$, ${}^{(\text{Interesting})}\mathscr{T}$, ${}^{(\text{Interesting})}\mathfrak{t}_\tau$, $\breve{G}$, $\breve{H}$, and $\mathcal{M}_{\text{Interesting}}$)**.** *Let* ${}^{(\text{Interesting})}\tau$, ${}^{(\text{Interesting})}\mathfrak{t}_\tau$, ${}^{(\text{Interesting})}\mathscr{T}$, $\breve{G}$, *and* $\breve{H}$ *be as in Def. 32.9. Then these quantities enjoy the following properties on* $\mathcal{M}_{\text{Interesting}}$.

Properties of ${}^{(\text{Interesting})}\tau$.

- *The following estimate holds:*

$$\|{}^{(\text{Interesting})}\tau\|_{C^{1,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C. \tag{32.39}$$

- *The following estimate holds:*

$$\frac{\partial}{\partial t}{}^{(\text{Interesting})}\tau \approx 1, \qquad \text{on } \mathcal{M}_{\text{Interesting}}. \tag{32.40}$$

- *For* $\tau \in [\tau_0, 0]$, *the level-set portions* ${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]}$ *defined in* (32.35) *are* $\mathbf{g}$*-spacelike, except along the singular boundary portion* $\mathcal{B}^{[0,\mathfrak{n}_0]} \subset {}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Singular}}$.

The behavior of μ ***on*** $\mathcal{M}_{\text{Interesting}}$.

- *For* $\tau \in [\tau_0, 0]$, *we have:*

$$\min_{{}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]}} \mu = -\tau. \tag{32.41}$$

Moreover, within ${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]}$, *the minimum value of* $-\tau$ *in* (32.41) *is achieved by* μ *precisely on the set* $\breve{\mathbb{M}}_{-\tau}^{[0,\mathfrak{n}_0]} \stackrel{def}{=} \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ *from definition* (32.2).
- *In particular, by the previous point and definition* (32.3), *it follows that within* $\mathcal{M}_{\text{Interesting}}$, μ *is positive, except along the singular boundary portion* $\mathcal{B}^{[0,\mathfrak{n}_0]} = \breve{\mathbb{M}}_0^{[0,\mathfrak{n}_0]}$, *which is a subset of* ${}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}$.

Properties of ${}^{(\text{Interesting})}\mathfrak{t}_\tau$.

- *For each* $\tau \in [\tau_0, 0]$, *we have:*

$$\|{}^{(\text{Interesting})}\mathfrak{t}_\tau\|_{C^{1,1}([-U_1,U_2]\times\mathbb{T}^2)} \leq C. \tag{32.42}$$

- *The level-set portions* ${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]}$ *defined in* (32.35) *have the following graph structure:*

$${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]} = \left\{(t,u,x^2,x^3) \mid (u,x^2,x^3) \in [-U_1,U_2]\times\mathbb{T}^2,\ t = {}^{(\text{Interesting})}\mathfrak{t}_\tau(u,x^2,x^3)\right\}. \tag{32.43}$$

Properties of ${}^{(\text{Interesting})}\mathscr{T}$.

- *The change of variables map* ${}^{(\text{Interesting})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\text{Interesting})}\tau, u, x^2, x^3)$ *is a diffeomorphism from* $\mathcal{M}_{\text{Interesting}}$ *onto* $[\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$.
- *The following estimate holds:*

$$\| {}^{(\text{Interesting})}\mathscr{T} \|_{C^{1,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C. \tag{32.44}$$

Properties of $\breve{G}$.

- *In* $\mathcal{M}_{\text{Interesting}}$, *we have:*

$$\breve{G}\,{}^{(\text{Interesting})}\tau = 1, \qquad \breve{G}u = \breve{G}x^2 = \breve{G}x^3 = 0. \tag{32.45}$$

In particular, $\breve{G}$ *is the partial derivative with respect to* ${}^{(\text{Interesting})}\tau$ *in the coordinate system* $({}^{(\text{Interesting})}\tau, u, x^2, x^3)$.
- *The following estimate holds:*

$$\|\breve{G}\|_{C^{0,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C. \tag{32.46}$$

Properties of $\breve{H}$.

- *In* $\mathcal{M}_{\text{Interesting}}$, *we have:*

$$\breve{H}u = 1, \qquad \breve{H}\,{}^{(\text{Interesting})}\tau = \breve{H}x^2 = \breve{H}x^3 = 0. \tag{32.47}$$

In particular, $\breve{H}$ *is the partial derivative with respect to* u *in the coordinate system* $({}^{(\text{Interesting})}\tau, u, x^2, x^3)$.
- *The following estimate holds:*

$$\|\breve{H}\|_{C^{0,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C. \tag{32.48}$$

- *Every* u*-parametrized integral curve of* $\breve{H}$ *is defined on the interval* $[-U_1, U_2]$, *and the images of these integral curves are contained in* $\mathcal{M}_{\text{Interesting}}$.
- *For each fixed* $\tau \in [\tau_0, 0]$, *every* u*-parametrized integral curve of* $\breve{H}$ *in the level-set* ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$ *intersects the torus* $\breve{\mathbf{T}}_{-\tau,0} \subset \breve{\mathbb{X}}_0^{[\tau_0,0]}$ *in precisely one point. Moreover,* $\breve{\mathbf{T}}_{-\tau,0} \subset {}^{(\text{Interesting})}\Sigma_{\tau}^{[-\frac{1}{2}U_{☢}, \frac{1}{2}U_{☢}]}$.

Properties of $\mathcal{M}_{\text{Interesting}}$. *The following results hold.*

- $\mathcal{M}_{\text{Interesting}}$ *is a compact subset of geometric coordinate space* $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$.
-

$$\mathcal{M}_{\text{Interesting}} = \bigcup_{\tau \in [\tau_0, 0]} {}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}. \tag{32.49}$$

- *For every pair of points* $q_1, q_2 \in [\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$, *we have:*

$$\text{dist}_{\text{flat}}\left({}^{(\text{Interesting})}\mathscr{T}^{-1}(q_1), {}^{(\text{Interesting})}\mathscr{T}^{-1}(q_2)\right) \approx \text{dist}_{\text{flat}}(q_1, q_2), \tag{32.50}$$

where on both sides of (32.50), $\text{dist}_{\text{flat}}(A,B)$ *is the standard Euclidean distance between* A *and* B *in the flat space* $\mathbb{R} \times \mathbb{R} \times \mathbb{T}^2$.
- *(**Quasi-convexity**)* $\mathcal{M}_{\text{Interesting}}$ *is quasi-convex. That is, there is a constant* $C > 0$ *such that every pair of points* $p_1, p_2 \in \mathcal{M}_{\text{Interesting}}$ *is connected by a* C^1_{geo} *curve in* $\mathcal{M}_{\text{Interesting}}$ *whose length with respect to the standard flat Euclidean metric on geometric coordinate space* $\mathbb{R} \times \mathbb{R} \times \mathbb{T}^2$ *is* $\leq C\text{dist}_{\text{flat}}(p_1, p_2)$.
- *(**Sobolev embedding**) There is a constant* $C > 0$ *such that the following Sobolev embedding result holds for scalar functions* f *on* $\text{int}(\mathcal{M}_{\text{Interesting}})$:

$$\|f\|_{C^{0,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C\|f\|_{W^{1,\infty}_{\text{geo}}(\text{int}(\mathcal{M}_{\text{Interesting}}))}. \tag{32.51}$$

Proof.

Proof of (32.49): From definition (32.1a), we see that $\mathcal{M}_{\text{Left}} \subset {}^{(0)}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$. Hence, since ${}^{(0)}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ is foliated by the level-sets of ${}^{(0)}\tau$, it follows from definition (32.34) that $\mathcal{M}_{\text{Left}}$ is foliated by the level-sets of ${}^{(\text{Interesting})}\tau$, which have the range $[\tau_0, 0]$. Using similar reasoning based on definition (32.1c), we see that $\mathcal{M}_{\text{Right}}$ is foliated by the level-sets of ${}^{(\text{Interesting})}\tau$. Moreover, using (32.5)–(32.6), we see that $\mathcal{M}_{\text{Singular}}$ is foliated by the level-sets of μ, which have the range $[0, \mathfrak{m}_0] = [0, -\tau_0]$ in $\mathcal{M}_{\text{Singular}}$. Hence, from definition (32.34), we see that $\mathcal{M}_{\text{Singular}}$ is foliated by the level-sets of ${}^{(\text{Interesting})}\tau$, which have the range $[\tau_0, 0]$. From these facts and definition (32.1d), we conclude (32.49).

Proof of (32.43): In Prop. 32.5, we showed that the top boundary of $\mathcal{M}_{\text{Left}}$, which is contained in the level-set $\{{}^{(0)}\tau = 0\}$, is the hypersurface $\left\{(t, u, x^2, x^3) \mid (x^2, x^3) \in \mathbb{T}^2,\ \mathfrak{U}_{0,0}(x^2, x^3) \le u \le U_2, \text{ and } t = \mathfrak{t}_{0,0}(u, x^2, x^3)\right\}$. The proof, in conjunction with (15.28) and (15.37), also shows that $\breve{\mathbf{T}}_{0,0} = \left\{(t, u, x^2, x^3) \mid (x^2, x^3) \in \mathbb{T}^2,\ u = \mathfrak{U}_{0,0}(x^2, x^3) \text{ and } t = \mathfrak{t}_{0,0}(u, x^2, x^3)\right\}$, and that although $\breve{\mathbf{T}}_{0,0}$ is in the closure of $\mathcal{M}_{\text{Left}}$, it does not belong to $\mathcal{M}_{\text{Left}}$. The same arguments, together with definition (32.34), also yield that $\mathcal{M}_{\text{Left}}$ is foliated by the level sets of ${}^{(\text{Interesting})}\tau$, and that for each fixed $\tau \in [\tau_0, 0]$, we have $\left\{{}^{(\text{Interesting})}\tau = \tau\right\} \cap \mathcal{M}_{\text{Left}} = \left\{(t, u, x^2, x^3) \mid (x^2, x^3) \in \mathbb{T}^2,\ \mathfrak{U}_{-\tau,0}(x^2, x^3) < u \le U_2, \text{ and } t = \mathfrak{t}_{-\tau,0}(u, x^2, x^3)\right\}$. Similarly, $\mathcal{M}_{\text{Right}}$ is foliated by the level sets of ${}^{(\text{Interesting})}\tau$, and for each fixed $\tau \in [\tau_0, 0]$, we have $\left\{{}^{(\text{Interesting})}\tau = \tau\right\} \cap \mathcal{M}_{\text{Right}} = \left\{(t, u, x^2, x^3) \mid (x^2, x^3) \in \mathbb{T}^2,\ -U_1 \le u < \mathfrak{U}_{-\tau,\mathfrak{n}_0}(x^2, x^3), \text{ and } t = \mathfrak{t}_{-\tau,-\mathfrak{n}_0}(u, x^2, x^3)\right\}$. Moreover, from Prop. 32.5 (in particular (32.16) and (32.18)) and definition (32.34), we see that for each fixed $\tau \in [\tau_0, 0]$, we have $\left\{{}^{(\text{Interesting})}\tau = \tau\right\} \cap \mathcal{M}_{\text{Singular}} = \left\{\left(\check{T}_{-\tau}(u, x^2, x^3), u, x^2, x^3\right) \mid (x^2, x^3) \in \mathbb{T}^2,\ \mathfrak{U}_{-\tau,-\mathfrak{n}_0}(x^2, x^3) \le u \le \mathfrak{U}_{-\tau,0}(x^2, x^3)\right\}$. From these facts, (32.49), and definition (32.37), we conclude (32.43).

Proof of (32.40): From definitions (32.1a), (32.1c), and (32.34), and Lemma 15.6 in the cases $\mathfrak{n} = 0$ and $\mathfrak{n} = \mathfrak{n}_0$, we see that ${}^{(\text{Interesting})}\tau \in C^{2,1}_{\text{geo}}(\text{cl}(\mathcal{M}_{\text{Left}}))$ and ${}^{(\text{Interesting})}\tau \in C^{2,1}_{\text{geo}}(\text{cl}(\mathcal{M}_{\text{Right}}))$, where cl denotes set closure in geometric coordinate space. Similarly, from definition (32.34) and the estimate (32.21b), we see that ${}^{(\text{Interesting})}\tau \in C^{2,1}_{\text{geo}}(\mathcal{M}_{\text{Singular}})$. Moreover, Prop. 32.5 yields that $\mathcal{M}_{\text{Left}}$ and $\mathcal{M}_{\text{Singular}}$ have the common $C^{1,1}$ boundary $\breve{\mathbb{X}}_0^{[\tau_0,0]}$, and from the above observations, definition (32.34), and Lemma 15.3 with $\mathfrak{n} = 0$, we see that ${}^{(\text{Interesting})}\tau$ and its first partial derivatives with respect to the geometric coordinates (t, u, x^2, x^3) are continuous across this common boundary. Similar arguments yield that $\mathcal{M}_{\text{Right}}$ and $\mathcal{M}_{\text{Singular}}$ have the common $C^{1,1}$ boundary $\breve{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$, and that ${}^{(\text{Interesting})}\tau$ and its first partial derivatives with respect to the geometric coordinates (t, u, x^2, x^3) are continuous across this common boundary. From these facts, definition (32.36), and Rademacher's theorem, it follows that ${}^{(\text{Interesting})}\tau, {}^{(\text{Interesting})}\mathscr{T} \in W^{2,\infty}_{\text{geo}}(\text{int}(\mathcal{M}_{\text{Interesting}})) \cap C^1_{\text{geo}}(\mathcal{M}_{\text{Interesting}})$, and that the following estimates hold:

$$\|{}^{(\text{Interesting})}\tau\|_{W^{2,\infty}_{\text{geo}}(\text{int}(\mathcal{M}_{\text{Interesting}}))}, \|{}^{(\text{Interesting})}\mathscr{T}\|_{W^{2,\infty}_{\text{geo}}(\text{int}(\mathcal{M}_{\text{Interesting}}))} \le C, \tag{32.52}$$

$$\|{}^{(\text{Interesting})}\tau\|_{C^1(\mathcal{M}_{\text{Interesting}})}, \|{}^{(\text{Interesting})}\mathscr{T}\|_{C^1(\mathcal{M}_{\text{Interesting}})} \le C. \tag{32.53}$$

Next, we use definition (32.34) and the estimate (15.20) to deduce that $\frac{\partial}{\partial t}{}^{(\text{Interesting})}\tau|_{\mathcal{M}_{\text{Left}}} \approx 1$ and $\frac{\partial}{\partial t}{}^{(\text{Interesting})}\tau|_{\mathcal{M}_{\text{Right}}} \approx 1$. Similarly, since definitions (32.1b), (32.34), and Def. 4.5 imply that ${}^{(\text{Interesting})}\tau|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]}} = {}^{(\mathfrak{n})}\tau|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]}}$, we deduce from (15.20) that $\frac{\partial}{\partial t}{}^{(\text{Interesting})}\tau|_{\mathcal{M}_{\text{Singular}}} \approx 1$. From these bounds and (32.53), we conclude (32.40).

Proof that ${}^{(\text{Interesting})}\mathscr{T}$ is a diffeomorphism, proof that $\mathcal{M}_{\text{Interesting}}$ is compact, and proof of (32.42): From the definition (32.36) of ${}^{(\text{Interesting})}\mathscr{T}$, the estimates (32.40) and (32.53), and the inverse function theorem, we see that ${}^{(\text{Interesting})}\mathscr{T}$ is a local diffeomorphism on $\mathcal{M}_{\text{Interesting}}$. Also using (32.49) and the graph structure of ${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]}$ from (32.43), we see that ${}^{(\text{Interesting})}\mathscr{T}$ is injective on $\mathcal{M}_{\text{Interesting}}$ and that ${}^{(\text{Interesting})}\mathscr{T}(\mathcal{M}_{\text{Interesting}}) = [\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$. That is, ${}^{(\text{Interesting})}\mathscr{T}$ is a global diffeomorphism from $\mathcal{M}_{\text{Interesting}}$ onto $[\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$. The compactness of $\mathcal{M}_{\text{Interesting}}$ now follows since it is the image of the compact set $[\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$ under the inverse of the diffeomorphism ${}^{(\text{Interesting})}\mathscr{T}$.

Next, using (32.40), (32.52), and (32.53), we deduce that the inverse map satisfies $\|{}^{(\text{Interesting})}\mathscr{T}^{-1}\|_{C^1([\tau_0,0]\times[-U_1,U_2]\times\mathbb{T}^2)} \le C$ and $\|{}^{(\text{Interesting})}\mathscr{T}^{-1}\|_{W^{2,\infty}((\tau_0,0)\times(-U_1,U_2)\times\mathbb{T}^2)} \le C$. Thanks to the convexity of $(\tau_0, 0) \times (-U_1, U_2) \times \mathbb{T}^2$, standard Sobolev embedding also yields $\|{}^{(\text{Interesting})}\mathscr{T}^{-1}\|_{C^{1,1}([\tau_0,0]\times[-U_1,U_2]\times\mathbb{T}^2)} \le C\|{}^{(\text{Interesting})}\mathscr{T}^{-1}\|_{W^{2,\infty}((\tau_0,0)\times(-U_1,U_2)\times\mathbb{T}^2)} \le C$. Since ${}^{(\text{Interesting})}\mathfrak{t}_\tau$ is the first component function of ${}^{(\text{Interesting})}\mathscr{T}^{-1}$, we conclude (32.42).

Proof of (32.41) **and related properties of** μ: From (32.5)–(32.6), and definition (32.34), it follows that ${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Left}} = {}^{(0)}\widetilde{\Sigma}_\tau^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Left}}$, ${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Right}} = {}^{(\mathfrak{n}_0)}\widetilde{\Sigma}_\tau^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Right}}$, and ${}^{(\text{Interesting})}\Sigma_\tau^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Singular}} = \breve{\mathbb{M}}_{-\tau}^{[0,\mathfrak{n}_0]}$. The result (32.41) and the results stated just below (32.41) follow from these three identities, the fact that

${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]} = ({}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Left}}) \cup ({}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Right}}) \cup ({}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]} \cap \mathcal{M}_{\text{Singular}})$, the result (18.1) and the results stated just below (18.1), and (32.5) (which shows that $\mu \equiv -\tau$ along $\breve{\mathbb{M}}_{-\tau}^{[0,\mathfrak{n}_0]}$).

Proof of (32.50)**, of the quantitative quasi-convexity of** $\mathcal{M}_{\text{Interesting}}$**, and of** (32.51): Thanks to the estimate (32.40), the estimate $\|{}^{(\text{Interesting})}\mathscr{T}^{-1}\|_{C^1([\tau_0,0]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C$ proved above, and the convexity of $[\tau_0,0]\times[-U_1,U_2]\times\mathbb{T}^2$, we can use arguments similar to the ones given in the proof of Lemma 15.5 to conclude (32.50), the quasi-convexity of $\mathcal{M}_{\text{Interesting}}$, and the Sobolev embedding result (32.51).

Proof of (32.39) **and** (32.44): These estimates follow from (32.51) and (32.52).

Proof that the ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$ **are g-spacelike for** $\tau \in [\tau_0,0]$**, except along** $\mathcal{B}^{[0,\mathfrak{n}_0]}$: This follows from definition (32.34), Lemma 32.7 (note that $\mathcal{M}_{\text{Singular}}\backslash\mathcal{B}^{[0,\mathfrak{n}_0]} = \bigcup_{\mathfrak{m}\in(0,\mathfrak{m}_0]}\breve{\mathbb{M}}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$), (6.20a), (6.20c), (18.8a) (which implies that $-L\mu \approx 1$ on the support of ϕ), and (18.27), which collectively show that the vectorfield ${}^{(\mathfrak{n})}\widetilde{N}$ (which is $\mathbf{g}$-orthogonal to ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$) is $\mathbf{g}$-timelike in regions where $\mu > 0$. We clarify that $\mu > 0$ on $\mathcal{M}_{\text{Left}} \cup \mathcal{M}_{\text{Right}}$ by virtue of definitions (32.1a) and (32.1c), (18.1) and the results stated below (18.1), and the fact for each $\mathfrak{n} \in [0,\mathfrak{n}_0]$, we have that $\breve{X}\mu = -\mathfrak{n}$ along the μ-adapted torus $\breve{\mathbf{T}}_{0,\mathfrak{n}}$ in ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau),[-U_1,U_2]}$ where μ vanishes.

Proof of (32.45) **and** (32.47): These identities are straightforward to verify from Def. 32.9, Lemma 3.9, and equation (4.4a).

Proof of (32.46) **and** (32.48): Since we have already shown that $\breve{H}$ and $\breve{G}$ are coordinate partial derivatives in the coordinate system $({}^{(\text{Interesting})}\tau, u, x^2, x^3)$, the estimates (32.46) and (32.48) follow from (32.44) and the estimate

$$\|{}^{(\text{Interesting})}\mathscr{T}^{-1}\|_{C^{1,1}([\tau_0,0]\times[-U_1,U_2]\times\mathbb{T}^2)} \leq C$$

noted above.

Proof of the remaining properties of $\breve{H}$: Since (32.47) shows that $\breve{H}$ is the partial derivative with respect to u in the coordinate system $({}^{(\text{Interesting})}\tau, u, x^2, x^3)$, it trivially follows that every u-parametrized integral curve of $\breve{H}$ that starts in the level-set ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$ (for some $\tau \in [\tau_0,0]$) is defined on the interval $[-U_1,U_2]$ and remains in ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$.

Finally, we show that in ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$, every integral curve from the previous paragraph must intersect $\breve{\mathbf{T}}_{-\tau,0}$ in a unique point. To this end, we note that Prop. 32.5 implies that $\mathfrak{L}^{-1}$ is a $C_{\text{geo}}^{1,1}$ diffeomorphism from $\mathcal{M}_{\text{Singular}}$ onto $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$. In particular, considering the form (32.7) of $\mathfrak{L}$, and using (18.3b) and the fact that ${}^{(\text{Interesting})}\tau|_{\breve{\mathbb{X}}_0^{[\tau_0,0]}} = {}^{(0)}\tau|_{\breve{\mathbb{X}}_0^{[\tau_0,0]}}$, we see that $\breve{\mathbf{T}}_{-\tau,0} \subset \breve{\mathbb{X}}_0^{[\tau_0,0]} \cap {}^{(\text{Interesting})}\Sigma_{\tau}^{[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]} \subset \mathcal{M}_{\text{Singular}}$, and that along $\breve{\mathbf{T}}_{-\tau,0}$, u is a $C^{1,1}$ function of $(x^2,x^3) \in \mathbb{T}^2$ satisfying $|u| \leq \frac{1}{2}U_{☢}$. Combining these results and using that $\breve{H}u = 1$ and $\breve{H}x^2 = \breve{H}x^3 = 0$, we see that every integral curve from the previous paragraph must intersect $\breve{\mathbf{T}}_{-\tau,0}$. The uniqueness of the point follows from the fact that $\breve{X}\mu|_{\breve{\mathbf{T}}_{-\tau,0}} = 0$ and the fact that when $|u| \leq U_{☢}$, we have the estimate $\breve{H}\breve{X}\mu \approx 1$; this estimate follows from definition (32.38a), Lemma 5.5, Prop. 9.1, (18.5), (18.8a), and the estimates of Lemma 15.5 and Prop. 17.1.

□

33. Homeomorphism and diffeomorphism properties of Υ on $\mathcal{M}_{\text{Interesting}}$ and a description of the singular boundary in Cartesian coordinate space

In this section, we reveal the homeomorphism and diffeomorphism properties of the change of variables map $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ on the region $\mathcal{M}_{\text{Interesting}}$. We also reveal how the singular boundary $\mathcal{B}^{[0,\mathfrak{n}_0]}$ is embedded in Cartesian coordinate space under Υ, i.e., we exhibit various properties of the set $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, including its structure as a $\mathbf{g}$-null hypersurface in the Cartesian coordinate differential structure. We refer to Remark 33.3 for a discussion of interesting degeneracies in the vectorfield L that occur along $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$.

33.1. **Homeomorphism and diffeomorphism properties of** Υ **on** $\mathcal{M}_{\text{Interesting}}$**.** In the next proposition, we reveal the homeomorphism and diffeomorphism properties of the change of variables map Υ. The proposition is crucial for translating results that we have derived with respect to the geometric coordinates on $\mathcal{M}_{\text{Interesting}}$ into results with respect to the Cartesian coordinates on $\Upsilon(\mathcal{M}_{\text{Interesting}})$. This will become apparent in the proof of Theorem 34.1.

Proposition 33.1 (Homeomorphism and diffeomorphism properties of Υ on $\mathcal{M}_{\text{Interesting}}$)**.** *Assume the hypotheses and conclusions of Theorem 31.1 for* $\mathfrak{n} \in [0, \mathfrak{n}_0]$. *Recall that* $\mathcal{M}_{\text{Interesting}}$ *is the set defined in* (32.1d) *and depicted in Figs. 6 and 13. Then the change of variables map* $\Upsilon(t, u, x^2, x^3) = (t, x^1, x^2, x^3)$ *enjoys the following properties.*

- Υ *is a continuous, injective map on the set* $\mathcal{M}_{\text{Interesting}}$ *defined in* (32.1d)*, which is compact by Prop. 32.11. In particular,* Υ *is a homeomorphism from* $\mathcal{M}_{\text{Interesting}}$ *onto its image.*
- *The following estimates hold on* $\mathcal{M}_{\text{Interesting}}$*:*

$$\|\Upsilon\|_{C^{3,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C, \tag{33.1}$$

$$\det \frac{\partial \Upsilon(t, u, x^2, x^3)}{\partial(t, u, x^2, x^3)} = \mu \frac{c^2}{X^1} = -\{1 + \mathcal{O}(\mathring{\alpha})\} \mu. \tag{33.2}$$

- Υ *is a global diffeomorphism on the subset* $\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}$*, i.e., it is a diffeomorphism away from the singular boundary.*

Proof. Using Lemma 5.5, Prop. 9.1, and Prop. 17.1, we compute that $\|\Upsilon\|_{W^{4,\infty}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \lesssim 1$. From this estimate and (32.51), we further deduce that $\|\Upsilon\|_{C^{3,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \lesssim 1$ as desired.

(33.2) follows from same the arguments we used to prove (18.20). From these facts, the fact that μ is positive on $\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}$ (see Prop. 32.11), and the inverse function theorem, we deduce that Υ is a local diffeomorphism on $\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}$.

The rest of proof is similar to the proof of Prop. 18.4. We will silently use the following results from Prop. 32.11: the vectorfield $\breve{H}$ is the partial derivative with respect to u in the coordinate system $({}^{(\text{Interesting})}\tau, u, x^2, x^3)$, and the vectorfield $\breve{G}$ is the partial derivative with respect to ${}^{(\text{Interesting})}\tau$ in the coordinate system $({}^{(\text{Interesting})}\tau, u, x^2, x^3)$. Moreover, we will use the notation "$*$" to denote any quantity that is pointwise bounded in magnitude by $\mathcal{O}(\mathring{\alpha})$.

To complete the proof of the proposition, we must show that Υ is injective on $\mathcal{M}_{\text{Interesting}}$. In view of the diffeomorphism properties of ${}^{(\text{Interesting})}\mathscr{T}$ shown in Prop. 32.11, we see that it suffices to show that $\Upsilon \circ {}^{(\text{Interesting})}\mathscr{T}^{-1}$, which maps $({}^{(\text{Interesting})}\tau, u, x^2, x^3) \to (t, x^1, x^2, x^3)$, is injective on the domain $[\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$.

As an intermediate step, we will show that the map $({}^{(\text{Interesting})}\tau, u, x^2, x^3) \to ({}^{(\text{Interesting})}\tau, x^1, x^2, x^3)$ is injective on the domain $[\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$ and is a diffeomorphism away from the crease. Below we will show that the map $({}^{(\text{Interesting})}\tau, x^1, x^2, x^3) \to (t, x^1, x^2, x^3)$ is also injective, which will complete the proof. To achieve the intermediate step, we will show that for every fixed $(\tau, x^2, x^3) \in [\tau_0, 0] \times \mathbb{T}^2$, the map $u \to x^1(\tau, u, x^2, x^3)$ is strictly decreasing (here we stress that τ denotes a fixed value of ${}^{(\text{Interesting})}\tau$) on the domain $[-U_1, U_2]$ such that $\breve{H}x^1 < 0$ away from ${}^{(\text{Interesting})}\mathscr{T}\left(\partial\_\mathcal{B}^{[0,\mathfrak{n}_0]}\right)$, i.e., away from the image of the crease under ${}^{(\text{Interesting})}\mathscr{T}$. To this end, we first use Lemma 5.5, Prop. 9.1, (32.38a), and the estimates of Lemma 15.5 and Props. 17.1 and 32.11 to deduce the following estimates, where ϕ is the cut-off from Definition 4.1:

$$\breve{H}x^1 = \begin{cases} \mu(-1 + *), & \text{in } {}^{(\text{Interesting})}\mathscr{T}(\mathcal{M}_{\text{Left}}), \\ -(1 + *)\mu - (1 + *)\frac{\breve{X}\mu}{L\mu}, & \text{in } {}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Singular}}\right), \\ -(1 + *)\mu + \phi(1 + *)\frac{\mathfrak{n}_0}{L\mu}, & \text{in } {}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Right}}\right). \end{cases} \tag{33.3}$$

Recall now that by Prop. 32.11, we have $\mu > 0$ in ${}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Interesting}}\right)$, except along the singular boundary portion ${}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{B}^{[0,\mathfrak{n}_0]}\right)$, where ${}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{B}^{[0,\mathfrak{n}_0]}\right) \subset {}^{(\text{Interesting})}\mathscr{T}\left({}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}\right) \subset {}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Singular}}\right)$. Moreover, from Prop. 18.1 definitions (4.3c) and (32.1b), and Remark 32.2, it follows that in ${}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Singular}}\right)$, we have $L\mu \approx -1$ and $\breve{X}\mu < 0$, except along the subset ${}^{(\text{Interesting})}\mathscr{T}\left(\breve{\mathbb{X}}_0^{[\tau_0,0]}\right)$, where $\breve{X}\mu$ vanishes. Moreover, using (15.44), we see that along ${}^{(\text{Interesting})}\mathscr{T}\left(\breve{\mathbb{X}}_0^{[\tau_0,0]}\right)$, we have $\mu > 0$, except on the crease ${}^{(\text{Interesting})}\mathscr{T}\left(\partial\_\mathcal{B}^{[0,\mathfrak{n}_0]}\right) = {}^{(\text{Interesting})}\mathscr{T}\left(\breve{\mathbf{T}}_{0,0}\right)$. That is, ${}^{(\text{Interesting})}\mathscr{T}\left(\breve{\mathbf{T}}_{0,0}\right)$ is precisely the subset of ${}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Interesting}}\right)$ along which both μ and $\breve{X}\mu$ vanish. Note also that ${}^{(\text{Interesting})}\mathscr{T}\left(\breve{\mathbf{T}}_{0,0}\right) \subset {}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{B}^{[0,\mathfrak{n}_0]}\right) \subset {}^{(\text{Interesting})}\mathscr{T}\left({}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}\right)$. From these facts and (33.3), we see that within ${}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Singular}}\right)$, we have $\breve{H}x^1 < 0$, except along the crease ${}^{(\text{Interesting})}\mathscr{T}\left(\breve{\mathbf{T}}_{0,0}\right)$, which is a subset of ${}^{(\text{Interesting})}\mathscr{T}\left({}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}\right)$, where $\breve{H}x^1|_{{}^{(\text{Interesting})}\mathscr{T}(\breve{\mathbf{T}}_{0,0})} = 0$. In addition, Prop. 32.11 implies that every

u-parameterized integral curve of $\breve{H}$ in ${}^{(\text{Interesting})}\mathscr{T}\left({}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}\right)$ (i.e., in the level-set $\{{}^{(\text{Interesting})}\tau = 0\}$) intersects the crease at a unique value of $u \in [-\frac{1}{2}U_{\clubsuit}, \frac{1}{2}U_{\clubsuit}]$. In total, we have shown that along every u-parameterized integral curve of $\breve{H}$ in $[\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$, we have $\breve{H}x^1 < 0$, except at possibly one point on the integral curve. From this fact and the mean value theorem, we conclude that at every fixed $(\tau, x^2, x^3) \in [\tau_0, 0] \times \mathbb{T}^2$, the map $u \to x^1(\tau, u, x^2, x^3)$ is strictly decreasing. We have therefore shown that the map $({}^{(\text{Interesting})}\tau, u, x^2, x^3) \to ({}^{(\text{Interesting})}\tau, x^1, x^2, x^3)$ is injective on the domain $[\tau_0, 0] \times [-U_1, U_2] \times \mathbb{T}^2$ and is a C^1 diffeomorphism away from ${}^{(\text{Interesting})}\mathscr{T}\left(\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}\right)$.

Next, for use below, we use definitions (32.34) and (32.38b), Lemma 5.5, Prop. 9.1, and the estimates of Props. 17.1 and Prop. 32.11, to compute that in ${}^{(\text{Interesting})}\mathscr{T}\left(\mathcal{M}_{\text{Interesting}}\right)$, we have $\breve{G}x^1 \approx \frac{\partial}{\partial t}x^1 \approx Lx^1 - L^A\frac{\partial}{\partial x^A}x^1 \approx L^1 \approx 1$.

The rest of the proof now mirrors the proof of the injectivity of Υ on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\text{Boot}}],[-U_1,U_2]}$ provided by Prop. 18.4, where the estimate $\breve{G}x^1 \approx 1$ plays the role of the estimate (18.22) used in that proof. We will sketch the details. Specifically, the argument requires that we show that $\frac{1}{\partial_t {}^{(\text{Interesting})}\tau} > 0$ on $\mathcal{M}_{\text{Interesting}}$, except possibly when ${}^{(\text{Interesting})}\tau = 0$, where ∂_t is the Cartesian partial time derivative vectorfield. To prove this result, we first use (3.10), (3.13), (3.26a), (4.2), (4.4), (5.9a), Lemma 5.5, Prop. 9.1, definition (32.34), and the estimates of Lemma 15.5 and Props. 17.1 and 32.11 to compute that $\partial_t = L + (1 + *)X + *Y_{(2)} + *Y_{(3)}$ and that the following estimates hold, where ϕ is the cut-off from Definition 4.1:

$$\frac{1}{\partial_t {}^{(\text{Interesting})}\tau} \approx \begin{cases} 1, & \text{in } \mathcal{M}_{\text{Left}}, \\ \frac{1}{1 - \frac{\breve{X}\mu}{\mu}} = \frac{\mu}{\mu - \breve{X}\mu}, & \text{in } \mathcal{M}_{\text{Singular}}, \\ \frac{1}{1 + \frac{\mathfrak{n}_0\phi}{\mu}} = \frac{\mu}{\mu + \mathfrak{n}_0\phi}, & \text{in } \mathcal{M}_{\text{Right}}. \end{cases} \tag{33.4}$$

We now recall that $\breve{X}\mu|_{\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]}} = -\mathfrak{n}$ (and thus by (32.1b), $\breve{X}\mu \leq 0$ in $\mathcal{M}_{\text{Singular}}$), and that within $\mathcal{M}_{\text{Interesting}}$, μ vanishes precisely on the singular boundary $\mathcal{B}^{[0,\mathfrak{n}_0]}$, which is contained in ${}^{(\text{Interesting})}\Sigma_0^{[-U_1,U_2]}$, i.e., μ can vanish only when ${}^{(\text{Interesting})}\tau = 0$. From these facts, definitions (32.1a)–(32.1d), (32.49), and (33.4), it follows that within $\mathcal{M}_{\text{Interesting}}$, we have $\frac{1}{\partial_t {}^{(\text{Interesting})}\tau} > 0$, except possibly when ${}^{(\text{Interesting})}\tau = 0$, which is the desired result. This concludes the proof of the proposition.

□

33.2. **Description of the singular boundary in Cartesian coordinate space.** In the next proposition, we reveal how the singular boundary $\mathcal{B}^{[0,\mathfrak{n}_0]}$ is embedded in Cartesian coordinate space, i.e., we exhibit various properties of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$. Among the main conclusions is that $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ is ruled, in a degenerate sense made clear in the proposition and Remark 33.3, by integral curves of the $\mathbf{g}$-null vectorfield L.

Proposition 33.2 (Description of the singular boundary in Cartesian coordinate space)**.** *Assume the hypotheses and conclusions of Theorem 31.1 for* $\mathfrak{n} \in [0, \mathfrak{n}_0]$. *Let* Υ *be the change of variables map from geometric to Cartesian coordinates defined in* (5.1), *and let* $\mathfrak{S}$ *be the* $C^{1,1}$ *diffeomorphism defined in* (32.10). *Recall that* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *is a portion of the singular boundary in geometric coordinate space and that* $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}$ *denotes the crease, viewed as subsets of geometric coordinate space. Recall also that in Prop. 32.5, we showed that* $\mathfrak{S}$ *is a diffeomorphism from* $[0, \mathfrak{n}_0] \times \mathbb{T}^2$ *onto* $\mathcal{B}^{[0,\mathfrak{n}_0]}$. *Then the following conclusions hold.*

A homeomorphism onto $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ ***and a diffeomorphism onto*** $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]})$. $\Upsilon \circ \mathfrak{S}$ *is a* $C^{1,1}$ *injective map from* $[0, \mathfrak{n}_0] \times \mathbb{T}^2$ *onto* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ *- the image of* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *in Cartesian coordinate space - that satisfies:*

$$\frac{\partial[\Upsilon \circ \mathfrak{S}](\mathfrak{n}, x^2, x^3)}{\partial(\mathfrak{n}, x^2, x^3)} = \frac{\partial(t, x^1, x^2, x^3)}{\partial(\mathfrak{n}, x^2, x^3)} = \begin{pmatrix} \frac{\mathfrak{n}}{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu - (\frac{\partial}{\partial t}\mu)\frac{\partial}{\partial u}\breve{X}\mu} & * & * \\ \frac{\{1+*\}\mathfrak{n}}{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu - (\frac{\partial}{\partial t}\mu)\frac{\partial}{\partial u}\breve{X}\mu} & * & * \\ 0 & 1 & * \\ 0 & * & 1 \end{pmatrix}, \tag{33.5}$$

where $*$ *denotes terms of size* $\mathcal{O}(\mathring{\alpha})$, *and the denominator-terms satisfy* $(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu - (\frac{\partial}{\partial t}\mu)\frac{\partial}{\partial u}\breve{X}\mu \approx 1$. *Moreover, on* $(0, \mathfrak{n}_0] \times \mathbb{T}^2$, $\Upsilon \circ \mathfrak{S}$ *is an embedding, i.e., the differential of* $\Upsilon \circ \mathfrak{S}$ *is injective on* $(0, \mathfrak{n}_0] \times \mathbb{T}^2$, *and* $\Upsilon \circ \mathfrak{S}$ *is a diffeomorphism from* $(0, \mathfrak{n}_0] \times \mathbb{T}^2$ *onto its image* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]})$ *in Cartesian coordinate space. In addition, the following estimate* *holds:*

$$\|\Upsilon \circ \mathfrak{S}\|_{C^{1,1}([0,\mathfrak{n}_0]\times\mathbb{T}^2)} \leq C. \tag{33.6}$$

Furthermore, the map $(z, x^2, x^3) \to [\Upsilon \circ \mathfrak{S}](\sqrt{z}, x^2, x^3)$ *is a diffeomorphism from* $[0, \mathfrak{n}_0^2]\times\mathbb{T}^2$ *onto its image* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ *in Cartesian coordinate space, and the map is bounded in the norm* $\|\cdot\|_{C^{1,\frac{1}{2}}([0,\mathfrak{n}_0^2]\times\mathbb{T}^2)}$ *by* $\leq C$*. That is,* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]}) = \Upsilon \circ \mathfrak{S}\left([0, \mathfrak{n}_0^2] \times \mathbb{T}^2\right)$ *is a* $C^{1,\frac{1}{2}}$ *embedded sub-manifold-with-boundary of Cartesian coordinate space.*

A description of the g-spacelike embedded tori $\Upsilon(\breve{\mathbf{T}}_{0,-\mathfrak{n}})$*. For each fixed* $\mathfrak{n} \in [0, \mathfrak{n}_0]$*, the map* $(x^2, x^3) \to [\Upsilon\circ\mathfrak{S}](\mathfrak{n}, x^2, x^3)$ *is a diffeomorphism from* $\mathbb{T}^2$ *onto* $\Upsilon(\breve{\mathbf{T}}_{0,-\mathfrak{n}})$*, where* $\breve{\mathbf{T}}_{0,-\mathfrak{n}}$ *is the* μ*-adapted torus defined in* (4.3c)*. In particular, the differential of this map is injective, and* $\Upsilon(\breve{\mathbf{T}}_{0,-\mathfrak{n}})$ *is an embedded* $C^{1,1}$ *graph over* $\mathbb{T}^2$ *in Cartesian coordinate space such that at each* $q \in \breve{\mathbf{T}}_{0,-\mathfrak{n}}$*,* $\Upsilon(\breve{\mathbf{T}}_{0,-\mathfrak{n}})$ *is spacelike with respect to* $\mathbf{g}|_{\Upsilon(q)}$ *at the point* $\Upsilon(q)$*.*

The causal structure of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$*. The vectorfield* Q *defined by:*

$$Q \stackrel{def}{=} L - \frac{L\mu}{\breve{X}\mu}\breve{X} \tag{33.7}$$

enjoys the following properties:

- *Q is well-defined on and tangent to $\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, viewed as a subset of geometric coordinate space (which is the target of $\mathfrak{S}$).*
- *The integral curves of Q foliate $\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$ and satisfy $Qu < 0$ and $Qt > 0$ along $\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$. In particular, the integral curves are transversal to the characteristics $\mathcal{P}_u$.*
- *Along $\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, $\mathbf{g}(Q, V) = 0$ holds for every vectorfield $V = V^t\frac{\partial}{\partial t} + V^u\frac{\partial}{\partial u} + V^2\frac{\partial}{\partial 2} + V^3\frac{\partial}{\partial 3}$ on $\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, regardless of whether V is tangent to $\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$. In particular, on $\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, we have that $\mathbf{g}(Q,Q) = 0$.*
- *For every $q \in \mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, the pushforward vectorfield $[d_{\mathrm{geo}}\Upsilon(q)]\cdot Q(q)$, which is tangent to the hypersurface $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$ in Cartesian coordinate space, is equal to $L|_{\Upsilon(q)} = [L^\alpha\partial_\alpha]|_{\Upsilon(q)}$.*
- *For every $q \in \mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, $L|_{\Upsilon(q)}$ is $\mathbf{g}|_{\Upsilon(q)}$-orthogonal to the tangent space of $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$ at $\Upsilon(q)$.*
- *Q is the unique such vectorfield with the above properties.*

In particular, in the differential structure on spacetime induced by the Cartesian coordinates, there exist integral curves of the $\mathbf{g}$*-null vectorfield* $L = L^\alpha\partial_\alpha$ *that foliate* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ *and that are everywhere* $\mathbf{g}$*-orthogonal to* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$*. Considering also that the* $\Upsilon(\breve{\mathbf{T}}_{0,-\mathfrak{n}})$ *are* $\mathbf{g}$*-spacelike, we see that* $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]})$ *is a* $\mathbf{g}$*-null hypersurface in the Cartesian coordinate differential structure.*

Remark 33.3 (Non-uniqueness of the integral curves of $L^\alpha\partial_\alpha$ along $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$)**.** Note that along $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, even though the scalar functions $\breve{X}L^\beta = \mu XL^\beta$ remain bounded, the non-μ-weighted quantities $XL^\beta = X^\alpha\partial_\alpha L^\beta$ can blow up, due to the vanishing of μ there. Since $X^\alpha\partial_\alpha$ is a non-degenerate (i.e., everywhere non-zero and bounded) vectorfield in the Cartesian differential structure, it follows that the Cartesian partial derivatives $\partial_\alpha L^\beta$ can blow up along $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$. Hence, in the Cartesian differential structure, the vectorfield L does not have sufficient regularity to ensure uniqueness of its integral curves up to $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, i.e., standard uniqueness theorems would require Lipschitz regularity for L^α. This lack of uniqueness is the mechanism that allows for the existence of integral curves of L that foliate $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, even though integral curves of L also foliate the characteristics $\mathcal{P}_u$, which are distinct from $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$.

More precisely, consider any fixed point $p_0 \in \mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$, and let u_0 denote the eikonal function evaluated at p_0 (i.e., the u-coordinate of p_0 in geometric coordinates). By Prop. 33.2, there exists an interval I of u-values containing u_0 and a unique integral curve $\gamma_{p_0} : I \to \mathcal{B}^{[0,\mathfrak{n}_0]}\backslash\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$ of Q in geometric coordinate space satisfying:

$$\frac{\mathrm{d}}{\mathrm{d}u}\gamma_{p_0}(u) = Q \circ \gamma_p(u), \qquad \gamma_p(u_0) = p_0. \tag{33.8}$$

It also follows from Prop. 33.2 that the pushforward of the tangent vector $\frac{\mathrm{d}}{\mathrm{d}u}\gamma_{p_0}(u_0)$ under Υ is precisely $L|_{\Upsilon(p_0)}$. Similarly, since $p_0 \in \mathcal{P}_{u_0}$, L is tangent to $\mathcal{P}_{u_0}$, and $Lt = 1$, there is an interval J of t-values and a t-parameterized integral curve $\Lambda : J \to \mathcal{P}_{u_0}$ of L in geometric coordinate space such that the integral curve is *tangent to* $\mathcal{P}_{u_0}$ and such that $p_0 = \Lambda(t_0)$ for some $t_0 \in J$ (here, t_0 is the Cartesian time function evaluated at p_0). Hence, in Cartesian coordinate space, there are two distinct integral curves of L through $\Upsilon(p_0)$, namely $\Upsilon \circ \gamma$ and $\Upsilon \circ \Lambda$, each with the same tangent vector $L|_{\Upsilon(p_0)}$; see Fig. 14.

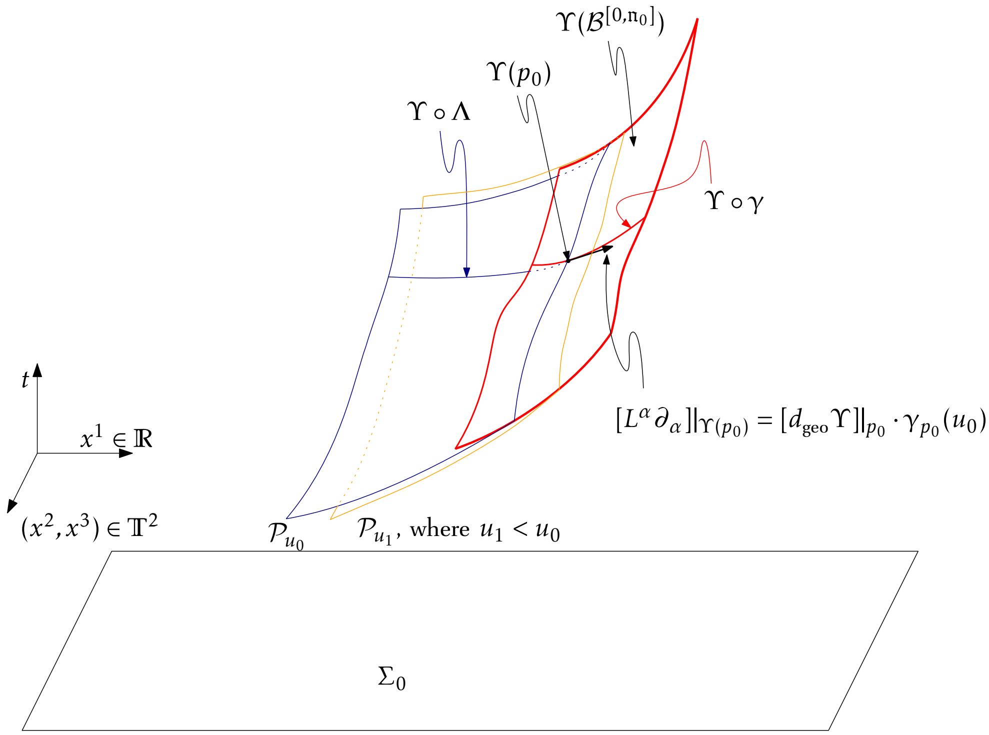


Figure 14. Non-uniqueness of integral curves of L along $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ in Cartesian coordinates.

Proof of Prop. 33.2. The injectivity of $\Upsilon \circ \mathfrak{S}$ and the estimate (33.6) follow from Props. 32.5 and 33.1. At the end of the proof, we will prove (33.5) and the denominator-term estimate $(\frac{\partial}{\partial u}\upmu)\frac{\partial}{\partial t}\breve{X}\upmu - (\frac{\partial}{\partial t}\upmu)\frac{\partial}{\partial u}\breve{X}\upmu \approx 1$. Taking these for granted for the time being, we deduce from the explicit form of (33.5) that the differential of $\Upsilon \circ \mathfrak{S}$ (with respect to $(\mathfrak{n}, x^2, x^3)$) is injective on $(0, \mathfrak{n}_0] \times \mathbb{T}^2$ and thus $\Upsilon \circ \mathfrak{S}$ is a diffeomorphism from $(0, \mathfrak{n}_0] \times \mathbb{T}^2$ onto its image, as is desired.

Proof of the properties of the map $(z, x^2, x^3) \to [\Upsilon \circ \mathfrak{S}](\sqrt{z}, x^2, x^3)$: We set $z \overset{\mathrm{def}}{=} \mathfrak{n}^2$. From the previous paragraph, (33.5) (note that the first column of the matrix on RHS (33.5) features a linear factor of $\mathfrak{n}$), and the chain rule, it follows that the map $(z, x^2, x^3) \to [\Upsilon \circ \mathfrak{S}](\sqrt{z}, x^2, x^3)$ is injective on $[0, \mathfrak{n}_0^2] \times \mathbb{T}^2$, that:

$$\frac{\partial[\Upsilon \circ \mathfrak{S}](\sqrt{z}, x^2, x^3)}{\partial(z, x^2, x^3)} = \begin{pmatrix} \frac{1}{2}(\frac{\partial}{\partial u}\upmu)\frac{\partial}{\partial t}\breve{X}\upmu - \frac{1}{2}(\frac{\partial}{\partial t}\upmu)\frac{\partial}{\partial u}\breve{X}\upmu & * & * \\ \frac{\{-\frac{1}{2}+*\}}{(\frac{\partial}{\partial u}\upmu)\frac{\partial}{\partial t}\breve{X}\upmu - (\frac{\partial}{\partial t}\upmu)\frac{\partial}{\partial u}\breve{X}\upmu} & * & * \\ 0 & 1 & * \\ 0 & * & 1 \end{pmatrix}, \tag{33.9}$$

and that $\frac{\partial[\Upsilon \circ \mathfrak{S}](\sqrt{z}, x^2, x^3)}{\partial(z, x^2, x^3)}$ can be expressed as $C^{0,1}$ function of $(\mathfrak{n}, x^2, x^3)$. Since the map $z \to \sqrt{z}$ is $C^{0,1/2}$, it follows that $\frac{\partial[\Upsilon \circ \mathfrak{S}](\sqrt{z}, x^2, x^3)}{\partial(z, x^2, x^3)}$ can be expressed as a $C^{0,1/2}$ function of (z, x^2, x^3). Since (33.9) implies that the differential of the map $(z, x^2, x^3) \to [\Upsilon \circ \mathfrak{S}](\sqrt{z}, x^2, x^3)$ is injective (when $\mathring{\alpha}$ is sufficiently small), we conclude that the map is a $C^{1,\frac{1}{2}}$ diffeomorphism from $[0, \mathfrak{n}_0^2] \times \mathbb{T}^2$ onto $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, as is desired.

Proof of the properties of $\Upsilon(\check{\mathbf{T}}_{0,-\mathfrak{n}})$: From the explicit form of (33.5), we see that the last two columns of the matrix on RHS (33.5) are linearly independent when $\mathring{\alpha}$ is sufficiently small (even if $\mathfrak{n} = 0$). Also considering (32.11), we see that for any $\mathfrak{n} \in [0, \mathfrak{n}_0]$, the map $(x^2, x^3) \to [\Upsilon \circ \mathfrak{S}](\mathfrak{n}, x^2, x^3)$ is a diffeomorphism from $\mathbb{T}^2$ onto $\Upsilon(\check{\mathbf{T}}_{0,-\mathfrak{n}})$, and the map is $C^{1,1}$ by (33.6). Since Lemma 32.7 shows that each $\check{\mathbf{T}}_{0,-\mathfrak{n}}$ is $\mathbf{g}$-spacelike (in the differential structure of the geometric coordinates), and since Prop. 18.4 implies that $\Upsilon|_{\check{\mathbf{T}}_{0,-\mathfrak{n}}}$ is a diffeomorphism, we immediately conclude that $\Upsilon(\check{\mathbf{T}}_{0,-\mathfrak{n}})$ is $\mathbf{g}$-spacelike.

Proof of the properties of Q: The uniqueness statement made about Q follows from the already proven fact that the differential of $\Upsilon \circ \mathfrak{S}$ is injective on $(0, \mathfrak{n}_0] \times \mathbb{T}^2$ and our assumption that $[d_{\mathrm{geo}}\Upsilon(q)] \cdot Q(q)$ is equal to $[L^\alpha \partial_\alpha]|_{\Upsilon(q)}$.

Next, in view of (33.7), we see that Q is well-defined at points where $\check{X}\mu < 0$, a condition that is satisfied on $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$ by (32.3)–(32.4) and the fact that $\check{X}\mu|_{\check{\mathbf{T}}_{0,-\mathfrak{n}}} \equiv -\mathfrak{n}$. We also note that $Q\mu = 0$ (i.e., Q is tangent to the singular boundary). Furthermore, using Lemma 3.9, $\check{X}\mu|_{\check{\mathbf{T}}_{0,-\mathfrak{n}}} \equiv -\mathfrak{n}$, (18.3b), and (18.8a), we find that along $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$, we have $Qt = 1 > 0$ and $Qu = -\frac{L\mu}{\check{X}\mu} < 0$, as desired.

Next, using Lemma 3.9, we compute that $\mathbf{g}(Q, L) = \mu \frac{L\mu}{\check{X}\mu}$, $\mathbf{g}(Q, \check{X}) = -\mu + \mu^2 \frac{L\mu}{\check{X}\mu}$, and $\mathbf{g}(Q, \frac{\partial}{\partial x^A}) = 0$ for $A = 2, 3$. Note that on $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$, where $\mu = 0$, all of these inner products vanish. Since $\left\{L, \check{X}, \frac{\partial}{\partial 2}, \frac{\partial}{\partial 3}\right\}$ spans the geometric coordinate tangent space, it follows that along $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$, Q is $\mathbf{g}$-orthogonal to every vectorfield, as we claimed in the proposition.

Proof of properties tied to foliations of $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$ and $\Upsilon\left(\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}\right)$: Using (18.5), (18.8a), the fact that $\check{X}\mu|_{\check{\mathbf{T}}_{0,-\mathfrak{n}}} \equiv -\mathfrak{n}$, and (33.7), we find that $Q\check{X}\mu > 0$ on $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$. Hence, since (32.3) implies that the level-sets of $\check{X}\mu$ foliate $\mathcal{B}^{[0,\mathfrak{n}_0]} = \{(t, u, x^2, x^3) \mid \mu(t, u, x^2, x^3) = 0, -\mathfrak{n}_0 \leq \check{X}\mu(t, u, x^2, x^3) \leq 0, (x^2, x^3) \in \mathbb{T}^2\}$, and since $Q\check{X}\mu > 0$ implies that Q is transversal to the level-sets of $\check{X}\mu$, we deduce that the integral curves of Q foliate $\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$. Hence, the pushforward of $Q|_{\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}}$ by Υ is a vectorfield tangent to $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]})$, which is foliated by the integral curves. We now come to the key point: for any $q \in \mathcal{B}^{[0,\mathfrak{n}_0]} \backslash \partial_- \mathcal{B}^{[0,\mathfrak{n}_0]}$, the pushforward of $Q|_q$ by Υ is $[L^\alpha \partial_\alpha]|_{\Upsilon(q)}$. The reason is that the pushforward of $\check{X}|_q$ by Υ is $[\mu X^\alpha \partial_\alpha]|_{\Upsilon(q)}$, which, in the Cartesian differential structure, vanishes along $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$, where $\mu \equiv 0$.

To complete the proof of the proposition, it remains for us to prove (33.5). We first complement the vectorfields J and K defined in (32.26)–(32.27) with the following pair of vectorfields ($A = 2, 3$):

$$P_{(A)} \overset{\text{def}}{=} \frac{\partial}{\partial x^A} - \left(\frac{\partial}{\partial x^A}\mu\right) J + \left(\frac{\partial}{\partial x^A}\check{X}\mu\right) K. \tag{33.10}$$

As in (32.28), we compute (recalling that $K\check{X}\mu = -1$) that:

$$P_{(2)} x^2 = 1, \qquad\qquad P_{(2)}\mu = -P_{(2)}\check{X}\mu = Jx^3 = 0. \tag{33.11}$$

It follows from (33.11) that $P_{(2)}$ is the partial derivative with respect to x^2 in the coordinates $(\mathfrak{n}, x^2, x^3)$ on $[0, \mathfrak{n}_0] \times \mathbb{T}^2$, i.e., on the domain of $\Upsilon \circ \mathfrak{S}$. Similarly, $P_{(3)}$ is the partial derivative with respect to x^3 in the coordinates $(\mathfrak{n}, x^2, x^3)$ on $[0, \mathfrak{n}_0] \times \mathbb{T}^2$. Hence, $\{K, P_{(2)}, P_{(3)}\}$ are the coordinate partial derivative vectorfields on $[0, \mathfrak{n}_0] \times \mathbb{T}^2$, and using Lemma 5.4, (32.27), (32.29), and (33.10), we calculate that the Jacobian matrix on LHS (33.5) can be expressed as:

$$\begin{aligned}
\frac{\partial [\Upsilon \circ \mathfrak{S}](\mathfrak{n}, x^2, x^3)}{\partial(\mathfrak{n}, x^2, x^3)} &= \frac{\partial(t, x^1, x^2, x^3)}{\partial(\mathfrak{n}, x^2, x^3)} \\
&= \begin{pmatrix}
\dfrac{\frac{\frac{\partial}{\partial u}\mu}{\frac{\partial}{\partial t}\mu}}{\frac{\partial}{\partial u}\check{X}\mu - \frac{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\check{X}\mu}{\frac{\partial}{\partial t}\mu}} & P_{(2)}t & P_{(3)}t \\
\dfrac{-1}{\frac{\partial}{\partial u}\check{X}\mu - \frac{(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\check{X}\mu}{\frac{\partial}{\partial t}\mu}} \left\{\frac{\mu c^2}{X^1} - \left(\frac{\frac{\partial}{\partial u}\mu}{\frac{\partial}{\partial t}\mu}\right)\left(\frac{L^1 X^1 + L^2 X^2 + L^3 X^3}{X^1}\right)\right\} & P_{(2)}x^1 & P_{(3)}x^1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\end{aligned} \tag{33.12}$$

(33.5) and the estimate $(\frac{\partial}{\partial u}\mu)\frac{\partial}{\partial t}\breve{X}\mu - (\frac{\partial}{\partial t}\mu)\frac{\partial}{\partial u}\breve{X}\mu \approx 1$ now follow from (32.26)–(32.27), (33.10), (33.12), (3.10), (3.13), Lemma 5.5, Prop. 9.1, the L^∞ estimates of Prop. 17.1, (18.5), (18.8a), the facts that $\mu|_{\breve{\Upsilon}_{0,-\mathfrak{n}}} \equiv 0$ and $\breve{X}\mu|_{\breve{\Upsilon}_{0,-\mathfrak{n}}} \equiv -\mathfrak{n}$, and the fact that by (5.7b), $\breve{X} = \frac{\partial}{\partial u}$ along the singular boundary (where $\mu \equiv 0$).

We have therefore proved the proposition.

□

34. The main results

In this section, we state and prove the main theorem of the paper. The theorem provides an assimilated version of results we have already proved.

Theorem 34.1 (The development and structure of the singular boundary)**.** *Fix any of the compactly supported admissible simple isentropic plane symmetric "background" solutions $\mathcal{R}_{(+)}^{\mathrm{PS}}$ from Def. A.7 (recall that $\mathcal{R}_{(-)}$, v^2, v^3, s, Ω, S, $\mathcal{C}$, and $\mathcal{D}$ vanish for these background solutions). Let $(\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)\big|_{\Sigma_0} \stackrel{def}{=} \left(\mathring{\mathcal{R}}_{(+)}, \mathring{\mathcal{R}}_{(-)}, \mathring{v}^2, \mathring{v}^3, \mathring{s}\right)$ be perturbed fluid data on the flat Cartesian hypersurface Σ_0, as in* (11.1)*, and let $u|_{\Sigma_0} = -x^1$ be the initial condition of the eikonal function, as in* (3.1) *and* (A.7b)*. Assume the hypotheses and conclusions of Theorem 31.1. In particular, assume the assumptions* (**A1**)–(**A5**) *stated in the theorem, which include the assumption that $N_{\mathrm{top}} \geq 24$, as well as the assumption that the quantity $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$ defined in* (11.4) *is sufficiently small, where $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$ is a Sobolev norm of the perturbation of the fluid data away from the background solution. Then the corresponding solution exhibits the following properties, where U_0, U_1, U_2, τ_0, $\mathfrak{n}_0$, $\mathfrak{m}_0$, $\mathring{\alpha}$, $\mathring{\epsilon}$, etc. are the parameters from Theorem 31.1.*

Classical existence with respect to the geometric coordinates on $\mathcal{M}_{\mathrm{Interesting}}$.

- *There exists a compact region $\mathcal{M}_{\mathrm{Interesting}}$ in geometric coordinate space $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2$, which is defined in Def. 32.1, depicted in Figs. 6 and 13, and which has the properties revealed by Props. 32.5 and 32.11. $\mathcal{M}_{\mathrm{Interesting}}$ is contained in $\bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, where the ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ are the developments from Theorem 31.1. Moreover, the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]}$, described below, is contained in the top boundary of $\mathcal{M}_{\mathrm{Interesting}}$.*
- *The fluid solution wave-variables $\vec{\Psi} = (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$ (see* (2.11a)*), the eikonal function u, μ, L^i, and all of the auxiliary quantities (such as the null second fundamental form χ) constructed out of these quantities exist classically with respect to the geometric coordinates (t, u, x^2, x^3) on all of $\mathcal{M}_{\mathrm{Interesting}}$,* ***including the singular boundary portion $\mathcal{B}^{[0,\mathfrak{n}_0]}$ described below****. In particular, with respect to the geometric coordinates, the fluid variables are solutions to equations* (2.6a)–(2.6c) *<u>and</u> the equations of Theorem 2.15 on $\mathcal{M}_{\mathrm{Interesting}}$.*
- *The following quantities extend as solutions to the compact set $\mathcal{M}_{\mathrm{Interesting}}$ as elements of the following spacetime Hölder spaces[63] with respect to the geometric coordinates, and their corresponding spacetime Hölder norms on $\mathcal{M}_{\mathrm{Interesting}}$ are bounded by $\leq C$:*
  - $\vec{\Psi}, \Omega^i, S^i, \mathcal{C}^i, \mathcal{D} \in C^{3,1}_{\mathrm{geo}}(\mathcal{M}_{\mathrm{Interesting}})$
  - $\Upsilon \in C^{3,1}_{\mathrm{geo}}(\mathcal{M}_{\mathrm{Interesting}})$
  - $L^i, \mu \in C^{2,1}_{\mathrm{geo}}(\mathcal{M}_{\mathrm{Interesting}})$
- *$\mathcal{M}_{\mathrm{Interesting}}$ is foliated by the level-sets of a time function ${}^{(\mathrm{Interesting})}\tau$, which satisfies $\|{}^{(\mathrm{Interesting})}\tau\|_{C^{1,1}_{\mathrm{geo}}(\mathcal{M}_{\mathrm{Interesting}})} \leq C$ and has the range $[\tau_0, 0] \stackrel{def}{=} [-\mathfrak{m}_0, 0]$ on $\mathcal{M}_{\mathrm{Interesting}}$. That is, $\mathcal{M}_{\mathrm{Interesting}} = \bigcup_{\tau\in[\tau_0,0]} {}^{(\mathrm{Interesting})}\Sigma_\tau^{[-U_1,U_2]}$, where for $\tau \in [\tau_0, 0]$, ${}^{(\mathrm{Interesting})}\Sigma_\tau^{[-U_1,U_2]} \stackrel{def}{=} \left\{(t, u, x^2, x^3) \in \mathbb{R} \times [-U_1, U_2] \times \mathbb{T}^2 \mid {}^{(\mathrm{Interesting})}\tau(t, u, x^2, x^3) = \tau\right\}$.*
- *The L^∞ estimates of Prop. 17.1 hold on $\mathcal{M}_{\mathrm{Interesting}}$ with ε replaced by $C\mathring{\epsilon}$. Moreover, on each development ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ with $\mathfrak{n} \in [0, \mathfrak{n}_0]$, the solution enjoys the energy estimates guaranteed by Theorem 31.1.*
- *For $\tau \in [\tau_0, 0]$, we have:*

$$\min_{{}^{(\mathrm{Interesting})}\Sigma_\tau^{[-U_1,U_2]}} \mu = -\tau. \tag{34.1}$$

[63]Actually, thanks to the estimates offered by Prop. 17.1, the solution enjoys additional regularity in directions tangent to the characteristics compared to what we have stated here; we have stated simpler, sub-optimal regularity conclusions only to avoid cluttering the presentation.

Moreover, within ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$*, the minimum value of* $-\tau$ *in* (34.1) *is achieved by* μ *precisely on the set* $\breve{\mathbb{M}}_{-\tau}^{[0,\mathfrak{n}_0]} \stackrel{def}{=} \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$ *from definition* (32.2)*, which is a three-dimensional* $C^{2,1}$ *embedded sub-manifold contained in* ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$ *with* $C^{1,1}$ *boundary components* $\breve{\mathbf{T}}_{-\tau,0}$ *and* $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}_0}$*. In particular, within* $\mathcal{M}_{\text{Interesting}}$*,* μ *vanishes precisely along* $\breve{\mathbb{M}}_{0}^{[0,\mathfrak{n}_0]}$*, which by Def. 32.4 is equal to the singular boundary portion* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *and which is contained in* ${}^{(\text{Interesting})}\Sigma_{0}^{[-U_1,U_2]}$*.*

- *The change of variables map* ${}^{(\text{Interesting})}\mathscr{T}(t,u,x^2,x^3) = ({}^{(\text{Interesting})}\tau,u,x^2,x^3)$ *defined in* (32.36) *is a diffeomorphism from* $\mathcal{M}_{\text{Interesting}}$ *onto its image* $[\tau_0,0]\times[-U_1,U_2]\times\mathbb{T}^2$ *satisfying* $\|{}^{(\text{Interesting})}\mathscr{T}\|_{C^{1,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C$*.*

<u>The geometric coordinate description of the singular boundary</u>.

- *The* ***singular boundary*** *portion* $\mathcal{B}^{[0,\mathfrak{n}_0]} = \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{0,-\mathfrak{n}}$ *from Def. 32.4 is contained in* $\partial\mathcal{M}_{\text{Interesting}}$*, and* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *is a* 3*-dimensional* $C^{2,1}$*-embedded sub-manifold-with-boundary of geometric coordinate space. Its two boundary components are its future boundary* $\breve{\mathbf{T}}_{0,-\mathfrak{n}_0}$ *and its past boundary* $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]} = \breve{\mathbf{T}}_{0,0}$*, which refer to as the* ***crease*** *(see definition* (32.4)*).*
- *The boundary components* $\breve{\mathbf{T}}_{0,-\mathfrak{n}_0}$ *and* $\partial_-\mathcal{B}^{[0,\mathfrak{n}_0]}$ *are* $C^{1,1}$ *embedded* 2*-dimensional tori in geometric coordinate space that are spacelike with respect to the acoustical metric* $\mathbf{g}$ *(see definition* (2.15a)*).*

<u>The Cartesian coordinate description of the singularity formation in $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\right)$</u>.

- *On* $\mathcal{M}_{\text{Interesting}}$*, the change of variables map* $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ *is an injection onto its image* $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\right)$ *in Cartesian coordinate space* $\mathbb{R}_t\times\mathbb{R}_{x^1}\times\mathbb{T}^2$ *verifying* $\|\Upsilon\|_{C^{3,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C$*. In particular,* Υ *is a homeomorphism from the compact set* $\mathcal{M}_{\text{Interesting}}$ *onto its image. Moreover, on* $\mathcal{M}_{\text{Interesting}}\backslash\mathcal{B}^{[0,\mathfrak{n}_0]}$ *(where* $\mathcal{B}^{[0,\mathfrak{n}_0]}$ *is the singular boundary portion from Def. 32.4),* Υ *is a diffeomorphism.*
- *On* $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\backslash\mathcal{B}^{[0,\mathfrak{n}_0]}\right)$*, the solution exists classically with respect to the Cartesian coordinates.*
- *The following lower bound holds in* $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\cap\{(t,u,x^2,x^3)\mid |u|\leq U_{☢}\}\right)$*:*

$$|X\mathcal{R}_{(+)}| \geq \frac{\mathring{\delta}_*}{\mu|\bar{c}_{;\rho}+1|}, \tag{34.2}$$

where $\mathring{\delta}_* > 0$ *is the data-parameter from* (11.6)*,* $\bar{c}_{;\rho} \stackrel{def}{=} c_{;\rho}(\rho=0,s=0)$ *is* $c_{;\rho}$ *evaluated at the trivial solution,* $\bar{c}_{;\rho}+1$ *is a* <u>*non-zero*</u> *constant by assumption, and the* Σ_t*-tangent vectorfield* X *has Euclidean length satisfying* $\sqrt{\sum_{a=1}^{3}(X^a)^2} = 1+\mathcal{O}(\mathring{\alpha})$*, where* $\mathring{\alpha}$ *is the small parameter from Sect. 10.2. In particular, if* $q\in\Upsilon\left(\mathcal{B}^{[0,\mathfrak{n}_0]}\right)$*, then since* $\mathcal{B}^{[0,\mathfrak{n}_0]}\subset\mathcal{M}_{\text{Interesting}}\cap\left\{(t,u,x^2,x^3)\mid |u|\leq\frac{1}{2}U_{☢}\right\}$ *by* (32.3) *and* (18.3b)*, and since* $\mu=0$ *along* $\Upsilon\left(\mathcal{B}^{[0,\mathfrak{n}_0]}\right)$*, it follows that* $|X\mathcal{R}_{(+)}|(q')\to\infty$ *as* $q'\to q$ *in* $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\backslash\mathcal{B}^{[0,\mathfrak{n}_0]}\right)$*. Similarly, the following lower bounds hold in* $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\cap\{(t,u,x^2,x^3)\mid |u|\leq U_{☢}\}\right)$*, where* ρ *is the logarithmic density (see* (2.3)*):*

$$|X\rho| \geq \frac{\mathring{\delta}_*}{4\mu|\bar{c}_{;\rho}+1|}, \qquad |Xv^1| \geq \frac{\mathring{\delta}_*}{4\mu|\bar{c}_{;\rho}+1|}. \tag{34.3}$$

- *(Regular behavior*[64] *along the characteristics). The derivatives of* $\vec{\Psi}$*,* Ω^i*, and* S^i *up to order* $N_{\text{top}}-11$ *with respect to the vectorfields in the* $\mathcal{P}_u$*-tangent commutation set* $\mathscr{P}$ *defined in* (3.16)*, and the derivatives of* $\mathcal{C}^i$ *and* $\mathcal{D}$ *up to order* $N_{\text{top}}-12$ *with respect to the elements of* $\mathscr{P}$ *are* L^∞*-bounded on* $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\right)$*. Finally, for* $\alpha=0,1,2,3$ *and* $A=2,3$*, the derivatives of* $\mathbf{g}_{ab}Y^a_{(A)}\partial_\alpha v^b$ *up to order* $N_{\text{top}}-11$ *with respect to the elements of* $\mathscr{P}$ *are* L^∞*-bounded on* $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\right)$*.*

<u>The Cartesian coordinate description of the singular boundary and the crease</u>.

[64] Here we have only highlighted some of the quantities that remain L^∞-bounded on $\Upsilon\left(\mathcal{M}_{\text{Interesting}}\right)$. We refer to Prop. 17.1 for more comprehensive results.

- $\Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ *is a* $C^{1,\frac{1}{2}}$ *embedded* $\mathbf{g}$*-null hypersurface in Cartesian coordinate space that is foliated by the integral curves of L; see Prop. 33.2 for a more detailed description.*
- *For* $\mathfrak{n} \in [0, \mathfrak{n}_0]$, $\Upsilon(\breve{\mathbf{T}}_{0,-\mathfrak{n}}) \subset \Upsilon(\mathcal{B}^{[0,\mathfrak{n}_0]})$ *is an embedded* $C^{1,1}$ *graph over* $\mathbb{T}^2$ *in Cartesian coordinate space such that at each* $q \in \breve{\mathbf{T}}_{0,-\mathfrak{n}}$, $\Upsilon(\breve{\mathbf{T}}_{0,-\mathfrak{n}})$ *is spacelike with respect to* $\mathbf{g}|_{\Upsilon(q)}$ *at the point* $\Upsilon(q)$.
- *In particular, considering the case* $\mathfrak{n} = 0$ *in the previous point, we see that the image of the crease under* Υ*, namely* $\Upsilon(\breve{\mathbf{T}}_{0,0})$*, is an embedded* $C^{1,1}$ *graph over* $\mathbb{T}^2$ *in Cartesian coordinate space that is* $\mathbf{g}$*-spacelike at each of its points.*

Proof.

Proof that $\mathcal{M}_{\text{Interesting}} \subset \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$: This result follows from Def. 32.1 and the fact that $\breve{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,0]} \subset {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, which in turn follows from the decomposition ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]} = \bigcup_{\tau\in[\tau_0,0]} {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$, the fact that $\breve{\mathbf{T}}_{-\tau,-\mathfrak{n}} \subset {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_1,U_2]}$ (see Def. 15.32), and (15.44) with $\tau_{\text{Boot}} = \mathfrak{m}_{\text{Boot}} = 0$. In the remainder of the proof, we will silently use the fact that $\mathcal{M}_{\text{Interesting}} \subset \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$ and the following consequence of Theorem 31.1: at fixed $\mathfrak{n} \in [0, \mathfrak{n}_0]$, all results proved in the paper prior to Theorem 31.1 hold with $\tau_{\text{Boot}} = 0$.

Proof of classical existence with respect to the geometric coordinates on $\mathcal{M}_{\text{Interesting}}$: Theorem 31.1 yields that for fixed $\mathfrak{n} \in [0, \mathfrak{n}_0]$, the solution exists classically with the respect to the geometric coordinates on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$. Since $\mathcal{M}_{\text{Interesting}} \subset \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, we immediately conclude classical existence with respect to the geometric coordinates on $\mathcal{M}_{\text{Interesting}}$.

Proof of the Hölder bounds: To derive the Hölder bounds $\|\vec{\Psi}\|_{C^{3,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C$, $\|\Omega^i\|_{C^{3,1}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C$, etc., we first use Lemma 15.6 with $\tau_{\text{Boot}} = 0$ (i.e., we use the lemma on ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$), the fact that $\mathcal{M}_{\text{Interesting}} \subset \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$, and Rademacher's theorem to deduce that $\|\vec{\Psi}\|_{W^{4,\infty}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C$, $\|\Omega^i\|_{W^{4,\infty}_{\text{geo}}(\mathcal{M}_{\text{Interesting}})} \leq C$, etc. From these bounds and the Sobolev embedding result (32.51), we conclude the desired Hölder bounds.

Proof of the properties of the time function ${}^{(\text{Interesting})}\tau$ **and the map** ${}^{(\text{Interesting})}\mathscr{T}(t, u, x^2, x^3)$: We derived these results in Prop. 32.11.

Proof of the properties of Υ **and classical existence with respect to the Cartesian coordinates on the domain** $\Upsilon\left(\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}\right)$: We derived the properties of Υ in Prop. 33.1. Since the proposition in particular shows that Υ is a global diffeomorphism $\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}$, and since we have already shown classical existence with respect to the geometric coordinates on $\mathcal{M}_{\text{Interesting}}$, we conclude that the solution exists classically with respect to the Cartesian coordinates on the domain $\Upsilon\left(\mathcal{M}_{\text{Interesting}} \backslash \mathcal{B}^{[0,\mathfrak{n}_0]}\right)$ in Cartesian coordinate space.

Proof of the regular behavior along the characteristics: These results follow from the last item stated in Theorem 31.1.

Proof of the properties of $\mathcal{B}^{[0,\mathfrak{n}_0]}$**,** $\breve{\mathbf{T}}_{0,-\mathfrak{n}_0}$**, and** $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]} = \breve{\mathbf{T}}_{0,0}$: We derived these results in in Prop. 32.5.

Proof of (34.1) **and related properties of** μ: In Prop. 32.11, we proved (34.1) and showed that within ${}^{(\text{Interesting})}\Sigma_{\tau}^{[-U_1,U_2]}$, the minimum value of $-\tau$ in (34.1) is achieved by μ precisely on the set $\breve{\mathbb{M}}_{-\tau}^{[0,\mathfrak{n}_0]} \overset{\text{def}}{=} \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{-\tau,-\mathfrak{n}}$.

Proof of the lower bounds (34.2) **and** (34.3): These estimates follow from the estimates (31.3) and (31.4), which hold in $\bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_{☢},U_{☢}]}\right)$, and the fact that $\Upsilon\left(\mathcal{M}_{\text{Interesting}} \cap \{(t, u, x^2, x^3) \mid |u| \leq U_{☢}\}\right) \subset \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \Upsilon\left({}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_{☢},U_{☢}]}\right)$ (since $\mathcal{M}_{\text{Interesting}} \subset \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} {}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,0],[-U_1,U_2]}$).

Proof of the Cartesian coordinate description of the singular boundary and the crease: We derived these results in Prop. 33.2.

□

Acknowledgements

We are grateful to John Anderson for providing helpful feedback on an earlier version of this paper.

Appendix A. Simple isentropic plane-symmetric solutions

In this appendix, we show that there exists a large family of shock-forming simple isentropic plane-symmetric solutions whose induced data on late-time rough hypersurfaces satisfy the assumptions stated in Sect. 11. As we will see, simple isentropic plane-wave solutions are characterized by $\mathring{\epsilon} = 0$, where $\mathring{\epsilon}$ is the data-size parameter featured in the assumptions of Sect. 11. In Appendix B, we combine the results of this appendix with Cauchy stability arguments to show that there exists an open set of data satisfying the assumptions of Sect. 11. By "open set," we mean open relative to the topologies corresponding to the norm $\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}$ on Σ_0 defined in (11.4), where $N_{\text{top}} \geq 24$.

A.1. **Plane symmetric and simple plane-symmetric solutions.** We begin with a quick presentation of the isentropic compressible Euler equations in plane-symmetry. By *isentropic plane-symmetric solutions*, we mean those solutions with the following properties: ρ and v^1 are functions of only t and x^1, "the symmetry breaking fluid variables" satisfy $v^2, v^3 \equiv 0$, and s is constant. For convenience, we will assume that $s \equiv 0$, though that is not essential for our main results, i.e., we could have easily handled solutions with $s \equiv s_0$, where s_0 is a constant. It is straightforward to see that for such solutions, the fluid vorticity $\operatorname{curl} v$ also vanishes. In Appendix B, we will view our isentropic plane-symmetric solutions as "background" solutions in three spatial dimensions with trivial dependence on the (x^2, x^3) coordinates. However, in this appendix, to shorten the presentation, we will completely suppress the variables (x^2, x^3) and instead view the plane-symmetric solutions as solutions in $1+1$ dimensions.

Throughout this appendix, we adorn symbols related to the background plane-symmetric solutions by a "PS." For example, we denote the logarithmic density by ρ^{PS}, and we set $v^{\text{PS}} \stackrel{\text{def}}{=} v^1$. Although the eikonal function also depends on the background solution through the dependence of the coefficients of the eikonal equation (3.1) on the fluid, we will continue to denote it by "u" instead of "u^{PS}." This is consistent with the point of view we take in Appendix B during our discussion of Cauchy stability, where for convenience, we will "fix" geometric coordinate space $\mathbb{R}_t \times \mathbb{R}_u \times \mathbb{T}^2_{x^2,x^3}$ and consider families of solutions that exist with respect to the geometric coordinates on a common domain. Note that, although for general solutions, the maps $(t, u, x^2, x^3) \to x^i(t, u)$ from geometric coordinates to a Cartesian spatial coordinate depend on the fluid solution, to avoid clutter, in this appendix, we will not adorn x^1 with a "PS" subscript. However, we do adorn the corresponding change of variables map: $\Upsilon_{\text{PS}}(t, u) \stackrel{\text{def}}{=} (t, x^1)$.

A.1.1. *The Riemann invariants in isentropic plane-symmetry.* In plane-symmetry with $s \equiv 0$, we define the Riemann invariants to be the following functions of v^{PS} and ρ^{PS}:

$$\mathcal{R}^{\text{PS}}_{(+)} \stackrel{\text{def}}{=} v^{\text{PS}} + F^{\text{PS}}(\rho^{\text{PS}}), \qquad \mathcal{R}^{\text{PS}}_{(-)} \stackrel{\text{def}}{=} v^{\text{PS}} - F^{\text{PS}}(\rho^{\text{PS}}), \tag{A.1}$$

where $F^{\text{PS}} = F^{\text{PS}}(z)$ is defined to be the solution to the following ODE initial value problem:

$$F^{\text{PS}}(0) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}z} F^{\text{PS}}(z) = c^{\text{PS}}(z), \tag{A.2}$$

where we recall that by our assumption (2.5), we have $c^{\text{PS}}(0) = 1$. On RHS (A.2) and throughout, $c^{\text{PS}}(z) \stackrel{\text{def}}{=} c(z, 0)$, where $c(z, 0)$ is the speed of sound (see (1.3)), viewed as a function of the logarithmic density z and the entropy s, evaluated at $s = 0$. We note that v^{PS} and ρ^{PS} can respectively be expressed in terms of the Riemann invariants as follows:

$$v^{\text{PS}} = \frac{1}{2}\left(\mathcal{R}^{\text{PS}}_{(+)} + \mathcal{R}^{\text{PS}}_{(-)}\right), \qquad \rho^{\text{PS}} = (F^{\text{PS}})^{-1} \circ \left\{\frac{1}{2}\left(\mathcal{R}^{\text{PS}}_{(+)} - \mathcal{R}^{\text{PS}}_{(-)}\right)\right\}, \tag{A.3}$$

where $(F^{\text{PS}})^{-1}$ is the inverse function of F^{PS}. Note that by (A.2) and (2.5), $(F^{\text{PS}})^{-1}$ is well-defined and smooth in a neighborhood of 0. Note that the Riemann invariants (A.1) agree with the almost Riemann invariants defined in (2.7). Much like in the bulk of the paper, when we are deriving estimates for the fluid, it is understood that all fluid variables are to be viewed as functions of the Riemann invariants via (A.3).

A.1.2. *The compressible Euler equations in terms of the Riemann invariants in isentropic plane-symmetry.* Due to the isentropic plane-symmetry, the Riemann invariants are in fact invariant along the characteristics. More precisely, it is straightforward to verify that for smooth isentropic plane-symmetric solutions, the compressible Euler equations (2.6a)–(2.6c) are equivalent to the following 2×2 system of quasilinear transport equations:

$$L^{\text{PS}} \mathcal{R}^{\text{PS}}_{(+)} = 0, \qquad \underline{L}^{\text{PS}} \mathcal{R}^{\text{PS}}_{(-)} = 0, \tag{A.4}$$

where:

$$L^{\mathrm{PS}} \stackrel{\text{def}}{=} \partial_t + (v^{\mathrm{PS}} + c^{\mathrm{PS}})\partial_1, \qquad \underline{L}^{\mathrm{PS}} \stackrel{\text{def}}{=} \partial_t + (v^{\mathrm{PS}} - c^{\mathrm{PS}})\partial_1. \tag{A.5}$$

We denote the initial data for (A.4) by:

$$\mathcal{R}_{(+)}^{\mathrm{PS}}|_{\Sigma_0} = \mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}, \qquad \mathcal{R}_{(-)}^{\mathrm{PS}}|_{\Sigma_0} = \mathring{\mathcal{R}}_{(-)}^{\mathrm{PS}}. \tag{A.6}$$

In the rest of the appendix, we will consider only *simple* isentropic plane-symmetric solutions, defined to be solutions with $\mathcal{R}_{(-)}^{\mathrm{PS}} \equiv 0$. From (A.4), it follows that such solutions arise from initial data with $\mathring{\mathcal{R}}_{(-)}^{\mathrm{PS}} \equiv 0$.

A consequence of standard methods going back Riemann's famous work [64] is that any non-trivial compactly supported initial datum $\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}$ launches a shock-forming simple isentropic plane-symmetric solution to (A.4). In the forthcoming subsections, we will construct compactly supported data along Σ_0 that launch "admissible" (see Def. A.7) solutions whose perturbations we study in Appendix B. To this end, we find it useful to revisit our construction of the acoustical geometry so that we can capitalize on the many simplifications that occur in simple isentropic plane-symmetry.

A.2. **The acoustic geometry and explicit solution formulas in simple isentropic plane-symmetry.**

A.2.1. *Definitions and identities.* We begin by defining the eikonal function in plane-symmetry to be the solution to the following transport equation initial value problem:

$$L^{\mathrm{PS}} u = 0, \tag{A.7a}$$

$$u|_{\Sigma_0} = -x^1. \tag{A.7b}$$

Note that the initial condition stated in (A.7b) is the same as the one (3.1) we assumed in the bulk of the paper. It is straightforward to check (cf. (7.5)) that in plane-symmetry, $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = -(L^{\mathrm{PS}}u)\underline{L}^{\mathrm{PS}}u$, and that u solves (A.7a)–(A.7b) if and only if it solves (3.1). In particular, in plane-symmetry, the (fully nonlinear) eikonal equation (3.1) is equivalent to (A.7a), which is linear in u because the operator L^{PS} can be defined by equation (A.5), which does not depend on u.

We now define the inverse foliation density as follows:

$$\upmu^{\mathrm{PS}} \stackrel{\text{def}}{=} -\frac{1}{c^{\mathrm{PS}}\partial_1 u}. \tag{A.8}$$

One can check that in isentropic plane-symmetry, the quantity "$\upmu^{\mathrm{PS}}$" defined by (A.8) is equal to the quantity "$\upmu$" defined in (3.2).

Straightforward calculations yield that in isentropic plane-symmetry, with L^{PS} as in (A.5), we have:

$$\underline{L}^{\mathrm{PS}} = L^{\mathrm{PS}} + 2X^{\mathrm{PS}}, \qquad X^{\mathrm{PS}} = -c^{\mathrm{PS}}\partial_1, \qquad \breve{X}^{\mathrm{PS}} \stackrel{\text{def}}{=} \upmu^{\mathrm{PS}} X^{\mathrm{PS}}, \tag{A.9}$$

where, given our construction of u in (A.7a)–(A.7b), L^{PS}, X^{PS}, $\breve{X}^{\mathrm{PS}}$ coincide with the vectorfields defined in Def. 3.8, while $\underline{L}^{\mathrm{PS}}$ coincides with the vectorfield defined in Def. 7.1.

Next, using (A.7b) and definition A.8, we compute that the following identities hold on Σ_0:

$$\upmu^{\mathrm{PS}}\big|_{\Sigma_0} = \frac{1}{c^{\mathrm{PS}}|_{\Sigma_0}} \qquad \Longrightarrow \qquad (c^{\mathrm{PS}}\upmu^{\mathrm{PS}})\big|_{\Sigma_0} = 1. \tag{A.10}$$

As in the bulk of the paper, we define (t,u) to be the *geometric coordinates*, and we denote the corresponding geometric coordinate vectorfields as $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right\}$. In plane-symmetry, $L^{\mathrm{PS}}x^2 = L^{\mathrm{PS}}x^3 = X^{\mathrm{PS}}x^2 = X^{\mathrm{PS}}x^2 = 0$ and thus, by Lemma 5.5, we have:

$$L^{\mathrm{PS}} = \frac{\partial}{\partial t}, \qquad \breve{X}^{\mathrm{PS}} = \frac{\partial}{\partial u}. \tag{A.11}$$

It follows that:

$$[L^{\mathrm{PS}}, \breve{X}^{\mathrm{PS}}] = 0. \tag{A.12}$$

We will often silently use (A.11)–(A.12) in the rest of this appendix.

A.2.2. *Explicit solution formula in geometric coordinates.* From (A.11), we see that the transport equation (A.4) for $\mathcal{R}_{(+)}^{\text{PS}}$ takes the following form in geometric coordinates:

$$\frac{\partial}{\partial t}\mathcal{R}_{(+)}^{\text{PS}}(t,u) = 0. \tag{A.13}$$

From (A.13) and (A.6), we see (recalling that $\mathring{\mathcal{R}}_{(-)}^{\text{PS}} \equiv 0$ by assumption) that in geometric coordinates, the solution to (A.13) is:

$$\mathcal{R}_{(+)}^{\text{PS}}(t,u) = \mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u). \tag{A.14}$$

A.2.3. *The evolution equation for* μ^{PS}. The following lemma provides the evolution equation for μ^{PS}.

Lemma A.1 (Transport equation satisfied by μ^{PS})**.** *For simple isentropic plane-symmetric solutions, the inverse foliation density satisfies the following transport equation, where* $\dot{c}^{\text{PS}} = \dot{c}^{\text{PS}}(\rho) \overset{def}{=} \frac{d}{d\rho}c^{\text{PS}}(\rho)$ *is the derivative of the speed of sound with respect to the logarithmic density:*

$$L^{\text{PS}}(c^{\text{PS}}\mu^{\text{PS}}) = H, \tag{A.15}$$

$$H \overset{def}{=} -\frac{1}{2}\left\{\frac{\dot{c}^{\text{PS}}}{c^{\text{PS}}} + 1\right\}\breve{X}^{\text{PS}}\mathcal{R}_{(+)}^{\text{PS}}. \tag{A.16}$$

Moreover,

$$L^{\text{PS}}L^{\text{PS}}\mu^{\text{PS}} = L^{\text{PS}}L^{\text{PS}}(c^{\text{PS}}\mu^{\text{PS}}) = 0. \tag{A.17}$$

Finally, we have the following identities:

$$H = \breve{X}^{\text{PS}}\mathfrak{H}, \tag{A.18a}$$

$$\mathfrak{H} = \mathfrak{H}[\mathcal{R}_{(+)}^{\text{PS}}] \overset{def}{=} -\dot{F}^{\text{PS}} \circ (F^{\text{PS}})^{-1} \circ \left(\frac{1}{2}\mathcal{R}_{(+)}^{\text{PS}}\right) - \frac{1}{2}\mathcal{R}_{(+)}^{\text{PS}} + \overline{c^{\text{PS}}}, \tag{A.18b}$$

$$\frac{d}{d\mathcal{R}_{(+)}^{\text{PS}}}\mathfrak{H}[\mathcal{R}_{(+)}^{\text{PS}}] = -\frac{1}{2}\frac{\ddot{F}^{\text{PS}}}{\dot{F}^{\text{PS}}} \circ (F^{\text{PS}})^{-1} \circ \left(\frac{1}{2}\mathcal{R}_{(+)}^{\text{PS}}\right) - \frac{1}{2}, \tag{A.18c}$$

where $\dot{F}^{\text{PS}} = \dot{F}^{\text{PS}}(\rho) \overset{def}{=} \frac{d}{d\rho}F^{\text{PS}}(\rho) = c(\rho)$, $\overline{c^{\text{PS}}} \overset{def}{=} \dot{F}^{\text{PS}}(\rho)|_{\rho=0}$, $(F^{\text{PS}})^{-1}$ *is the inverse of the map* $\rho \to F^{\text{PS}}(\rho)$, $\ddot{F}^{\text{PS}} = \ddot{F}^{\text{PS}}(\rho) = \dot{c}^{\text{PS}}(\rho) \overset{def}{=} \frac{d}{d\rho}c(\rho)$, $\mathfrak{H}'[\mathcal{R}_{(+)}^{\text{PS}}] \overset{def}{=} \frac{d}{d\mathcal{R}_{(+)}^{\text{PS}}}\mathfrak{H}[\mathcal{R}_{(+)}^{\text{PS}}]$, *and* $\circ$ *denotes the composition of functions.*

Remark A.2. Note that $\mathfrak{H}[0] = 0$. We will use this basic fact in Sect. A.5.

Proof of Lemma A.1. The transport equation (A.15) follows from (3.44), (3.46), and our assumption that the solution is isentropic, simple, and plane-symmetric, which in particular implies that $L^{\text{PS}}c^{\text{PS}} = 0$. Equation (A.17) follows from differentiating (A.15) with L^{PS} and using (A.12) along with $L^{\text{PS}}\mathcal{R}_{(+)}^{\text{PS}} = 0$. (A.18a)–(A.18c) follow from the chain rule, (A.2), and (A.3). □

A.2.4. *Recalling the non-degeneracy condition.* As in (2.4), we will assume the following non-degeneracy condition:

$$\frac{\overline{\dot{c}^{\text{PS}}}}{\overline{c^{\text{PS}}}} + 1 \neq 0, \tag{A.19}$$

where LHS (A.19) denotes the factor in braces on RHS (A.16) evaluated at the trivial solution $\rho^{\text{PS}} \equiv 0$ (which, in the present context, is equivalent to $\mathcal{R}_{(+)}^{\text{PS}} \equiv 0$). In view of our normalization assumption (2.5) (which implies that $\overline{c^{\text{PS}}} = 1$) and (A.18c), it follows that (A.19) is equivalent to:

$$-2\frac{d}{d\mathcal{R}_{(+)}^{\text{PS}}}\mathfrak{H}[\mathcal{R}_{(+)}^{\text{PS}}]|_{\mathcal{R}_{(+)}^{\text{PS}}=0} = \overline{\dot{c}^{\text{PS}}} + 1 \neq 0. \tag{A.20}$$

Note that (A.20) is equivalent to RHS (A.18c) being non-zero when it is evaluated at $\mathcal{R}_{(+)}^{\text{PS}} = 0$, and that it implies the invertibility of the map $\mathcal{R}_{(+)}^{\text{PS}} \to \mathfrak{H}[\mathcal{R}_{(+)}^{\text{PS}}]$ in a neighborhood of the origin (i.e., near $\mathcal{R}_{(+)}^{\text{PS}} = 0$). As we discussed in Sect. 3.13, for any equation of state except for that of a Chaplygin gas, there are always background densities $\overline{\varrho} > 0$ such that (A.20) holds.

Next, we note that in simple isentropic plane-symmetry, the quantity H defined in (A.16) can be viewed as a function of $\mathcal{R}_{(+)}^{\text{PS}}$ and $\breve{X}^{\text{PS}}\mathcal{R}_{(+)}^{\text{PS}}$. To emphasize this point of view, we use the notation $H[\mathcal{R}_{(+)}^{\text{PS}}, \breve{X}^{\text{PS}}\mathcal{R}_{(+)}^{\text{PS}}]$. Moreover, by (A.14), in geometric coordinates, $H[\mathcal{R}_{(+)}^{\text{PS}}, \breve{X}^{\text{PS}}\mathcal{R}_{(+)}^{\text{PS}}]$ is a function of only u. To simplify the presentation, we will emphasize this point of view with the shorthand notation $H(u)$, i.e.,

$$H[\mathcal{R}_{(+)}^{\text{PS}}, \breve{X}^{\text{PS}}\mathcal{R}_{(+)}^{\text{PS}}](t,u) \stackrel{\text{def}}{=} H[\mathcal{R}_{(+)}^{\text{PS}}(t,u), \breve{X}^{\text{PS}}\mathcal{R}_{(+)}^{\text{PS}}(t,u)] = H[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u), \frac{\partial}{\partial u}\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)] \stackrel{\text{def}}{=} H(u). \tag{A.21}$$

In a similar vein, $c^{\text{PS}} = \dot{F}^{\text{PS}}$ can be viewed as a function of $\mathcal{R}_{(+)}^{\text{PS}}$, where $\mathcal{R}_{(+)}^{\text{PS}}$ is a function of u alone, and we will use the following shorthand notation:

$$c^{\text{PS}}[\mathcal{R}_{(+)}^{\text{PS}}](t,u) \stackrel{\text{def}}{=} \dot{F}^{\text{PS}}[\mathcal{R}_{(+)}^{\text{PS}}(t,u)] = \dot{F}^{\text{PS}}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)] \stackrel{\text{def}}{=} c^{\text{PS}}(u). \tag{A.22}$$

A.2.5. *Explicit expressions for various solution variables.* The following corollary is a straightforward consequence of the prior discussion in this appendix, and we therefore omit the simple proof.

Corollary A.3 (Explicit expressions for μ^{PS}, $L^{\text{PS}}\mu^{\text{PS}}$, and $\partial_1\mathcal{R}_{(+)}^{\text{PS}}$). *For simple isentropic plane-symmetric solutions, the following identities hold relative to the geometric coordinates:*

$$L^{\text{PS}}\left\{c^{\text{PS}}(u)\mu^{\text{PS}}(t,u)\right\} = H(u) = \frac{d}{du}\mathfrak{H}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)] = -\frac{1}{2}\left\{\frac{\ddot{F}^{\text{PS}}}{\dot{F}^{\text{PS}}} \circ (F^{\text{PS}})^{-1} \circ \left(\frac{1}{2}\mathcal{R}_{(+)}^{\text{PS}}\right) + 1\right\}\frac{d}{du}\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u), \tag{A.23a}$$

$$c^{\text{PS}}(u)\mu^{\text{PS}}(t,u) = 1 + tH(u) = 1 + t\frac{d}{du}\mathfrak{H}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)] = 1 + t\mathfrak{H}'[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)]\frac{d}{du}\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u), \tag{A.23b}$$

$$\breve{X}^{\text{PS}}\mu^{\text{PS}}(t,u) = \frac{t\frac{d^2}{du^2}\mathfrak{H}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)]}{\dot{F}^{\text{PS}}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)]} - \left\{\frac{\frac{d}{du}\dot{F}^{\text{PS}}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)]}{\dot{F}^{\text{PS}}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)]}\right\}\left\{1 + t\frac{d}{du}\mathfrak{H}[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)]\right\}, \tag{A.24}$$

$$[\partial_1\mathcal{R}_{(+)}^{\text{PS}}](t,u) = -\frac{1}{c^{\text{PS}}(u)\mu^{\text{PS}}(t,u)}\frac{\partial}{\partial u}\mathcal{R}_{(+)}^{\text{PS}}(u) = -\frac{\frac{d}{du}\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)}{1 + tH(u)} = -\frac{\frac{d}{du}\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)}{1 + t\mathfrak{H}'[\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)]\frac{d}{du}\mathring{\mathcal{R}}_{(+)}^{\text{PS}}(u)}. \tag{A.25}$$

A.3. **The rough time function in simple isentropic plane-symmetry.** In simple isentropic plane-symmetry, using (A.11), we can write the transport equation (4.4a) for ${}^{(\mathfrak{n})}\tau_{\text{PS}}(t,u)$ as follows:

$${}^{(\mathfrak{n})}\breve{W}^{\text{PS}(\mathfrak{n})}\tau_{\text{PS}}(t,u) = \frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau_{\text{PS}}(t,u) + \phi(u)\frac{\mathfrak{n}}{\frac{\partial}{\partial t}\mu^{\text{PS}}(t,u)}\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{\text{PS}}(t,u) = 0, \tag{A.26}$$

where $\phi(u) = \psi\left(\frac{u}{U_{\clubsuit}}\right)$ is the cut-off from Definition 4.1, and the differential operator on LHS (A.26) is the adapted rough coordinate vectorfield $\frac{\widetilde{\partial}}{\partial u}$, i.e., we have the following identities (see (5.15)):

$$\frac{\widetilde{\partial}}{\widetilde{\partial} u} = {}^{(\mathfrak{n})}\breve{W}^{\text{PS}} = \frac{\partial}{\partial u} + \phi\frac{\mathfrak{n}}{\frac{\partial}{\partial t}\mu^{\text{PS}}}\frac{\partial}{\partial t}. \tag{A.27}$$

We also note that the initial condition (4.4b) is equivalent to:

$${}^{(\mathfrak{n})}\tau_{\text{PS}}|_{\{\breve{X}^{\text{PS}}\mu^{\text{PS}}=-\mathfrak{n}\}} = -\mu^{\text{PS}}|_{\{\breve{X}^{\text{PS}}\mu^{\text{PS}}=-\mathfrak{n}\}}. \tag{A.28}$$

A.4. **Admissible background simple isentropic plane-symmetric solutions.** In this section, we construct a large class of "admissible" simple isentropic plane-symmetric solutions to (A.4) that, on a rough hypersurface, induce data satisfying the assumptions stated in Sect. 11. The main results are Theorem A.4 and Cor. A.6. We formalize the notion of "admissible" in Def. A.7.

In Theorem A.4, we will state some Sobolev estimates for the solution that involve high order $\frac{\partial}{\partial u}$ derivatives. The availability of these estimates will simplify our discussion of Cauchy stability in Appendix B, although with additional effort, it would have been possible for us to prove the needed Cauchy stability result without such high order $\frac{\partial}{\partial u}$

estimates. We now define the corresponding Sobolev norms. Given a scalar function $f = f(t,u)$ of the geometric coordinates, $N \in \mathbb{N}$, and real numbers $u_1 \leq u_2$, we define:

$$\|f\|_{H_u^N(\Sigma_t^{[u_1,u_2]})} \stackrel{\text{def}}{=} \sqrt{\sum_{M=0}^{N} \int_{u_1}^{u_2} \left| \left(\frac{\partial}{\partial u}\right)^M f(t,u) \right|^2 \mathrm{d}u}. \tag{A.29}$$

If $M \in \mathbb{N}$ and $f = f(t,u)$ is a scalar function with bounded derivatives up to order M on a subset $\mathcal{S} \subset \mathbb{R}_t \times \mathbb{R}_u$, then we define:

$$\|f\|_{C_{\text{geo}}^M(\mathcal{S})} \stackrel{\text{def}}{=} \sum_{M_1+M_2 \leq M} \sup_{p \in \mathcal{S}} \left| \left(\frac{\partial}{\partial t}\right)^{M_1} \left(\frac{\partial}{\partial u}\right)^{M_2} f(p) \right|. \tag{A.30}$$

Moreover, if $M \in \mathbb{N}$ and $f = f(u)$ is a scalar function with bounded derivatives up to order M on a subset $\mathcal{S} \subset \mathbb{R}$, then we define:

$$\|f\|_{C_u^M(\mathcal{S})} \stackrel{\text{def}}{=} \sum_{M' \leq M} \sup_{p \in \mathcal{S}} \left| \left(\frac{\partial}{\partial u}\right)^{M'} f(p) \right|. \tag{A.31}$$

As at end of Sect. 8.1.2, we extend the above norms to array- or matrix-valued functions.

A.5. **Construction of the initial data that lead to admissible shock-forming solutions.** In this section, we will exhibit a large family of profiles that, upon rescaling, yield initial data that launch admissible solutions.

A.5.1. *Assumptions on the "seed" profile.* To start, we fix any scalar "seed profile" $\mathring{\varphi}$ with the following properties (it is straightforward to show that such functions exist):

- $\mathring{\varphi} = \mathring{\varphi}(u)$ is compactly supported in an interval $[-U_1, U_2]$ of u-values, where $U_1, U_2 > 1$.
- $\mathring{\varphi} \in H_u^{N_{\text{top}}+1}(\Sigma_0^{[-U_1,U_2]})$ for some integer $N_{\text{top}} \geq 24$.
- $\frac{d}{du}\mathring{\varphi}(u)$ has a unique, non-degenerate minimum at $u = 0$. In particular, $\frac{d^3}{du^3}\mathring{\varphi}(0) > 0$.
- Modifying $\mathring{\varphi}$ by multiplying it by a constant and composing it with a linear map of the form $u \to zu$ for some constant $z \in \mathbb{R}$ if necessary, we assume that (the modified $\mathring{\varphi}$) satisfies:

$$\frac{d}{du}\mathring{\varphi}(u)|_{u=0} = -1, \qquad \frac{d^2}{du^2}\mathring{\varphi}(u)|_{u=0} = 0, \tag{A.32}$$

that there is a constant $\mathfrak{b}$ satisfying:

$$\mathfrak{b} > 0 \tag{A.33}$$

such that $\frac{d^3}{du^3}\mathring{\varphi}(0) = \mathfrak{b}$ and such that $\frac{1}{2}\mathfrak{b} \leq \frac{d^3}{du^3}\mathring{\varphi}(u) \leq 2\mathfrak{b}$ when $|u| \leq 1$, and that there is a constant $\mathfrak{p}$ satisfying:

$$0 \leq \mathfrak{p} < 1, \tag{A.34}$$

such that

$$\frac{d}{du}\mathring{\varphi}(u) > -\mathfrak{p}, \qquad \text{if } |u| \geq 1. \tag{A.35}$$

See Fig. 15 for the graph (with u increasing from right to left) of a representative seed profile.

A.5.2. *Construction of one-parameter families of initial data for $\mathcal{R}_{(+)}^{\text{PS}}$.* Let $\mathring{\varphi}$ be as in Sect. A.5.1. Given a real parameter $\mathfrak{a} > 0$, we define:

$$\mathring{\varphi}_{\mathfrak{a}}(u) \stackrel{\text{def}}{=} \mathfrak{a}\mathring{\varphi}(u), \tag{A.36}$$

$$(\mathring{\mathcal{R}}_{(+)}^{\text{PS}})_{\mathfrak{a}}(u) \stackrel{\text{def}}{=} \mathfrak{H}^{-1}[\mathring{\varphi}_{\mathfrak{a}}(u)], \tag{A.37}$$

where $\mathfrak{H}^{-1}$ is the inverse function of of the map $\mathcal{R}_{(+)}^{\text{PS}} \to \mathfrak{H}[\mathcal{R}_{(+)}^{\text{PS}}]$ define by (A.18b).

Taking into account Remark A.2 and (A.20), we deduce from Taylor expansions and the standard Sobolev calculus that if $\mathfrak{a}$ is sufficiently small, then the following conclusions hold:

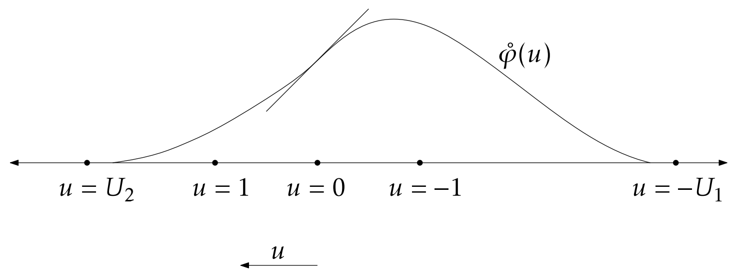


Figure 15. The graph of a representative "seed profile."

- $(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}$ is compactly supported in $[-U_1, U_2]$ and satisfies the following bounds:

$$\left\| \frac{d^M}{du^M} (\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}} \right\|_{L^\infty(\Sigma_0^{[-U_1,U_2]})} \lesssim \mathfrak{a}, \qquad M = 0, 1, 2, 3, 4, \tag{A.38a}$$

$$\left\| (\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}} \right\|_{H_u^{N_{\mathrm{top}}+1}(\Sigma_0^{[-U_1,U_2]})} \lesssim \mathfrak{a}. \tag{A.38b}$$

- $\frac{d}{du}\mathfrak{H}[(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u)]$ has a unique, negative, non-degenerate minimum at $u = 0$. In particular, $\frac{d^3}{du^3}\mathfrak{H}[(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u)]|_{u=0} > 0$.
- There is a differentiable function $\mathcal{F} : [-U_1, U_2] \to \mathbb{R}$ such that $\|\mathcal{F}\|_{C^1_{\mathrm{geo}}(\Sigma_0^{[-U_1,U_2]})} \lesssim 1$ and such that:

$$\frac{d}{du}\mathfrak{H}[(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u)] = -\mathfrak{a} + \frac{1}{2}\mathfrak{a}\mathfrak{b}u^2 + \mathcal{F}(u)\mathfrak{a}u^3. \tag{A.39}$$

- For $j = 1, 2, 3$, there exists a continuous function $F_j : [-U_1, U_2] \to (0, \infty)$ such that:

$$F_j(u) \approx 1, \qquad u \in [-U_1, U_2], \tag{A.40}$$

  and such that for $M = 0, 1, 2$, we have:

$$\frac{d^M}{du^M}\left(\frac{d}{du}\mathfrak{H}[(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u)] + \mathfrak{a}\right) = F_{M+1}(u)\mathfrak{a}u^{2-M}, \qquad \text{if } |u| \le 1, \tag{A.41a}$$

$$\frac{d}{du}\mathfrak{H}[(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u)] \ge -\mathfrak{p}\mathfrak{a}, \qquad \text{if } |u| \ge 1, \tag{A.41b}$$

where $\mathfrak{p} \in [0, 1)$ is the non-negative constant in (A.35). In the above relations, all implicit constants are independent of $\mathfrak{a}$ and u, though they depend on $\mathring{\varphi}$ and its derivatives.

A.6. **Admissible solutions.** We now prove the main results of Appendix A, i.e., we show that when $\mathfrak{a}$ is sufficiently small, the initial datum $(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}$ launches a simple isentropic plane-symmetric solution that satisfies all the assumptions we stated in stated in Sect. 11.

A.6.1. *Some preliminary definitions.* Given any initial data function $\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}} = \mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}(u)$, as in (A.6), we define:

$$\mathring{\delta}_*^{\mathrm{PS}} = \mathring{\delta}_*^{\mathrm{PS}}[\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}] \stackrel{\mathrm{def}}{=} \max_{u\in[-U_1,U_2]} [H(u)]_- = \max_{u\in[-U_1,U_2]} \left[\mathfrak{H}'[\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}(u)]\frac{d}{du}\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}}(u)\right]_-, \tag{A.42a}$$

$$T_{\mathrm{Shock}}^{\mathrm{PS}} \stackrel{\mathrm{def}}{=} \frac{1}{\mathring{\delta}_*^{\mathrm{PS}}}, \tag{A.42b}$$

where $[z]_- \stackrel{\mathrm{def}}{=} \max\{-z, 0\}$. In view of (A.16) and (A.21), we see that for simple isentropic plane-wave solutions, $\mathring{\delta}_*^{\mathrm{PS}}$ coincides with the quantity defined in (11.6). Moreover, from Cor. A.3, we see that $T_{\mathrm{Shock}}^{\mathrm{PS}}$ is the Cartesian time of first blowup of $\partial_1\mathcal{R}_{(+)}^{\mathrm{PS}}$.

A.6.2. *The existence of admissible simple isentropic plane-symmetric solutions.* We now state and prove the main results of this appendix.

Theorem A.4 (The existence of admissible simple isentropic plane-symmetric solutions)**.** *Let $\mathring{\varphi}$ be a "seed profile" function with the properties stated in Sect. A.5. In particular, $\mathring{\varphi}$ is supported in the interval $[-U_1, U_2]$ of u-values (where by* (A.7b)*, $u = -x^1$ along Σ_0), and $\|\mathring{\varphi}\|_{H_u^{N_{\mathrm{top}}+1}(\Sigma_0^{[-U_1,U_2]})} < \infty$. There exist small constants $\mathring{\mathfrak{a}}_0 > 0$ and $\zeta > 0$, depending on $\mathring{\varphi}$ (including the constant $\mathfrak{b} > 0$ from* (A.33)*) and satisfying*[65] *$\zeta < \min\{1, 1-\mathfrak{p}\}$ (where $\mathfrak{p} \in [0,1)$ is the non-negative constant in* (A.34)–(A.35)*) such that if $0 < \mathfrak{a} \leq \mathfrak{a}_0$, then the following conclusions hold, where in all estimates, the constants (including the implicit ones corresponding to "$\lesssim$") are independent of $\mathfrak{a}$, ζ, t, and u (on the domains where the estimates are asserted to hold).*

Let $(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}$ be the initial datum, defined by (A.36)–(A.37)*, for the simple isentropic plane-symmetric compressible Euler equations, i.e., for* (A.4) *with $\mathcal{R}_{(-)}^{\mathrm{PS}} \equiv 0$. Let τ_0 be any negative real number such that:*

$$-\frac{1}{4} < -\frac{\zeta}{4} < \tau_0 < 0, \tag{A.43}$$

and define $U_{\maltese} > 0$ and $\mathfrak{n}_0 > 0$ by:

$$U_{\maltese} \stackrel{\mathit{def}}{=} \frac{\mathfrak{a}}{\mathfrak{b}^2}, \tag{A.44}$$

$$\mathfrak{n}_0 \stackrel{\mathit{def}}{=} \frac{|\tau_0|\mathfrak{a}}{16\mathfrak{b}}, \tag{A.45}$$

where $\mathfrak{b} > 0$ is the $\mathring{\varphi}$-dependent constant from Sect. A.5.1.

Classical existence with respect to the geometric coordinates*. With respect to the geometric coordinates (t, u), the solution $\mathcal{R}_{(+)}^{\mathrm{PS}}$ is a function of u alone (i.e., $\mathcal{R}_{(+)}^{\mathrm{PS}}(t,u) = \mathcal{R}_{(+)}^{\mathrm{PS}}(u) = (\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u)$), vanishes on the complement of the region $\{(t,u) \in \mathbb{R}\times\mathbb{R} \mid u \in [-U_1, U_2]\}$, and exists classically for $(t,u) \in [0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}$, where the parameter $\mathring{\delta}_*^{\mathrm{PS}} = \mathring{\delta}_*^{\mathrm{PS}}[(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}]$ defined by* (A.42a) *satisfies:*

$$\mathring{\delta}_*^{\mathrm{PS}} = \mathfrak{a}, \tag{A.46}$$

and:

$$T_{\mathrm{Shock}}^{\mathrm{PS}} \stackrel{\mathit{def}}{=} \frac{1}{\mathring{\delta}_*^{\mathrm{PS}}} = \frac{1}{\mathfrak{a}}. \tag{A.47}$$

Similarly, the null vectorfield L^{PS} and inverse foliation density μ^{PS} exist classically for $(t,u) \in [0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}$, and on the complement of $\{(t,u) \in \mathbb{R}\times\mathbb{R} \mid u \in [-U_1, U_2]\}$, we have $(L^{\mathrm{PS}})^1 = L^{\mathrm{PS}}x^1 = 1$ and $\mu^{\mathrm{PS}} = 1$. Finally, $\mu^{\mathrm{PS}} > 0$ on $\Big([0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}\Big) \setminus \{(T_{\mathrm{Shock}}^{\mathrm{PS}}, 0)\}$, and $\mu^{\mathrm{PS}}(T_{\mathrm{Shock}}^{\mathrm{PS}}, 0) = 0$.

Description of the crease relative to the geometric coordinates*. Relative to the geometric coordinates (t, u), the crease, which by definition is $\partial_-\mathcal{B}^{\mathrm{PS}} \stackrel{\mathit{def}}{=} \{(t,u) \mid \mu^{\mathrm{PS}}(t,u) = 0\} \cap \{(t,u) \mid \breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u) = 0\} \cap \Big([0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{\maltese}, U_{\maltese}]\Big)$, is equal to the single point $(T_{\mathrm{Shock}}^{\mathrm{PS}}, 0)$.*

The Cartesian coordinate description of the singularity formation up to the crease*.*

- *Let $\Upsilon_{\mathrm{PS}}(t,u) \stackrel{\mathit{def}}{=} (t, x^1)$ denote the change of variables map from geometric to Cartesian coordinates. Then Υ_{PS} is a homeo- (resp. diffeo)-morphism from $[0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}$ (resp. from $\Big([0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}\Big) \setminus \partial_-\mathcal{B}^{\mathrm{PS}}$) onto its image.*
- *The solution $\mathcal{R}_{(+)}^{\mathrm{PS}}$ exists classically with respect to the Cartesian coordinates on the subset $\Upsilon_{\mathrm{PS}}\Big(([0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}) \setminus \partial_-\mathcal{B}^{\mathrm{PS}}\Big)$ of Cartesian coordinate space $\mathbb{R}_t \times \mathbb{R}_{x^1}$.*
- *There exists a past neighborhood $\mathcal{N}$ of $\Upsilon_{\mathrm{PS}}(\partial_-\mathcal{B}^{\mathrm{PS}})$ in Cartesian coordinate space such that, with $\mathcal{R}_{(+)}^{\mathrm{PS}} = \mathcal{R}_{(+)}^{\mathrm{PS}}(t,u) = \mathcal{R}_{(+)}^{\mathrm{PS}}(u)$, we have:*

$$\Big|[\partial_1 \mathcal{R}_{(+)}^{\mathrm{PS}}] \circ \Upsilon_{\mathrm{PS}}^{-1}(t,x^1)\Big| = \frac{1}{\mu^{\mathrm{PS}}} \left|\frac{1}{c^{\mathrm{PS}}} \breve{X}^{\mathrm{PS}} \mathcal{R}_{(+)}^{\mathrm{PS}}\right| \circ \Upsilon_{\mathrm{PS}}^{-1}(t,x^1) \gtrsim \frac{\mathfrak{a}}{\mu^{\mathrm{PS}}}, \qquad (t,x^1) \in \mathcal{N}. \tag{A.48}$$

[65] Our assumption $\zeta < 1-\mathfrak{p}$ ensures that (11.21) is satisfied; see (A.43) and (A.78), and recall that in the bulk of the paper, we have $\mathfrak{m}_0 = -\tau_0$.

In particular, for any sequence of points $\{q_n\}_{n\in\mathbb{N}} \subset \Upsilon_{\mathrm{PS}}\Big(([0,T_{\mathrm{Shock}}^{\mathrm{PS}}]\times\mathbb{R})\setminus\partial_{-}\mathcal{B}^{\mathrm{PS}}\Big)$ *converging to the point* $\Upsilon_{\mathrm{PS}}(\partial_{-}\mathcal{B}^{\mathrm{PS}})$ *(i.e.,* $\Upsilon_{\mathrm{PS}}^{-1}(q_n)\to(T_{\mathrm{Shock}}^{\mathrm{PS}},0)$*), we have that* $|\partial_1\mathcal{R}_{(+)}^{\mathrm{PS}}|\circ\Upsilon_{\mathrm{PS}}^{-1}(q_n)\to\infty$ *as* $n\to\infty$*.*

Estimates for the solution and acoustic geometry with respect to the geometric coordinates*. For* $(t,u)\in[0,T_{\mathrm{Shock}}^{\mathrm{PS}}]\times\mathbb{R}$*, the speed of sound satisfies the following estimate:*

$$c^{\mathrm{PS}}(t,u)=c^{\mathrm{PS}}(u)=1+\mathcal{O}(\mathfrak{a}). \tag{A.49}$$

For $t\in[0,T_{\mathrm{Shock}}^{\mathrm{PS}}]$*, the following Sobolev estimates hold:*

$$\left\|\mathcal{R}_{(+)}^{\mathrm{PS}}\right\|_{H_u^{N_{\mathrm{top}}+1}(\Sigma_t)}\lesssim\mathfrak{a}, \tag{A.50a}$$

$$\left\|(L_{(\mathrm{Small})}^{\mathrm{PS}})^1\right\|_{H_u^{N_{\mathrm{top}}+1}(\Sigma_t)}\lesssim\mathfrak{a}, \tag{A.50b}$$

$$\left\|\frac{\partial}{\partial t}\mu^{\mathrm{PS}}\right\|_{H_u^{N_{\mathrm{top}}}(\Sigma_t)}\lesssim\mathfrak{a}, \tag{A.50c}$$

$$\left\|\mu^{\mathrm{PS}}\right\|_{H_u^{N_{\mathrm{top}}}(\Sigma_t)}\lesssim 1. \tag{A.50d}$$

For $M=0,1,2,3,4$ *and* $t\in[0,T_{\mathrm{Shock}}^{\mathrm{PS}}]$*, the following estimates hold:*

$$\left\|(\breve{X}^{\mathrm{PS}})^M\mathcal{R}_{(+)}^{\mathrm{PS}}\right\|_{L^\infty(\Sigma_t)}\lesssim\mathfrak{a}, \tag{A.51a}$$

$$\left\|(\breve{X}^{\mathrm{PS}})^M(L_{(\mathrm{Small})}^{\mathrm{PS}})^1\right\|_{L^\infty(\Sigma_t)}\lesssim\mathfrak{a}. \tag{A.51b}$$

Similarly, for $M=0,1,2,3$ *and* $t\in[0,T_{\mathrm{Shock}}^{\mathrm{PS}}]$*, we have:*

$$\left\|L^{\mathrm{PS}}(\breve{X}^{\mathrm{PS}})^M\mu^{\mathrm{PS}}\right\|_{L^\infty(\Sigma_t)}=\frac{1}{2}\left\|(\breve{X}^{\mathrm{PS}})^M\left\{\frac{1}{c^{\mathrm{PS}}}\left(\frac{(c^{\mathrm{PS}})'}{c^{\mathrm{PS}}}+1\right)\breve{X}^{\mathrm{PS}}\mathcal{R}_{(+)}^{\mathrm{PS}}\right\}\right\|_{L^\infty(\Sigma_0)}\lesssim\mathfrak{a}, \tag{A.52a}$$

$$\begin{aligned}\left\|(\breve{X}^{\mathrm{PS}})^M\mu^{\mathrm{PS}}\right\|_{L^\infty(\Sigma_t)}&\leq\left\|(\breve{X}^{\mathrm{PS}})^M\left\{\frac{1}{c^{\mathrm{PS}}}\right\}\right\|_{L^\infty(\Sigma_0)}+\frac{1}{2}\frac{1}{\mathring{\delta}_*^{\mathrm{PS}}}\left\|(\breve{X}^{\mathrm{PS}})^M\left\{\frac{1}{c^{\mathrm{PS}}}\left(\frac{(c^{\mathrm{PS}})'}{c^{\mathrm{PS}}}+1\right)\breve{X}^{\mathrm{PS}}\mathcal{R}_{(+)}^{\mathrm{PS}}\right\}\right\|_{L^\infty(\Sigma_0)}\\&\lesssim 1.\end{aligned} \tag{A.52b}$$

Estimates tied to the change of variables map Υ_{PS}*. For* $t\in[0,T_{\mathrm{Shock}}^{\mathrm{PS}}]$*, the Cartesian spatial coordinate* $x^1=x^1(t,u)$ *satisfies the following estimates:*

$$-U_2+t\leq\min_{\Sigma_t^{[-U_1,U_2]}}x^1\leq\max_{\Sigma_t^{[-U_1,U_2]}}x^1\leq U_1+t. \tag{A.53}$$

Moreover,

$$d_{\mathrm{geo}}\Upsilon_{\mathrm{PS}}(t,u)\overset{def}{=}\frac{\partial\Upsilon_{\mathrm{PS}}(t,u)}{\partial(t,u)}=\begin{pmatrix}1 & 0\\(L^{\mathrm{PS}})^1 & -c^{\mathrm{PS}}\mu^{\mathrm{PS}}\end{pmatrix}, \tag{A.54}$$

and the following estimates hold for $t\in[0,T_{\mathrm{Shock}}^{\mathrm{PS}}]$*:*

$$\left\|d_{\mathrm{geo}}\Upsilon_{\mathrm{PS}}\right\|_{C_{\mathrm{geo}}^{N_{\mathrm{top}}-1}(\Sigma_t)}\lesssim 1, \tag{A.55a}$$

$$\left\|\frac{\partial}{\partial t}d_{\mathrm{geo}}\Upsilon_{\mathrm{PS}}\right\|_{H_u^{N_{\mathrm{top}}}(\Sigma_t)},\left\|\frac{\partial}{\partial t}d_{\mathrm{geo}}\Upsilon_{\mathrm{PS}}\right\|_{C_{\mathrm{geo}}^{N_{\mathrm{top}}-1}(\Sigma_t)}\lesssim\mathfrak{a}. \tag{A.55b}$$

Properties of the rough time functions ${}^{(\mathfrak{n})}\tau_{\mathrm{PS}}$ ***and the location of their level-sets****. We define:*

$$\Delta^{\mathrm{PS}}\overset{def}{=}\frac{|\tau_0|}{16}T_{\mathrm{Shock}}^{\mathrm{PS}}=\frac{|\tau_0|}{16\mathfrak{a}}. \tag{A.56}$$

For $\tau \in [2\tau_0, \frac{1}{2}\tau_0]$ *and* $\mathfrak{n} \in [0, \mathfrak{n}_0]$*, there exist functions* $\mathrm{t}^{\mathrm{PS}}_{\tau,\mathfrak{n}} : \mathbb{R} \to [T^{\mathrm{PS}}_{\mathrm{Shock}} - 2\zeta T^{\mathrm{PS}}_{\mathrm{Shock}}, T^{\mathrm{PS}}_{\mathrm{Shock}} - 2\Delta^{\mathrm{PS}}]$*, depending on* τ *and* $\mathfrak{n}$ *and strictly increasing with respect to* τ*, such that* $\|\mathrm{t}^{\mathrm{PS}}_{\tau,\mathfrak{n}}\|_{C^3_u((-\infty,\infty))} \lesssim 1$ *and such that the rough time function* ${}^{(\mathfrak{n})}\tau_{\mathrm{PS}}$ *exists as a* C^3 *function of* (t,u) *on the domain* ${}^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}$ *defined by:*

$$ {}^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)} = \Big\{(t,u) \mid \mathrm{t}^{\mathrm{PS}}_{2\tau_0,\mathfrak{n}}(u) \leq t \leq \mathrm{t}^{\mathrm{PS}}_{\frac{1}{2}\tau_0,\mathfrak{n}}(u),\, u \in (-\infty,\infty)\Big\}, \tag{A.57} $$

and such that relative to the geometric coordinates, the level-sets ${}^{(\mathfrak{n})}\widetilde{\Sigma}^{\mathrm{PS}}_{\tau} \stackrel{def}{=} \{(t,u) \in \mathbb{R}\times\mathbb{R} \mid {}^{(\mathfrak{n})}\tau_{\mathrm{PS}}(t,u) = \tau\}$ *are the following graphical surfaces:*

$$ {}^{(\mathfrak{n})}\widetilde{\Sigma}^{\mathrm{PS}}_{\tau} = \Big\{(t,u) \mid t = \mathrm{t}^{\mathrm{PS}}_{\tau,\mathfrak{n}}(u),\, u \in \mathbb{R}\Big\}. \tag{A.58} $$

In particular,

$$ \bigcup_{\tau\in[2\tau_0,\frac{1}{2}\tau_0]} {}^{(\mathfrak{n})}\widetilde{\Sigma}^{\mathrm{PS}}_{\tau} \subset \bigcup_{t\in[T^{\mathrm{PS}}_{\mathrm{Shock}}-2\zeta T^{\mathrm{PS}}_{\mathrm{Shock}}, T^{\mathrm{PS}}_{\mathrm{Shock}}-2\Delta^{\mathrm{PS}}]} \Sigma_t. \tag{A.59} $$

Moreover, the following estimates hold for $(t,u) \in {}^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}$*:*

$$ \frac{31}{32}\mathring{\delta}^{\mathrm{PS}}_{*} \leq \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{\mathrm{PS}}(t,u) \leq \frac{33}{32}\mathring{\delta}^{\mathrm{PS}}_{*}, \tag{A.60} $$

$$ \left|\frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau_{\mathrm{PS}}\right| \leq 2\mathfrak{n}_0 = \frac{|\tau_0|\mathfrak{a}}{8\mathfrak{b}}. \tag{A.61} $$

Properties of ${}^{(\mathfrak{n})}\mathscr{T}_{\mathrm{PS}}$ ***and*** ${}^{(\mathfrak{n})}\mathbf{J}_{\mathrm{PS}}$. *The change of variables map* ${}^{(\mathfrak{n})}\mathscr{T}_{\mathrm{PS}}$ *defined by:*

$$ {}^{(\mathfrak{n})}\mathscr{T}_{\mathrm{PS}}(t,u) \stackrel{def}{=} ({}^{(\mathfrak{n})}\tau_{\mathrm{PS}}, u) \tag{A.62} $$

is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}$ *onto its image, which is* $[2\tau_0, \frac{1}{2}\tau_0] \times (-\infty,\infty)$*, and it satisfies:*

$$ \|{}^{(\mathfrak{n})}\mathscr{T}_{\mathrm{PS}}\|_{C^3_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)})} \lesssim 1, \tag{A.63} $$

$$ \frac{31}{32}\mathring{\delta}^{\mathrm{PS}}_{*} \leq \det\Big(d_{\mathrm{geo}}{}^{(\mathfrak{n})}\mathscr{T}_{\mathrm{PS}}\Big) = \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{\mathrm{PS}} \leq \frac{33}{32}\mathring{\delta}^{\mathrm{PS}}_{*}, \qquad \text{on } {}^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}. \tag{A.64} $$

In addition, for every $\mathfrak{n} \in [0,\mathfrak{n}_0]$*, the Jacobian matrix* ${}^{(\mathfrak{n})}\mathbf{J}_{\mathrm{PS}} \stackrel{def}{=} \frac{\partial(\mu^{\mathrm{PS}}, \breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{({}^{(\mathfrak{n})}\tau_{\mathrm{PS}},u)}$ *(see also* (5.4b)*) is invertible for every* $q \stackrel{def}{=} (\tau,u) \in [2\tau_0,\frac{1}{2}\tau_0] \times [-U_{\maltese}, U_{\maltese}]$ *and satisfies:*

$$ \sup_{q_1,q_2\in[2\tau_0,\frac{1}{2}\tau_0]\times[-U_{\maltese},U_{\maltese}]} \left|{}^{(\mathfrak{n})}\mathbf{J}^{-1}_{\mathrm{PS}}(q_1){}^{(\mathfrak{n})}\mathbf{J}_{\mathrm{PS}}(q_2) - \mathrm{ID}\right|_{\mathrm{Euc}} \leq \frac{1}{4}, \tag{A.65} $$

where $|\cdot|_{\mathrm{Euc}}$ *is the standard Frobenius norm on matrices (equal to the square root of the sum of the squares of the matrix entries) and* ID *denotes the* 2×2 *identity matrix.*

Properties of $\breve{\mathscr{M}}_{\mathrm{PS}}$. *We define the map* $\breve{\mathscr{M}}_{\mathrm{PS}}$ *from geometric coordinates to "*$(\mu, \breve{X}\mu)$*-space" and its Jacobian* ${}^{(\breve{\mathscr{M}}_{\mathrm{PS}})}\mathbf{J}$ *as follows (see also* (5.3a)–(5.3b)*):*

$$ \breve{\mathscr{M}}_{\mathrm{PS}}(t,u,x^2,x^3) \stackrel{def}{=} (\mu, \breve{X}^{\mathrm{PS}}\mu), \tag{A.66a} $$

$$ {}^{(\breve{\mathscr{M}}_{\mathrm{PS}})}\mathbf{J}(t,u) \stackrel{def}{=} \frac{\partial\breve{\mathscr{M}}(t,u)}{\partial(t,u)} = \frac{\partial(\mu,\breve{X}^{\mathrm{PS}}\mu)}{\partial(t,u)}. \tag{A.66b} $$

Then there is a $C > 1$ *such that for* $\mathfrak{n} \in [0,\mathfrak{n}_0]$*,* ${}^{(\breve{\mathscr{M}}_{\mathrm{PS}})}\mathbf{J}$ *is invertible for* $(t,u) \in [T^{\mathrm{PS}}_{\mathrm{Shock}} - 2\zeta T^{\mathrm{PS}}_{\mathrm{Shock}}, T^{\mathrm{PS}}_{\mathrm{Shock}}] \times [-U_{\maltese}, U_{\maltese}]$ *and satisfies the following bounds:*

$$ -C \leq \det{}^{(\breve{\mathscr{M}})}\mathbf{J} \leq -\frac{1}{C}, \qquad \text{on } [T^{\mathrm{PS}}_{\mathrm{Shock}} - 2\zeta T^{\mathrm{PS}}_{\mathrm{Shock}}, T^{\mathrm{PS}}_{\mathrm{Shock}}] \times [-U_{\maltese}, U_{\maltese}], \tag{A.67} $$

$$\sup_{p_1,p_2\in[T^{\mathrm{PS}}_{\mathrm{Shock}}-2\zeta T^{\mathrm{PS}}_{\mathrm{Shock}},T^{\mathrm{PS}}_{\mathrm{Shock}}]\times[-U_{☢},U_{☢}]}\left|{}^{(\breve{\mathscr{M}}_{\mathrm{PS}})}\mathbf{J}(p_1)\,{}^{(\breve{\mathscr{M}}_{\mathrm{PS}})}\mathbf{J}^{-1}(p_2)-\mathrm{ID}\right|_{\mathrm{Euc}}\leq\frac{1}{4}, \tag{A.68}$$

where $|\cdot|_{\mathrm{Euc}}$ *is the standard Frobenius norm on matrices (equal to the square root of the sum of the squares of the matrix entries) and* ID *denotes the* 2×2 *identity matrix.*

Behavior of μ^{PS} ***in the interesting region****. The following estimates hold for* $t\in[0,T^{\mathrm{PS}}_{\mathrm{Shock}}]$*:*

$$-\frac{33}{32}\mathring{\delta}^{\mathrm{PS}}_*\leq\min_{\Sigma_t^{[-U_{☢},U_{☢}]}}L^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u)\leq\max_{\Sigma_t^{[-U_{☢},U_{☢}]}}L^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u)\leq-\frac{31}{32}\mathring{\delta}^{\mathrm{PS}}_*. \tag{A.69}$$

Moreover, for $(t,u)\in[0,T^{\mathrm{PS}}_{\mathrm{Shock}}]\times[-U_{☢},U_{☢}]$*, the following estimates hold:*

$$\mu^{\mathrm{PS}}(t,u)=\{1+\mathcal{O}(\mathfrak{a})\}\frac{\mathfrak{b}}{2}u^2+\{1+\mathcal{O}(\mathfrak{a})\}\mathfrak{a}(T^{\mathrm{PS}}_{\mathrm{Shock}}-t), \tag{A.70}$$

$$L^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u)=-\{1+\mathcal{O}(\mathfrak{a})\}\mathfrak{a}, \tag{A.71}$$

$$\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u)=\{1+\mathcal{O}(\mathfrak{a})\}\mathfrak{b}u+\mathcal{O}(\mathfrak{a}^2)(T^{\mathrm{PS}}_{\mathrm{Shock}}-t), \tag{A.72}$$

$$L^{\mathrm{PS}}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u)=\mathcal{O}(\mathfrak{a}^2), \tag{A.73}$$

$$\breve{X}^{\mathrm{PS}}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u)=\{1+\mathcal{O}(\mathfrak{a})\}\mathfrak{b}+\mathcal{O}(\mathfrak{a})(T^{\mathrm{PS}}_{\mathrm{Shock}}-t). \tag{A.74}$$

In addition, for $t\in[T^{\mathrm{PS}}_{\mathrm{Shock}}-T^{\mathrm{PS}}_{\mathrm{Shock}}\zeta,T^{\mathrm{PS}}_{\mathrm{Shock}}]$ *and* $\mathfrak{n}\in[0,\mathfrak{n}_0]$*, we have:*[66]

$$\{\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}=-\mathfrak{n}\}\cap\Sigma_t^{[-U_{☢},U_{☢}]}\subset\Sigma_t^{[-\frac{1}{4}U_{☢},\frac{1}{4}U_{☢}]}, \tag{A.75}$$

$$\min_{\Sigma_t^{[-U_{☢},U_{☢}]}\setminus\Sigma_t^{[-\frac{1}{2}U_{☢},\frac{1}{2}U_{☢}]}}|\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}+\mathfrak{n}|\geq\frac{\mathfrak{b}U_{☢}}{8}, \tag{A.76}$$

$$\begin{aligned}\frac{\mathfrak{b}}{2}&\leq\min_{\Sigma_t^{[-U_{☢},U_{☢}]}}\left\{{}^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}\,{}^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}\mu^{\mathrm{PS}},\ {}^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}},\ \breve{X}^{\mathrm{PS}}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}},\ \frac{\partial}{\partial u}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}-\frac{(\frac{\partial}{\partial u}\mu^{\mathrm{PS}})\frac{\partial}{\partial t}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}}{\frac{\partial}{\partial t}\mu^{\mathrm{PS}}}\right\}\\&\leq\max_{\Sigma_t^{[-U_{☢},U_{☢}]}}\left\{{}^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}\,{}^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}\mu^{\mathrm{PS}},\ {}^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}},\ \breve{X}^{\mathrm{PS}}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}},\ \frac{\partial}{\partial u}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}-\frac{(\frac{\partial}{\partial u}\mu^{\mathrm{PS}})\frac{\partial}{\partial t}\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}}{\frac{\partial}{\partial t}\mu^{\mathrm{PS}}}\right\}\leq 2\mathfrak{b}.\end{aligned} \tag{A.77}$$

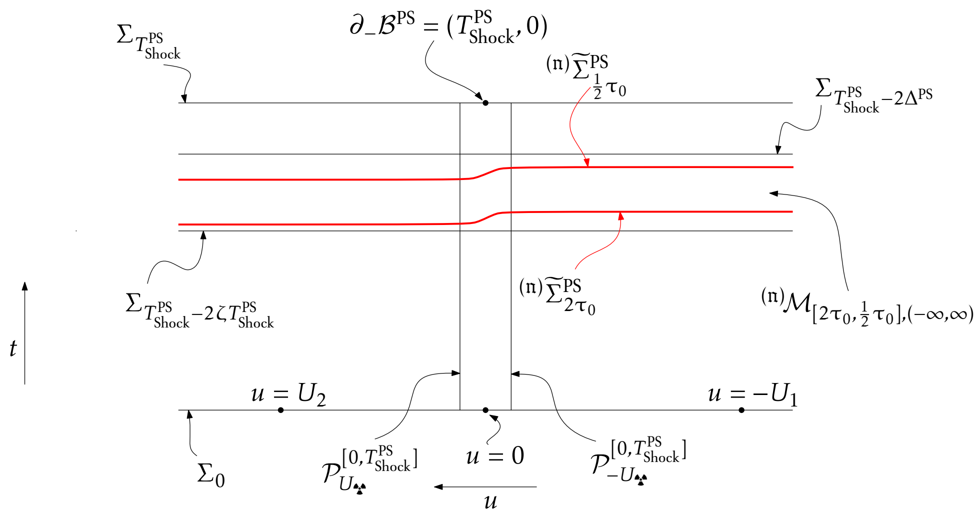
μ^{PS} ***is uniformly positive away from the interesting region****. With* $\mathfrak{p}\in[0,1)$ *denoting the non-negative constant on RHS* (A.41b)*, we have the following estimate:*

$$\min_{\{(t,u)\ |\ t\in[0,T^{\mathrm{PS}}_{\mathrm{Shock}}],|u|\geq U_{☢}\}}\mu^{\mathrm{PS}}(t,u)\geq\frac{1}{2}(1-\mathfrak{p}). \tag{A.78}$$

Remark A.5 (Generalizations of Theorem A.4)**.** Before proving the theorem, we first make a series of remarks on how it could be extended.

- In Theorem A.4, we chose to follow the solution up to the Cartesian time of first blowup because we believe that the results could help prepare the reader for the more difficult analysis in the bulk of the paper. However, we could have "stopped the analysis" before then; our main goal in the theorem was to construct the rough time functions ${}^{(\mathfrak{n})}\tau_{\mathrm{PS}}$ and to describe the state of the solution near the hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}^{\mathrm{PS}}_{\tau_0}$ so that we can use these results in Appendix B, in our study of Cauchy stability.
- Our definition (A.45) of $\mathfrak{n}_0$ is such that $\mathfrak{n}_0$ decreases as $|\tau_0|\downarrow 0$. This is highly non-optimal and is an artifact of our insistence (out of convenience) that the level-sets ${}^{(\mathfrak{n})}\widetilde{\Sigma}^{\mathrm{PS}}_{\tau_0}$ should be contained in the rectangular shaped domain $[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-2\Delta^{\mathrm{PS}}]\times\mathbb{R}$ in geometric coordinate space; by studying the solution on a larger (curved) subset of geometric coordinate space, we could have shown that $\mathfrak{n}_0$ can be chose to be independent of all sufficiently small $|\tau_0|$. In a similar vein, with modest additional effort, we could have shown that $U_{☢}$ can be chosen to be independent of all sufficiently small $\mathfrak{a}$.

[66]It might appear that there are fewer quantities in braces in (A.77) compared to (11.18), but this is not true. The reason is that in simple isentropic plane-symmetry, some of the quantities in braces in (11.18) are equal to each other.



- One could generalize Theorem A.4 in a straightforward fashion to allow for much more general initial data of simple isentropic plane-symmetric type. For example, one could consider seed profile functions $\mathring{\varphi}$ of multi-bump type, leading to the formation distinct shocks that are separated in space. One could also consider two-parameter families of rescaled seed profile functions of the form $\mathring{\varphi}_{\mathfrak{a}_1;\mathfrak{a}_2}(u) \stackrel{\mathrm{def}}{=} \mathfrak{a}_1\mathring{\varphi}(\mathfrak{a}_2 u)$, which would allow one to produce small-amplitude shock-forming solutions for initial data with small derivatives (i.e., when $\mathfrak{a}_1$ is small and $\mathfrak{a}_1\mathfrak{a}_2$ is even smaller) or large derivatives (i.e., when $\mathfrak{a}_1$ is small and $\mathfrak{a}_1\mathfrak{a}_2$ is large). Moreover, the assumption that $\mathring{\varphi}$ is compactly supported can easily be eliminated.
- One could also prove an analog of Theorem A.4 for isentropic plane-symmetric solutions that are not simple, i.e., when both Riemann invariants in the system (A.4) are non-vanishing. In this way, one could produce shocks along the characteristics of L^{PS} (i.e., the blowup of $\partial_1\mathcal{R}_{(+)}$, as in Theorem A.4) as well as along the characteristics of $\underline{L}^{\mathrm{PS}}$ (i.e., the blowup of $\partial_1\mathcal{R}_{(-)}$).
- In Theorem A.4, we followed the solution up to the crease $\partial_-\mathcal{B}^{\mathrm{PS}}$. With minor additional effort, we could have followed the solution up to a portion of the singular boundary that contains a neighborhood of the crease. We have omitted such results because we have already derived them away from plane-symmetry, in Theorem 34.1.

Proof of Theorem A.4. In each step of the proof, we will silently adjust the smallness of the positive constants $\mathfrak{a}_0$ and ζ without explicitly mentioning it each time.

Proof of classical existence with respect to the geometric coordinates for $(t,u) \in [0, T^{\mathrm{PS}}_{\mathrm{Shock}}] \times \mathbb{R}$: The facts that $\mathcal{R}^{\mathrm{PS}}_{(+)} = \mathcal{R}^{\mathrm{PS}}_{(+)}(u)$ and that $\mathcal{R}^{\mathrm{PS}}_{(+)}$ vanishes on the complement of the region $\{(t,u) \in \mathbb{R}\times\mathbb{R} \mid u \in [-U_1, U_2]\}$ follow from the evolution equation $\frac{\partial}{\partial t}\mathcal{R}^{\mathrm{PS}}_{(+)}(t,u) = 0$ (see (A.4) and (A.11)) and our assumption that the initial data are supported in the u-interval $[-U_1, U_2]$. The identity (A.46) follows from definition (A.42a), (A.18a)–(A.18c), and the properties of $\mathfrak{H}[(\mathring{\mathcal{R}}^{\mathrm{PS}}_{(+)})_{\mathfrak{a}}(u)]$ described in Sect. A.5. The properties of $\mathcal{R}^{\mathrm{PS}}_{(+)}$, L^{PS}, and μ^{PS} follow easily from our assumptions on the support of $\mathring{\varphi}$, the fact that relative to the geometric coordinates, $\mathcal{R}^{\mathrm{PS}}_{(+)}$ depends only on u, equations (A.3) and (A.5), the normalization assumption (2.5) (which implies that $c^{\mathrm{PS}} = 1 + \mathcal{R}^{\mathrm{PS}}_{(+)}\mathrm{f}(\mathcal{R}^{\mathrm{PS}}_{(+)})$, where f is smooth), and the explicit solution formulas provided by Cor. A.3. In particular, from the identity (A.23b), definition (A.42a), the identity (A.46), definition (A.47), and our assumption that $\frac{d}{du}\mathfrak{H}[(\mathring{\mathcal{R}}^{\mathrm{PS}}_{(+)})_{\mathfrak{a}}(u)]$ has a unique, negative, non-degenerate minimum at $u = 0$ (see Sect. A.5.2), it follows that $\mu^{\mathrm{PS}} > 0$ on $\left([0, T^{\mathrm{PS}}_{\mathrm{Shock}}] \times \mathbb{R}\right) \backslash \{(T^{\mathrm{PS}}_{\mathrm{Shock}}, 0)\}$, and that $\mu^{\mathrm{PS}}(T^{\mathrm{PS}}_{\mathrm{Shock}}, 0) = 0$.

Proof of (A.49)–(A.52b)**:** These estimates are straightforward consequences of the fact that in geometric coordinates, $\mathcal{R}^{\mathrm{PS}}_{(+)}$ depends only on u, (A.5), (A.11), (A.23b), the normalization assumption (2.5) (which implies that $c^{\mathrm{PS}} = 1 + \mathcal{R}^{\mathrm{PS}}_{(+)}\mathrm{f}(\mathcal{R}^{\mathrm{PS}}_{(+)})$, where f is smooth), the data-estimates (A.38a)–(A.38b), and the standard Sobolev calculus.

Proof (A.53)**:** Note that by (A.5), $\frac{\partial}{\partial t}x^1 = L^{\mathrm{PS}}x^1 = v^{\mathrm{PS}} + c^{\mathrm{PS}}$. Considering also that $\mathcal{R}_{(+)}^{\mathrm{PS}}$ vanishes on the complement of $\{(t,u) \in \mathbb{R}\times\mathbb{R} \mid u \in [-U_1, U_2]\}$, we see that for $(t,u) \in [0, T_{\mathrm{Shock}}^{\mathrm{PS}}]\times((-\infty, -U_1] \cup [U_2, \infty))$, we have $v^{\mathrm{PS}}(t,u)+c^{\mathrm{PS}}(t,u) = 1$. Recalling also that $x^1(0,u) = -u$ (see (A.7b)), we see that for $t \in [0, T_{\mathrm{Shock}}^{\mathrm{PS}}]$, we have $x^1(t,-U_1) = U_1 + t$ and $x^1(t,U_2) = -U_2 + t$. Moreover, since (A.9) and (A.11) imply that $\frac{\partial}{\partial u}x^1 = \check{X}^{\mathrm{PS}}x^1 = -c^{\mathrm{PS}}\mu^{\mathrm{PS}} \leq 0$, we conclude that for $(t,u) \in [0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_1, U_2]$, we have $x^1(t,U_2) \leq x^1(t,u) \leq x^1(t,-U_1)$. Combining these results, we conclude (A.53).

Proof of (A.54) **and** (A.55a)–(A.55b)**:** Since $\Upsilon_{\mathrm{PS}}(t,u) = (t,x^1)$, (A.54) follows easily from (A.5), (A.9), and (A.11).

The estimates (A.55a)–(A.55b) follow from (A.54), the fact that relative to the geometric coordinates, $\mathcal{R}_{(+)}^{\mathrm{PS}}$ depends only on u, the estimates (A.50a)–(A.50d), and standard Sobolev calculus.

Proof of the remaining properties of Υ_{PS}**:** We now prove that the map $\Upsilon_{\mathrm{PS}}(t,u) = (t,x^1)$ is a diffeomorphism on $[0, T_{\mathrm{Shock}}^{\mathrm{PS}}) \times \mathbb{R}$. First, using (A.54), we compute that $\det d\Upsilon_{\mathrm{PS}} = -c^{\mathrm{PS}}\mu^{\mathrm{PS}}$. Also using (A.49) and the fact that $\mu^{\mathrm{PS}} > 0$ on $[0, T_{\mathrm{Shock}}^{\mathrm{PS}}) \times \mathbb{R}$, we see that $\det d\Upsilon_{\mathrm{PS}} < 0$ on $[0, T_{\mathrm{Shock}}^{\mathrm{PS}}) \times \mathbb{R}$, and since $\frac{\partial}{\partial u}x^1 = -c^{\mathrm{PS}}\mu^{\mathrm{PS}}$ (by (A.54)), we see that for $t \in [0, T_{\mathrm{Shock}}^{\mathrm{PS}})$, the map $u \to x^1(t,u)$ is strictly decreasing for $u \in \mathbb{R}$. We therefore find that Υ_{PS} is injective on $[0, T_{\mathrm{Shock}}^{\mathrm{PS}}) \times \mathbb{R}$ and that it is a diffeomorphism on the same domain. Moreover, since $\mu^{\mathrm{PS}}(T_{\mathrm{Shock}}^{\mathrm{PS}}, u)$ vanishes only at the origin $u = 0$ (where the crease is located), the map $u \to x^1(T_{\mathrm{Shock}}^{\mathrm{PS}}, u)$ is strictly decreasing for $u \in \mathbb{R}$. It follows that Υ_{PS} is a homeomorphism on $[0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}$, as is desired.

Proof of (A.69)**,** (A.70)–(A.74)**,** (A.75)–(A.76)**, and** (A.77)**:** These estimates are straightforward to derive via the standard Sobolev calculus, Taylor expansions, the identities (A.23a)–(A.23b), the properties of the function $\mathfrak{H}[(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u)] = \mathring{\varphi}_{\mathfrak{a}}(u)$ stated in Sect. A.5, the estimates (A.49)–(A.52b), and the definitions (A.44)–(A.45) of $U_{☢}$ and $\mathfrak{n}_0$.

Proof of (A.48)**:** We first use (A.25), (A.39), (A.20), (A.49), (A.51a), and (A.44) to deduce, via Taylor expanding, that for $(t,u) \in [0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]$, we have:

$$[\mu^{\mathrm{PS}}\partial_1\mathcal{R}_{(+)}^{\mathrm{PS}}](t,u) = \frac{1}{c^{\mathrm{PS}}}\check{X}^{\mathrm{PS}}\mathcal{R}_{(+)}^{\mathrm{PS}} = \{1 + \mathcal{O}(\mathfrak{a})\}\frac{d}{du}(\mathring{\mathcal{R}}_{(+)}^{\mathrm{PS}})_{\mathfrak{a}}(u) = \frac{-2\mathfrak{a}}{\overline{c^{\mathrm{PS}}} + 1} + \mathcal{O}(\mathfrak{a}^2). \tag{A.79}$$

From (A.79), the non-degeneracy assumption (A.20), and the fact that the crease has geometric coordinates $(t,u) = (T_{\mathrm{Shock}}^{\mathrm{PS}}, 0)$, we conclude that if $\mathfrak{a}$ is sufficiently small, then (A.48) holds with $\mathcal{N} \stackrel{\mathrm{def}}{=} \Upsilon\left([0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]\right)$.

Proof of (A.78)**:** This estimate follows from (A.41b), (A.23b), (A.49), and (A.47).

An intermediate step - the map $(t,u) \to (\mu^{\mathrm{PS}}, \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})$ **is a local diffeomorphism:** Using (A.11), (A.43), (A.44), (A.47), and (A.71)–(A.74), we compute that for $(t,u) \in [T_{\mathrm{Shock}}^{\mathrm{PS}} - 2\zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]$, we have:

$$\frac{\partial(\mu^{\mathrm{PS}}, \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(t,u)} = \begin{pmatrix} -\mathfrak{a} & 0 \\ 0 & \mathfrak{b} \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\mathfrak{a}^2) & \mathcal{O}(\mathfrak{a}) \\ \mathcal{O}(\mathfrak{a}^2) & \mathcal{O}(\mathfrak{a}) + \mathcal{O}(\zeta) \end{pmatrix}. \tag{A.80}$$

Using (A.80), we deduce that if $(t,u) \in [T_{\mathrm{Shock}}^{\mathrm{PS}} - 2\zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]$, and if $\mathfrak{a}$ and ζ are sufficiently small, then $\frac{\partial(\mu^{\mathrm{PS}}, \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(t,u)}$ is invertible, and moreover, that:

$$\max_{(t_1,u_1),(t_2,u_2) \in [T_{\mathrm{Shock}}^{\mathrm{PS}} - 2\zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]} \left| \left(\frac{\partial(\mu^{\mathrm{PS}}, \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(t,u)}|_{(t_1,u_1)} \right)^{-1} \frac{\partial(\mu^{\mathrm{PS}}, \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(t,u)}|_{(t_2,u_2)} - \mathrm{ID} \right|_{\mathrm{Euc}} \leq \frac{1}{4}, \tag{A.81}$$

where $|\cdot|_{\mathrm{Euc}}$ is the standard Frobenius norm on matrices (equal to the square root of the sum of the squares of the matrix entries) and ID denotes the 2×2 identity matrix. These estimates, together with (A.17) and (A.52a)–(A.52b), imply that the map $(t,u) \to (\mu^{\mathrm{PS}}, \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})$ is a C^2 diffeomorphism from the compact, convex set $[T_{\mathrm{Shock}}^{\mathrm{PS}} - 2\zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]$ onto its image, which we denote by $\mathcal{I}$, i.e.,

$$\mathcal{I} \stackrel{\mathrm{def}}{=} \left\{ \left(\mu^{\mathrm{PS}}(t,u), \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u) \right) \mid (t,u) \in [T_{\mathrm{Shock}}^{\mathrm{PS}} - 2\zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}] \right\}. \tag{A.82}$$

Let $\widetilde{\mathcal{I}}$ be the following subset of $\mathcal{I}$, where Δ^{PS} is defined in (A.56):

$$\widetilde{\mathcal{I}} \stackrel{\mathrm{def}}{=} \left\{ \left(\mu^{\mathrm{PS}}(t,u), \check{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u) \right) \mid (t,u) \in [T_{\mathrm{Shock}}^{\mathrm{PS}} - \zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}} - 3\Delta^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}] \right\}. \tag{A.83}$$

Proof of the properties of the $^{(\mathfrak{n})}\tau_{\mathrm{PS}}$ and the location of their level-sets: Let $\widetilde{\mathcal{I}}$ be as in (A.83). Considering (A.70)–(A.74), recalling that τ_0 is allowed to be any negative real number satisfying (A.43), that $\mathfrak{n}_0$ is defined by (A.45), and that Δ^{PS} is defined in (A.56), we find that if $\mathfrak{a}$ and ζ are sufficiently small, then:

$$\begin{aligned}\left[\frac{1}{2}|\tau_0|, 2|\tau_0|\right] \times [-\mathfrak{n}_0, 0] &\subset \left[\frac{|\tau_0|^2\mathfrak{a}^2}{64\mathfrak{b}^3} + \frac{1}{4}|\tau_0|, \frac{1}{2}\zeta\right] \times [-\mathfrak{n}_0, 0] = \left[\frac{4\mathfrak{n}_0^2}{\mathfrak{b}} + \frac{1}{4}|\tau_0|, \frac{1}{2}\zeta\right] \times [-\mathfrak{n}_0, 0] \\ &\subset \bigcup_{\mathfrak{n}\in[-\frac{\zeta\mathfrak{a}}{2\mathfrak{b}}, \frac{\zeta\mathfrak{a}}{2\mathfrak{b}}]} \left[\frac{4\mathfrak{n}^2}{\mathfrak{b}} + \frac{1}{4}|\tau_0|, \frac{1}{2}\zeta\right] \times \{-\mathfrak{n}\} \subset \widetilde{\mathcal{I}}.\end{aligned} \tag{A.84}$$

In particular, (A.84) shows that for $\mathfrak{n} \in [0, \mathfrak{n}_0]$, along the portion of the level-set $\{(t,u) \mid \breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u) = -\mathfrak{n}\}$ that is contained in $[T_{\mathrm{Shock}}^{\mathrm{PS}} - \zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}} - 3\Delta^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]$, μ^{PS} ranges over an interval that contains $\left[\frac{1}{2}|\tau_0|, 2|\tau_0|\right]$.

We now study the initial value problem for the rough time function $^{(\mathfrak{n})}\tau_{\mathrm{PS}}$ (see Definition 4.5). Fix $\mathfrak{n} \in [0, \mathfrak{n}_0]$, where $\mathfrak{n}_0$ is defined by (A.45). From the diffeomorphism properties of the map $(t,u) \to (\mu^{\mathrm{PS}}, \breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})$ established above, (A.82), and (A.84), we see that for every $\tau \in [2\tau_0, \frac{1}{2}\tau_0]$, there exists a unique point $q_\tau \in \{(t,u) \mid \breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u) = -\mathfrak{n}\} \cap [T_{\mathrm{Shock}}^{\mathrm{PS}} - \zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}} - 3\Delta^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}]$ such that $\mu^{\mathrm{PS}}(q_\tau) = -\tau$ and such that the map $\tau \to (t_{q_\tau}, u_{q_\tau})$ is C^2, where (t_{q_τ}, u_{q_τ}) are the geometric coordinates of q_τ. Note that q_τ is a point on the "initial" data-hypersurface for $^{(\mathfrak{n})}\tau_{\mathrm{PS}}(t,u)$ (see Definition 4.5). In view of Definition 4.5, we see that $^{(\mathfrak{n})}\tau_{\mathrm{PS}}(t_{q_\tau}, u_{q_\tau}) = -\mu^{\mathrm{PS}}(t_{q_\tau}, u_{q_\tau}) = \tau$, i.e., the initial value of $^{(\mathfrak{n})}\tau_{\mathrm{PS}}$ at q_τ is τ. Much like in the bulk of the paper, we will use the notation $^{\mathrm{PS}}\breve{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0, \frac{1}{2}\tau_0]}$ to denote the union of the points q_τ as τ varies over the interval $[2\tau_0, \frac{1}{2}\tau_0]$, i.e.,

$$\begin{aligned}{}^{\mathrm{PS}}\breve{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0, \frac{1}{2}\tau_0]} &\overset{\mathrm{def}}{=} \left\{(t,u) \mid -\frac{1}{2}\tau_0 \leq \mu^{\mathrm{PS}}(t,u) \leq -2\tau_0\right\} \cap \left\{(t,u) \mid \breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}}(t,u) = -\mathfrak{n}\right\} \\ &\quad \cap [T_{\mathrm{Shock}}^{\mathrm{PS}} - \zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times [-U_{☢}, U_{☢}].\end{aligned} \tag{A.85}$$

Let $\gamma_{q_\tau} : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the u-parameterized integral curve of $^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}} = \frac{\widetilde{\partial}}{\widetilde{\partial} u}$ (recall that $\frac{\widetilde{\partial}}{\widetilde{\partial} u} u = 1$) that emanates from q_τ, where the target is geometric coordinate space, i.e., $\gamma_{q_\tau}(u_{q_\tau}) = (t_{q_\tau}, u_{q_\tau})$, where $\gamma_{q_\tau}(u)$ belongs to geometric coordinate space for u belonging to the interval of existence of γ_{q_τ}. Below we will show that for $\tau \in [2\tau_0, \frac{1}{2}\tau_0]$, we have:

$$\gamma_{q_\tau}(\mathbb{R}) \subset [T_{\mathrm{Shock}}^{\mathrm{PS}} - 2\zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}} - 2\Delta^{\mathrm{PS}}] \times \mathbb{R}, \tag{A.86}$$

which shows in particular that the entire integral curve is contained in the region of classical existence with respect to the geometric coordinates and is temporally separated from the Cartesian time of first blowup by at least $2\Delta^{\mathrm{PS}}$. Moreover, (A.77) implies that these integral curves are transversal to $^{\mathrm{PS}}\breve{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0, \frac{1}{2}\tau_0]}$. It follows that the map $(\tau, u) \to (t, u)$ is an injection from $[2\tau_0, \frac{1}{2}\tau_0] \times (-\infty, \infty)$ onto a subset of $[0, T_{\mathrm{Shock}}^{\mathrm{PS}}] \times \mathbb{R}$, where the image component function $t = \gamma_{q_\tau}^0(u)$ is defined to be the Cartesian time coordinate of the point $\gamma_{q_\tau}(u)$. Considering also that (A.26) implies $^{(\mathfrak{n})}\tau_{\mathrm{PS}}$ is constant along the integral curves $u \to \gamma_{q_\tau}(u)$, we see that the map $(\tau, u) \to (t, u)$ is precisely the map from adapted rough coordinates to geometric coordinates, and that its inverse is the map $^{(\mathfrak{n})}\mathscr{T}_{\mathrm{PS}}(t,u) = (^{(\mathfrak{n})}\tau_{\mathrm{PS}}, u)$ from (A.62). In addition, from this reasoning and (A.86), we also conclude (A.59).

We now prove (A.86). First, using (A.83)–(A.84), we deduce that for $\tau \in [2\tau_0, \frac{1}{2}\tau_0]$, we have:

$$t_{q_\tau} \in [T_{\mathrm{Shock}}^{\mathrm{PS}} - \zeta T_{\mathrm{Shock}}^{\mathrm{PS}}, T_{\mathrm{Shock}}^{\mathrm{PS}} - 3\Delta^{\mathrm{PS}}]. \tag{A.87}$$

Next, in view of (A.27), we see that if $0 \leq \mathfrak{n} \leq \mathfrak{n}_0$, then as u varies over $\mathbb{R}$, we can bound the total change in the Cartesian time coordinate t along γ_{q_τ}, denoted by $\Delta_{\gamma_{q_\tau}}^0$, as follows:

$$\begin{aligned}|\Delta_{\gamma_{q_\tau}}^0| &\leq \int_{u'=-\infty}^{\infty} \left|\frac{\widetilde{\partial}}{\widetilde{\partial} u} t\right| \mathrm{d}u' = \int_{u'=-\infty}^{\infty} \left|\frac{\mathfrak{n}\phi}{\frac{\partial}{\partial t}\mu^{\mathrm{PS}}}\right| \mathrm{d}u' = \int_{u'=-U_{☢}}^{U_{☢}} \left|\frac{\mathfrak{n}\phi}{\frac{\partial}{\partial t}\mu^{\mathrm{PS}}}\right| \mathrm{d}u' \leq 2\int_{|u'|\leq \frac{\mathfrak{a}}{\mathfrak{b}^2}} \frac{\mathfrak{n}}{\mathfrak{a}} \mathrm{d}u' \\ &= \frac{4\mathfrak{n}}{\mathfrak{b}^2} \leq \frac{4\mathfrak{n}_0}{\mathfrak{b}^2} = \frac{\mathfrak{a}|\tau_0|}{4\mathfrak{b}^3},\end{aligned} \tag{A.88}$$

where to obtain the second "=" and the next-to-last "≤" on RHS (A.88), we used the properties of ϕ from Definition 4.1, the estimate (A.69) for $\frac{\partial}{\partial t}\mu^{\mathrm{PS}} = L^{\mathrm{PS}}\mu^{\mathrm{PS}}$, and the definition (A.44) of $U_{☢}$, and to obtain the last "=" on RHS (A.88), we used definition (A.45). From (A.87) and (A.88), we see that for any point $\gamma_{q_\tau}(u) \in {}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{PS}$, we can bound its Cartesian time coordinate t as follows:

$$\begin{aligned} t = \gamma_{q_\tau}^0(u) &\in [T_{\text{Shock}}^{PS} - \zeta T_{\text{Shock}}^{PS} - |\Delta_{\gamma_{q_\tau}}^0|, T_{\text{Shock}}^{PS} - 3\Delta^{PS} + |\Delta_{\gamma_{q_\tau}}^0|] \\ &\subset [T_{\text{Shock}}^{PS} - \zeta T_{\text{Shock}}^{PS} - \frac{\mathfrak{a}|\tau_0|}{4\mathfrak{b}^3}|, T_{\text{Shock}}^{PS} - 3\Delta^{PS} + \frac{\mathfrak{a}|\tau_0|}{4\mathfrak{b}^3}]. \end{aligned} \tag{A.89}$$

From (A.43), (A.56), and (A.89), we conclude that if $\mathfrak{a}$ and ζ are sufficiently small, then the desired result (A.86) holds.

Next we prove (A.60)–(A.61). We first note that in simple isentropic plane-symmetry, $\frac{\partial}{\partial t}$ commutes with $\frac{\partial}{\partial u} + \phi(u)\frac{\mathfrak{n}}{\frac{\partial}{\partial t}\mu^{PS}(t,u)}\frac{\partial}{\partial t}$ and thus, by (A.23a) and (A.26), $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}$ satisfies the following transport equation:

$$\left\{\frac{\partial}{\partial u} + \phi(u)\frac{\mathfrak{n}}{\frac{\partial}{\partial t}\mu^{PS}(t,u)}\frac{\partial}{\partial t}\right\}\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}(t,u) = 0. \tag{A.90}$$

Moreover, since the same arguments used to prove (15.10) imply that $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}} = -\frac{\partial}{\partial t}\mu^{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$, we deduce from (A.69) and (A.75) the following "data-estimates:"

$$\frac{31}{32}\mathfrak{a} = \frac{31}{32}\mathring{\delta}_*^{PS} \le \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}} \le \frac{33}{32}\mathring{\delta}_*^{PS} = \frac{33}{32}\mathfrak{a}. \tag{A.91}$$

We emphasize that (A.86) implies that the data-hypersurfaces ${}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}$ and the integral curves of $\frac{\partial}{\partial u} + \phi(u)\frac{\mathfrak{n}}{\frac{\partial}{\partial t}\mu^{PS}}\frac{\partial}{\partial t}$ emanating from them are contained in the region of classical existence. The bounds stated in (A.60) now follow from the transport equation (A.90) for $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}$ and the data-estimates (A.91). To derive (A.61), we first use (A.26) to deduce the pointwise bound $\left|\frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau_{PS}(t,u)\right| \le \phi(u)\frac{\mathfrak{n}}{\left|\frac{\partial}{\partial t}\mu^{PS}(t,u)\right|}\left|\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}(t,u)\right|$. Also using (A.91) and (A.69) and the fact that $\phi \ge 0$ is supported on $\{|u| \le U_{\clubsuit}\}$ and bounded by 1, we further deduce that $|\frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau_{PS}(t,u)| \le 2\mathfrak{n} \le 2\mathfrak{n}_0$. From this bound and (A.45), we conclude (A.61).

We now exhibit the diffeomorphism properties of the map $(\tau,u) \to (t,u)$ and its inverse ${}^{(\mathfrak{n})}\mathscr{T}_{PS}$. Straightforward calculations show that $\det\frac{\widetilde{\partial}(t,u)}{\widetilde{\partial}(\tau,u)} = \frac{\widetilde{\partial}}{\partial\tau}t$. By the chain rule, we see that $\frac{\widetilde{\partial}}{\partial\tau}t = \frac{1}{\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau}$ and thus the estimate (A.60) implies that the matrix is $\frac{\widetilde{\partial}(t,u)}{\widetilde{\partial}(\tau,u)}$ is invertible. In view of the injectivity established shortly after (A.86), we conclude that the map $(\tau,u) \to (t,u)$ is a diffeomorphism from $[2\tau_0,\frac{1}{2}\tau_0]\times(-\infty,\infty)$ onto its image ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}^{PS}$, and that its inverse map ${}^{(\mathfrak{n})}\mathscr{T}_{PS}(t,u) = ({}^{(\mathfrak{n})}\tau,u)$ is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}^{PS}$ onto $[2\tau_0,\frac{1}{2}\tau_0]\times(-\infty,\infty)$. Moreover, from the identity $\det\frac{\widetilde{\partial}({}^{(\mathfrak{n})}\tau_{PS},u)}{\widetilde{\partial}(t,u)} = \frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}$ and the estimate (A.60), we also conclude (A.64).

Now that we have shown that ${}^{(\mathfrak{n})}\mathscr{T}_{PS}$ is a diffeomorphism on ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}^{PS}$, we can derive C^3 estimates for the map, i.e., we can control the up-to-third-order derivatives of ${}^{(\mathfrak{n})}\tau_{PS}$. Our argument has some commonalities with our proof of Lemma 15.1 and relies on the bound $\|\mu^{PS}\|_{C_{geo}^3({}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}^{PS})} \lesssim 1$, which follows from (A.17) and (A.52a)–(A.52b). To proceed, we first recall that ${}^{(\mathfrak{n})}\tau|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}} = -\mu^{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$. We also note that the same arguments used to prove (15.10) imply that $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau_{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}} = -\frac{\partial}{\partial t}\mu^{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$ and $\frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau_{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}} = -\frac{\partial}{\partial u}\mu^{PS}|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$. Next, we note that the vectorfield $\check{V} \stackrel{\text{def}}{=} (\check{X}^{PS}\check{X}^{PS}\mu^{PS})\frac{\partial}{\partial t} - (\frac{\partial}{\partial t}\check{X}^{PS}\mu^{PS})\check{X}^{PS}$ satisfies $\check{V}\check{X}^{PS}\mu^{PS} = 0$ and thus is tangent to the curve ${}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}$. From this fact, the identities for ${}^{(\mathfrak{n})}\tau|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$, $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$, and $\frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$ noted above, and the estimate $\|\mu^{PS}\|_{C_{geo}^3({}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}^{PS})} \lesssim 1$ noted above, we deduce that ${}^{(\mathfrak{n})}\tau|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$, $\frac{\partial}{\partial t}{}^{(\mathfrak{n})}\tau|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$, $\frac{\partial}{\partial u}{}^{(\mathfrak{n})}\tau|_{{}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}}$ and the up-to-second-order derivatives of all three of these "data functions" with respect to $\check{V}$ are pointwise bounded by $\le C$ along ${}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}$. Using these bounds, (A.74), and the transport equation (A.26) to solve for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial u}$ in terms of $\check{V}$ and ${}^{(\mathfrak{n})}\check{W}^{PS}$ along ${}^{PS}\check{\mathbb{X}}_{-\mathfrak{n}}^{[2\tau_0,\frac{1}{2}\tau_0]}$, and using the bound $\|\mu^{PS}\|_{C_{geo}^3({}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)}^{PS})} \lesssim 1$ mentioned above, we deduce that $\|^{(\mathfrak{n})}\tau_{\mathrm{PS}}\|_{C^3_{\mathrm{geo}}(^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)})} \lesssim 1$. Using this "data bound," we can differentiate the transport equation (A.26) up to three times with the elements of $\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right\}$ and use the aforementioned bound $\|\mu^{\mathrm{PS}}\|_{C^3_{\mathrm{geo}}(^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)})} \lesssim 1$ and (A.69) (which implies that $-L\mu^{\mathrm{PS}} \approx \mathfrak{a}$ on the support of ϕ), as well as a standard argument based on Grönwall's inequality (with respect to u, since the transport operator $^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}$ on LHS (A.26) satisfies $^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}u = 1$), thereby deducing that $\|^{(\mathfrak{n})}\breve{W}^{\mathrm{PS}}\|_{C^3_{\mathrm{geo}}(^{(\mathfrak{n})}\mathcal{M}^{\mathrm{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],(-\infty,\infty)})} \lesssim 1$. From this bound and definition (A.62), we conclude (A.63).

Next, we note that the function $\mathfrak{t}^{\mathrm{PS}}_{\tau,\mathfrak{n}}$ from the statement of the theorem is the first component of the map $u \to {}^{(\mathfrak{n})}\mathscr{T}^{-1}_{\mathrm{PS}}(\tau, u)$. Moreover, (A.86) implies that $\mathfrak{t}^{\mathrm{PS}}_{\tau,\mathfrak{n}}(\mathbb{R}) \in [T^{\mathrm{PS}}_{\mathrm{Shock}} - 2\zeta T^{\mathrm{PS}}_{\mathrm{Shock}}, T^{\mathrm{PS}}_{\mathrm{Shock}} - 2\Delta^{\mathrm{PS}}]$. Since (A.63) and the diffeomorphism properties of $^{(\mathfrak{n})}\mathscr{T}_{\mathrm{PS}}$ imply that $\|^{(\mathfrak{n})}\mathscr{T}^{-1}_{\mathrm{PS}}\|_{C^3([2\tau_0,\frac{1}{2}\tau_0]\times(-\infty,\infty))} \lesssim 1$, we conclude that for $\tau \in [2\tau_0, \frac{1}{2}\tau_0]$, $\|\mathfrak{t}^{\mathrm{PS}}_{\tau,\mathfrak{n}}\|_{C^3_u((-\infty,\infty))} \lesssim 1$ and that the level-sets $^{(\mathfrak{n})}\widetilde{\Sigma}^{\mathrm{PS}}_{\tau}$ are the graphical surfaces in (A.58).

Proof of (A.65)**:** Using the chain rule relation $\frac{\partial(\mu^{\mathrm{PS}},\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(\tau,u)} = \frac{\partial(\mu^{\mathrm{PS}},\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(t,u)} \cdot \left(\frac{\partial(\tau,u)}{\partial(t,u)}\right)^{-1}$ and the estimates (A.80) and (A.60)–(A.61), we compute that for $(\tau, u) \in [2\tau_0, \frac{1}{2}\tau_0] \times [-U_{☢}, U_{☢}]$, we have:

$$\frac{\partial(\mu^{\mathrm{PS}}, \breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(\tau, u)} = \begin{pmatrix} -1 & 0 \\ 0 & \mathfrak{b} \end{pmatrix} + \mathcal{E}, \tag{A.92}$$

where the entries of the "error matrix" $\mathcal{E} \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix}$ satisfy the following estimates:

$$|\mathcal{E}_{11}| \leq \frac{1}{8}, \qquad |\mathcal{E}_{12}|, |\mathcal{E}_{21}|, |\mathcal{E}_{22}| = \mathcal{O}(\mathfrak{a}) + \mathcal{O}(\zeta). \tag{A.93}$$

From (A.92) and (A.93), we deduce that if $(\tau, u) \in [2\tau_0, \frac{1}{2}\tau_0] \times [-U_{☢}, U_{☢}]$, and if $\mathfrak{a}$ and ζ are sufficiently small, then the Jacobian matrix $^{(\mathfrak{n})}\mathbf{J}_{\mathrm{PS}}(\tau, u) \stackrel{\mathrm{def}}{=} \frac{\partial(\mu^{\mathrm{PS}},\breve{X}^{\mathrm{PS}}\mu^{\mathrm{PS}})}{\partial(\tau,u)}$ is invertible, and that for every pair of points $(\tau_1, u_1), (\tau_2, u_2) \in [2\tau_0, \frac{1}{2}\tau_0] \times [-U_{☢}, U_{☢}]$, we have $\left|^{(\mathfrak{n})}\mathbf{J}^{-1}_{\mathrm{PS}}(\tau_1, u_1)^{(\mathfrak{n})}\mathbf{J}_{\mathrm{PS}}(\tau_2, u_2) - \mathrm{ID}\right|_{\mathrm{Euc}} \leq \frac{1}{4}$, which is the desired bound (A.65).

Proof of the properties of $\mathscr{M}_{\mathrm{PS}}$ and (A.66a)–(A.66b)**,** (A.67)**, and** (A.68)**:** These results follow from the "intermediate step" mentioned above, including the estimates (A.80)–(A.81).

□

Corollary A.6 (The data-assumptions of Sect. 11.2 are satisfied)**.** *The solutions provided by Theorem A.4 induce data on* $^{(\mathfrak{n})}\widetilde{\Sigma}^{[-U_1,U_2]}_{\tau_0}$ *that satisfy all the assumptions stated in Sect. 11.2, where:*

1. $\bar{\varrho} > 0$ *is a fixed constant density (see Def. 2.4);*
2. U_1*,* U_2*,* $U_{☢}$*,* τ_0*, and* $\mathfrak{n}_0$ *are as in the statement of the theorem;*
3. *The parameters* $\mathring{\alpha}^{\mathrm{PS}}$*,* $\mathring{\delta}^{\mathrm{PS}}_*$*,* $\mathring{\delta}^{\mathrm{PS}}$*,* $\mathring{\mathfrak{e}}$*,* M^{PS}_2*, and* $\mathfrak{m}^{\mathrm{PS}}_1$ *can be chosen to satisfy the following relations, where the implicit constants can depend on the function* $\mathring{\phi}$ *from the statement of the theorem:*
   - $\mathring{\alpha}^{\mathrm{PS}} \lesssim \mathfrak{a}$
   - $\mathring{\delta}^{\mathrm{PS}}_* = \mathfrak{a}$
   - $\mathring{\delta}^{\mathrm{PS}} \lesssim \mathfrak{a}$
   - $\mathring{\mathfrak{e}} = 0$
   - $M^{\mathrm{PS}}_2 = \min\{\frac{\mathfrak{b}}{2}, \frac{1}{2\mathfrak{b}}\}$ *(note that by* (A.43)–(A.45)*, the assumption* (10.8) *is satisfied)*
   - $\mathfrak{m}^{\mathrm{PS}}_1 = \frac{1}{2}(1 - \mathfrak{p})$*;*

   The following strict improvements[67] *of estimates stated in stated in Sect. 11.2 hold, where "strict" means that the estimates guaranteed by Theorem A.4 have* smaller-in-magnitude *constants than the corresponding estimates stated in Sect. 11.2:*
4. - (A.69) *is an improved version of* (11.22)*;*
   - (A.65) *is an improved version of* (11.23)*;*
   - (A.68) *is an improved version of* (11.24)*;*
   - *Considering* (A.43)*,* (A.47)*, and* (A.56)*, we see that* (A.59) *yields an improved version of* (11.17a)*;*

[67] We exploit these strict improvements in Appendix B, where we use Cauchy stability to show that the assumptions of Sect. 11.2 are satisfied by perturbations of the solutions from Theorem A.4.

- (A.53) *is an improved version of* (11.17b).

Proof. All aspects of the corollary, aside from the estimate (11.14b), can be deduced by comparing the results provided by Theorem A.4 with the assumptions stated in Sect. 11.2, and using the following facts:

- The solution vanishes on the complement of the region $\{(t,u) \in \mathbb{R} \times \mathbb{R} \mid u \in [-U_1, U_2]\}$, and in particular is trivial along the null hypersurface $\mathcal{P}_{-U_1}^{[0,\infty)}$.
- By (A.59), the hypersurface ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{\text{PS}}$ is contained in the region of classical existence relative to the geometric coordinates (where the estimates of the theorem hold).
- Relative to the geometric coordinates, $\mathcal{R}_{(+)}^{\text{PS}}$ depends only on u.
- Along ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{\text{PS}}$, t is a function of u (see (A.58)).

The estimate (11.14b) (with $\mathring{\varepsilon} = 0$, $\mathring{\delta}_*^{\text{PS}}$ in the role of $\mathring{\delta}_*$, $\breve{X}^{\text{PS}}$ in the role of $\breve{X}$, and μ^{PS} in the role of μ) follows from the facts noted above, (A.18b), (A.23a)–(A.23b), and the fact that in the context of Theorem A.4, we have $L^{\text{PS}} = \frac{\partial}{\partial t}$, $\breve{X}^{\text{PS}} = \frac{\partial}{\partial u}$, and $0 \leq t \leq \frac{1}{\mathring{\delta}_*^{\text{PS}}}$.

□

Definition A.7 (Admissible background solutions)**.** We refer to the compactly supported solutions furnished by Theorem A.4 as "admissible background solutions." We consider their parameters $\mathring{\alpha}^{\text{PS}}$, $\mathring{\delta}_*^{\text{PS}}$, etc. to be the ones guaranteed by Cor. A.6.

A.7. **The Cauchy stability region.** Fix any of the admissible "background" (shock-forming) simple isentropic plane-symmetric solutions from Def. A.7. Recall that these solutions are supported in the strip $\{(t,u) \in \mathbb{R} \times \mathbb{R} \mid u \in [-U_1, U_2]\}$. In this section, we discuss the behavior of the solution in a "Cauchy stability region," which is a large region close to the singularity where the background solution exists classically. In particular, we describe Fig. 16, which will guide our discussion in Appendix B. In Appendix B, we will use Cauchy stability arguments to show that perturbations of the background solution stay close to it, in all relevant norms, in the Cauchy stability region. From the Cauchy stability estimates, it will follow that there exist open sets data – without symmetry – satisfying the assumptions stated in Sect. 11.2. In the rest of this section (and also in Appendix B), we view the background solutions as solutions in three spatial dimensions that are independent of the torus coordinates (x^2, x^3).

Let $\mathring{\delta}_*^{\text{PS}}$ denote the quantity (11.6) evaluated at the (shock-forming) background solution. In Theorem A.4, we showed that relative to the geometric coordinates, the first singular point for the background solution occurs at $(t,u) = (T_{\text{Shock}}^{\text{PS}}, 0)$, where $T_{\text{Shock}}^{\text{PS}} = \frac{1}{\mathring{\delta}_*^{\text{PS}}}$. We define:

$$U_0 \stackrel{\text{def}}{=} U_1 + \frac{18}{\mathring{\delta}_*^{\text{PS}}}. \tag{A.94}$$

For $0 \leq t_1 \leq t_2$, we define (see Def. 3.2):

$$\mathfrak{CS}_{\text{Main}}^{[t_1,t_2]} \stackrel{\text{def}}{=} \bigcup_{t \in [t_1,t_2]} \Sigma_t^{[-U_1,U_2]}, \tag{A.95a}$$

$$\mathfrak{CS}_{\text{Small}}^{[t_1,t_2]} \stackrel{\text{def}}{=} \bigcup_{t \in [t_1,t_2]} \Sigma_t^{[3t-U_0,-U_1]}, \tag{A.95b}$$

$$\begin{aligned} \underline{\mathcal{S}}^{[t_1,t_2]} &\stackrel{\text{def}}{=} \left\{(t,u,x^2,x^3) \mid 3t-u = U_0, t \in [t_1,t_2], \text{ and } (x^2,x^3) \in \mathbb{T}^2\right\} \\ &= \bigcup_{t \in [t_1,t_2]} \ell_{t,3t-U_0}. \end{aligned} \tag{A.95c}$$

Note that $\underline{\mathcal{S}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$ is the right boundary of $\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$; see Fig. 16.

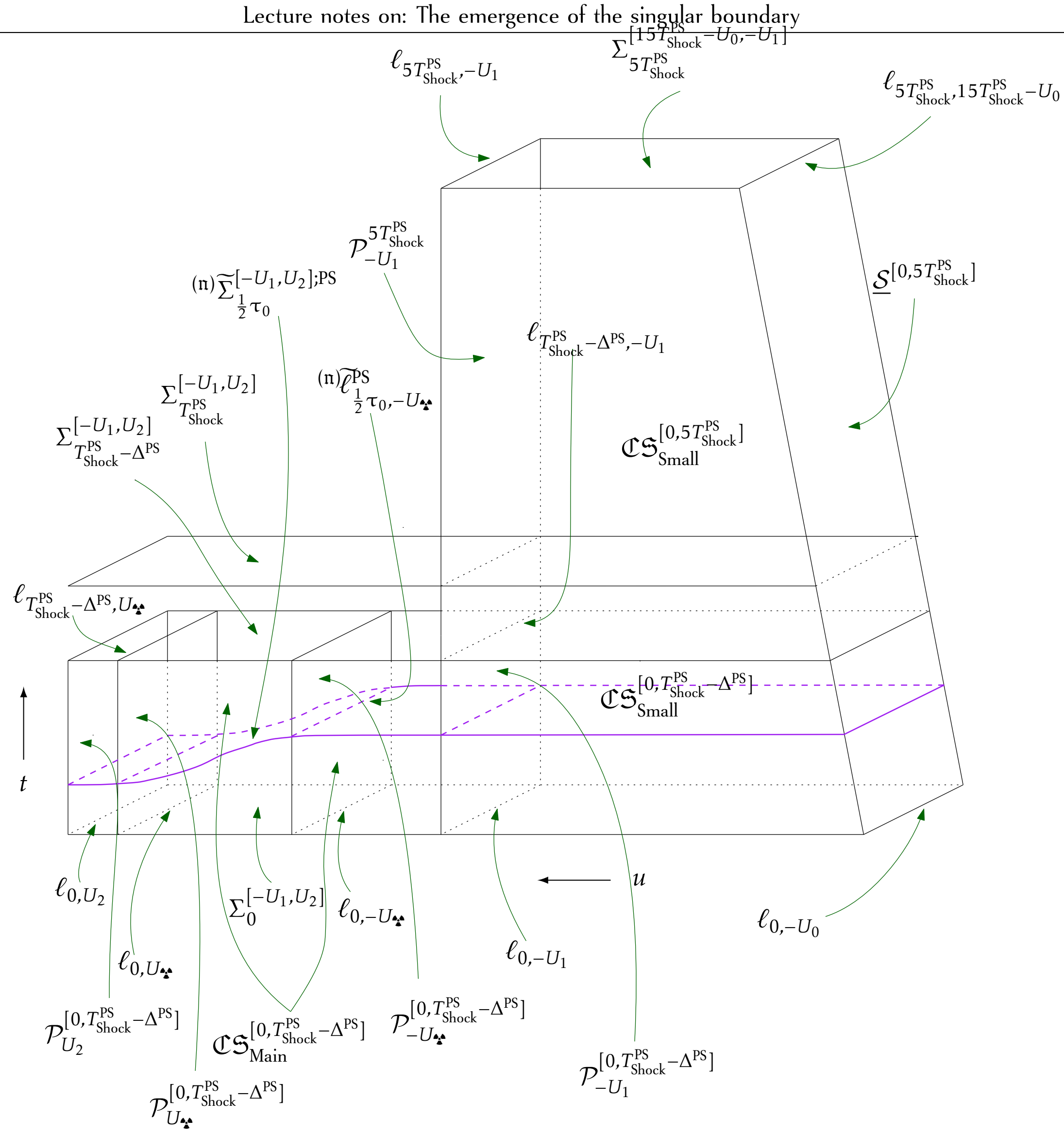


Figure 16. The Cauchy stability region $\mathfrak{CS}^{T^{\mathrm{PS}}_{\mathrm{Shock}};\Delta^{\mathrm{PS}}}$ in geometric coordinate space, not drawn to scale

Consider the "Cauchy stability region" $\mathfrak{CS}^{T^{\mathrm{PS}}_{\mathrm{Shock}};\Delta^{\mathrm{PS}}}$ depicted relative to the geometric coordinates (t,u,x^2,x^3) in Fig. 16, where $\Delta^{\mathrm{PS}} > 0$ is the small constant defined in (A.56). We decompose this region into various sub-regions:

$$\mathfrak{CS}^{T^{\mathrm{PS}}_{\mathrm{Shock}};\Delta^{\mathrm{PS}}} \overset{\mathrm{def}}{=} \mathfrak{CS}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-\Delta^{\mathrm{PS}}]}_{\mathrm{Main}} \cup \mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}}, \tag{A.96a}$$

$$\mathfrak{CS}^{[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}} = \mathfrak{CS}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-\Delta^{\mathrm{PS}}]}_{\mathrm{Small}} \cup \mathfrak{CS}^{[T^{\mathrm{PS}}_{\mathrm{Shock}}-\Delta^{\mathrm{PS}},5T^{\mathrm{PS}}_{\mathrm{Shock}}]}_{\mathrm{Small}} = \bigcup_{t\in[0,5T^{\mathrm{PS}}_{\mathrm{Shock}}]} \Sigma_t^{[3t-U_0,-U_1]}, \tag{A.96b}$$

$$\widehat{\mathfrak{CS}}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-\Delta^{\mathrm{PS}}]} \overset{\mathrm{def}}{=} \mathfrak{CS}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-\Delta^{\mathrm{PS}}]}_{\mathrm{Main}} \cup \mathfrak{CS}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-\Delta^{\mathrm{PS}}]}_{\mathrm{Small}} = \bigcup_{t\in[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-\Delta^{\mathrm{PS}}]} \Sigma_t^{[3t-U_0,U_2]}. \tag{A.96c}$$

For future use, we also define:

$$\widehat{\mathfrak{CS}}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-2\Delta^{\mathrm{PS}}]} \overset{\mathrm{def}}{=} \mathfrak{CS}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-2\Delta^{\mathrm{PS}}]}_{\mathrm{Main}} \cup \mathfrak{CS}^{[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-2\Delta^{\mathrm{PS}}]}_{\mathrm{Small}} = \bigcup_{t\in[0,T^{\mathrm{PS}}_{\mathrm{Shock}}-2\Delta^{\mathrm{PS}}]} \Sigma_t^{[3t-U_0,U_2]} \tag{A.97}$$

Note that (A.47) and (A.94) imply that the top boundary of $\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$, namely $\Sigma_{5T_{\text{Shock}}^{\text{PS}}}^{[15T_{\text{Shock}}^{\text{PS}}-U_0,-U_1]}$, has u-width equal to $3T_{\text{Shock}}^{\text{PS}} = \frac{3}{\mathring{\delta}_*^{\text{PS}}}$.

The background solutions provided by Theorem A.4 are smooth in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ and are trivial in the sub-region $\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$. That is, for the background solutions, in $\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$, both Riemann invariants identically vanish, $\mu \equiv 1$, etc. This ensures, in particular, that the surface portions $\underline{\mathcal{S}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$ are $\mathbf{g}$-spacelike with respect to the acoustical metric of the background. To see this, we compute that a future-directed normal (not of unit-length) to $\underline{\mathcal{S}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$ is $3\mathbf{B} - \frac{1}{\mu}L = 2L + 3X$, where we have used (3.24) and the fact that $\mu \equiv 1$ in the trivial region $\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$ (which contains $\underline{\mathcal{S}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$). We can therefore use Lemma 3.9 to compute that $\mathbf{g}(2L + 3X, 2L + 3X) = -3$, which indeed implies that $3\mathbf{B} - \frac{1}{\mu}L$ is $\mathbf{g}$-timelike along $\underline{\mathcal{S}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$.

To help prepare the reader for Appendix B, we now further discuss some aspects of Fig. 16 that follow from the conclusions of Theorem A.4. Let ${}^{(\mathfrak{n})}\tau_{\text{PS}}$ denote the rough time function of the background, let ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[u_1,u_2];\text{PS}}$ denote the level-set portions $\{{}^{(\mathfrak{n})}\tau_{\text{PS}} = \tau\} \cap \{u \in [u_1, u_2]\}$, and let ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u'}^{\text{PS}} = \{{}^{(\mathfrak{n})}\tau_{\text{PS}} = \tau\} \cap \{u = u'\}$ denote the background rough tori. The background solution, though smooth, is "about" to form a shock in $\mathfrak{CS}_{\text{Main}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$ near $u = 0$. Moreover, if $\tau_0 < 0$ and $\mathfrak{n}_0$ are sufficiently small as in the statement of the theorem, then (A.59) shows that for $\mathfrak{n} \in [0, \mathfrak{n}_0]$, the background rough time functions ${}^{(\mathfrak{n})}\tau_{\text{PS}}$ are defined on a subset of $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ such that for $\tau \in [2\tau_0, \frac{\tau_0}{2}]$, the level-set portions ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_0,U_2];\text{PS}}$ are contained in the subset $\bigcup_{t\in[T_{\text{Shock}}^{\text{PS}}-2\zeta T_{\text{Shock}}^{\text{PS}}, T_{\text{Shock}}^{\text{PS}}-2\Delta^{\text{PS}}]} \Sigma_t^{[-U_0,U_2]}$ of $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$. In particular, all the hypersurface portions ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_0,U_2];\text{PS}}$ are temporally separated from $\Sigma_{T_{\text{Shock}}^{\text{PS}}}$ by a distance at least equal to $2\Delta^{\text{PS}}$. Note that for the background solution, if we followed it all the way to the first singular point (which is contained in ${}^{(0)}\widetilde{\Sigma}_0^{[-U_0,U_2];\text{PS}}$), we would have ${}^{(0)}\widetilde{\Sigma}_0^{[-U_0,U_2];\text{PS}} = \Sigma_{T_{\text{Shock}}^{\text{PS}}}^{[-U_0,U_2]}$. This is because in plane-symmetry, by (A.26) with $\mathfrak{n} = 0$, ${}^{(0)}\tau_{\text{PS}}$ can be expressed as a function of t alone.

Appendix B. The existence of an open set of data satisfying the assumptions

By Cor. A.6, there exists a large family of isentropic plane-symmetric initial data on Σ_0 such that the corresponding background solutions (and corresponding parameters) induce data on the background-solution-dependent rough hypersurface ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2];\text{PS}}$ and null hypersurface portion $\mathcal{P}_{-U_1}^{[0,\frac{5}{\mathring{\delta}_*^{\text{PS}}}]}$ that satisfy the assumptions in Sects. 11.2–11.2.3 and the parameter-size assumptions of Sect. 10.2 with $\mathring{\epsilon} = 0$. In this section, we sketch a proof of Prop. B.2, which shows that if one perturbs – without symmetry, irrotationality, or isentropicity assumptions – the background initial data on Σ_0, then the corresponding perturbed solutions also induce data that satisfy the assumptions in Sects. 11.2.1–11.2.3 and the parameter-size assumptions of Sect. 10.2 with $\mathring{\epsilon}$ non-negative but small. In conjunction with Theorem 34.1, this shows that our main results hold for open sets of solutions.

Remark B.1 (We can choose the smallness of $|\tau_0|$)**.** Theorem A.4 implies that we can choose and fix the parameter $\tau_0 = -\mathfrak{m}_0 < 0$ to be as close to 0 as we want. For perturbed solutions, the smallness of $|\tau_0|$ corresponds to assuming that their initial rough hypersurfaces ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$ are close to the singularity of the background solution. While such smallness is not essential for our analysis of perturbed solutions, it is helpful because it allows us to simplify the proofs of various estimates in the bulk of the paper, i.e., it allows us to exploit that we only have to control perturbed solutions for $|\tau_0|$ amounts of rough time.

Proposition B.2 (Cauchy stability and the existence of open sets of data satisfying our assumptions)**.** *Fix any of the admissible "background" (shock-forming) simple isentropic plane-symmetric solutions from Def. A.7. Recall (see Cor. A.6) that* $\overline{\varrho}$, U_1, U_2, $U_{\clubsuit}$, $\mathfrak{n}_0$, $|\tau_0| = -\tau_0$, $\mathring{\alpha}^{\text{PS}}$, $\mathring{\delta}_*^{\text{PS}}$, $\mathring{\delta}^{\text{PS}}$, M_2^{PS}, *and* $\mathfrak{m}_1^{\text{PS}}$ *are* ***positive*** *parameters associated to the background solution and that* $U_0 \stackrel{def}{=} U_1 + \frac{18}{\mathring{\delta}_*^{\text{PS}}}$ *(see* (A.94)*). Recall that we can choose and fix* $|\tau_0|$ *(which is equal to* $\mathfrak{m}_0$*) to be as small as we want; see Remark B.1. Let* $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ *be the norm of the bona fide data perturbation defined in* (11.4)*. If* $\Delta^{\text{PS}} > 0$ *is the small constant defined in* (A.56)*, then for all sufficiently small* $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1} > 0$ *(where the required smallness depends on the background* *solution), the perturbed fluid solution, the eikonal function u, and all of the auxiliary geometric quantities constructed out of u (such as μ, L^i, and χ) exist classically in the Cauchy stability region $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}} = \mathfrak{CS}_{\text{Main}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]} \cup \mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}$ (which we view to be a fixed subset of geometric coordinate space) defined in* (A.96a). *Moreover, for $\mathfrak{n} \in [0,\mathfrak{n}_0]$ each perturbed time function ${}^{(\mathfrak{n})}\tau$ exists classically on a subset of $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ such that for $\tau \in [2\tau_0, \frac{1}{2}\tau_0]$, the level-set portions ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}$ are contained in the subset $\mathfrak{CS}_{\text{Main}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$ defined in* (A.95a). *In particular, the perturbed solution exists classically on ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],[-U_1,U_2]}$.*

Moreover, for the perturbed solution, we define $\mathring{\delta}_*$ by (11.6)*, we define $\mathfrak{m}_1$, $\mathring{\delta}$, and M_2 by:*[68]

$$\mathring{\alpha} \stackrel{def}{=} 2\mathring{\alpha}^{\text{PS}}, \qquad \mathring{\delta} \stackrel{def}{=} 2\mathring{\delta}^{\text{PS}}, \qquad M_2 \stackrel{def}{=} \frac{1}{2}M_2^{\text{PS}}, \qquad \mathfrak{m}_1 \stackrel{def}{=} \frac{1}{2}\mathfrak{m}_1^{\text{PS}}, \tag{B.1}$$

and we define the remaining parameters to be the same as for the background solution. Then the following estimate holds:

$$\mathring{\delta}_* = \mathring{\delta}_*^{\text{PS}} + \mathcal{O}\left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right), \tag{B.2}$$

and for these parameters, the perturbed solutions induce data on the perturbed rough hypersurface ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, the null hypersurface portion $\mathcal{P}_{-U_1}^{[0,4\mathring{\delta}_*]}$, and the perturbed rough tori ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$ that satisfy all of the assumptions of Sects. 11.2.1–11.2.3 with $\mathring{\epsilon} = \mathcal{O}\left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)$, where the implicit constants in "$\mathcal{O}(\cdot)$" depend on the background solution and $|\tau_0|$ (the implicit constants can blow up as $|\tau_0| \downarrow 0$).

Proof sketch. **Overview of the main ideas of the proof**. Because the background solutions satisfy the assumptions of Sects. 11.2.1–11.2.3 with $\mathring{\epsilon} = 0$, most aspects of the proposition follow from standard arguments based on Cauchy stability. The only non-standard aspects are the following, which we flesh out in Steps 1–5 below:

- (**Estimates - without derivative loss - on flat spacelike hypersurfaces and null hypersurfaces**) We need to show that in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$, we can control the solution with respect to the geometric coordinates (t,u,x^2,x^3) up to top-order, i.e., without losing a derivative relative to the data norm $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ defined in (11.4). This is essentially a much easier version of the proofs of the energy estimates of Props. 24.1, 24.2, 24.3, and 24.4, except there is one new conceptually important aspect of the tori energy estimates (i.e., the analog of the estimates of Prop. 24.3), described below. The main simplification compared to the bulk of the paper is that μ is uniformly positive in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ (see (B.3)), and this positivity allows one to use standard arguments based on Grönwall's inequality to show that the energies and null fluxes grow at most exponentially.
- (**Estimates - without derivative loss - on tori**) We need to adequately control the solution up to top-order on various tori. This control does not directly come from the energy estimates described in the previous step. We obtain the desired control by combining averaging arguments based on Chebychev's inequality with elliptic-hyperbolic identities in the spirit of (21.63).

Throughout this proof, we view $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ to be a fixed subset of geometric coordinate space. We will tacitly assume that $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ is sufficiently small. We will also freely use notation defined in Sect. A.7 and the results of Theorem A.4 for the background solution.

Step 1: Standard Cauchy stability with respect to the Cartesian coordinates in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ By Cauchy stability, if $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ is small enough, then the perturbed fluid solutions vary continuously with respect to the fluid data in the Cauchy stability region $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ depicted in Fig. 16 (see also (A.96a)). Similarly, the eikonal function u, which solves the fully nonlinear transport equation (3.1), varies continuously in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ with respect to the fluid data (here we are viewing u as a function of the Cartesian coordinates). Cauchy stability results of this type are standard if one measures continuity using topologies corresponding to Sobolev spaces respect to the Cartesian coordinates on portions of the hypersurfaces Σ_t of constant Cartesian time that are contained in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$; see [65, Proposition 9.17] for a detailed

[68]The parameters that serve as lower bounds in our PDE analysis are defined to be half the value of the corresponding background parameters, while parameters that serve as upper bounds are defined to be twice the value of the corresponding background parameter.

proof in the context of Einstein's equations. We emphasize that the present **Step 1 yields the continuous dependence of the solution only with respect to standard Sobolev norms defined through Cartesian partial derivatives**. In particular, this "Cartesian approach" does not yield the top-order regularity of the solution with respect to the geometric coordinates; in the remaining steps, we will obtain the desired top-order regularity in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$. We clarify that even though the eikonal function $u = u(t,x^1,x^2,x^3)$ varies slightly with the solution, domain of dependence considerations imply that $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ is always a development of the portion of the initial data on the fixed subset $\Sigma_0^{[-U_0,U_2]}$ of Σ_0.

Next, we highlight that all of the auxiliary geometric quantities constructed out of u, such as μ, L^i, and χ, also vary continuously[69] with respect to the fluid data. In particular, since the background solution's inverse foliation density μ^{PS} is strictly positive in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$, the perturbed solution satisfies:

$$\mu|_{\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}} \gtrsim 1, \tag{B.3}$$

where the implicit constants in (B.3) depend on the parameters of the background solution. Similarly, since $\mu^{\text{PS}}|_{\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}} = 1$, it is a standard consequence of Cauchy stability that the perturbed solution satisfies:

$$\mu|_{\mathfrak{CS}_{\text{Small}}^{[0,5T_{\text{Shock}}^{\text{PS}}]}} = 1 + \mathcal{O}\left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right). \tag{B.4}$$

Furthermore, if $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ is small, then, like the map $\Upsilon_{\text{PS}}(t,u) = (t,x^1)$ from Theorem A.4, the perturbed change of variables map $\Upsilon(t,u,x^2,x^3) = (t,x^1,x^2,x^3)$ defined in (5.1) is a diffeomorphism on $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$. Hence, from the chain rule and the standard Sobolev calculus, it follows that for the perturbed solution, we can view the following quantities as functions of the geometric coordinates (t,u,x^2,x^3), and for a neighborhood of the data of the background solution, they all vary continuously with respect to the fluid data in various Sobolev and Lebesgue norms corresponding to the geometric coordinates: all of the fluid variables, the eikonal function, all of the auxiliary quantities, such as μ, L^i, χ, etc. constructed out of the eikonal function (the auxiliary quantities solve various geometric PDEs), and the Cartesian coordinates (which we view to be functions of the geometric coordinates, where those functions depend on the perturbed solution). In particular, all the perturbed solutions exist and are smooth on the common domain $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$ in geometric coordinate space.

Next, we note that by Theorem A.4, for $\mathfrak{n} \in [0,\mathfrak{n}_0]$ (where $\mathfrak{n}_0$ is as in the theorem), the background change of variables map ${}^{(\mathfrak{n})}\mathscr{T}_{\text{PS}}$ is a diffeomorphism from the subset ${}^{(\mathfrak{n})}\mathcal{M}^{\text{PS}}_{[2\tau_0,\frac{1}{2}\tau_0],[-U_0,U_2]}$ of $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-2\Delta^{\text{PS}}]}$ onto $[2\tau_0,\frac{1}{2}\tau_0]\times[-U_0,U_2]\times\mathbb{T}^2$, where the set $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-2\Delta^{\text{PS}}]}$ is defined in (A.97) and is contained in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$. We clarify that here, we are lifting the map ${}^{(\mathfrak{n})}\mathscr{T}_{\text{PS}}$ to three spatial dimensions by the formula ${}^{(\mathfrak{n})}\mathscr{T}_{\text{PS}}(t,u,x^2,x^3) = \left({}^{(\mathfrak{n})}\tau_{\text{PS}}(t,u,x^2,x^3),u,x^2,x^3\right)$. Consider now the perturbed change of variables map ${}^{(\mathfrak{n})}\mathscr{T}(t,u,x^2,x^3) = \left({}^{(\mathfrak{n})}\tau(t,u,x^2,x^3),u,x^2,x^3\right)$ defined in (5.2), where the perturbed rough time function ${}^{(\mathfrak{n})}\tau$ solves the initial value problem (4.4a)–(4.4b). We clarify that by the diffeomorphism properties of Υ described in the previous paragraph, we can view the perturbed ${}^{(\mathfrak{n})}\tau$ and ${}^{(\mathfrak{n})}\mathscr{T}$ as functions of (t,u,x^2,x^3). By the results of Theorem A.4 and Cauchy stability, if $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ is sufficiently small, then for $\mathfrak{n} \in [0,\mathfrak{n}_0]$, ${}^{(\mathfrak{n})}\mathscr{T}$ is a diffeomorphism from a subset of the set $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$ (which is defined in (A.96c), contained in $\mathfrak{CS}^{T_{\text{Shock}}^{\text{PS}};\Delta^{\text{PS}}}$, and contains $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-2\Delta^{\text{PS}}]}$) onto $[2\tau_0,\frac{1}{2}\tau_0]\times[-U_0,U_2]\times\mathbb{T}^2$. That is, ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],[-U_0,U_2]} \subset \widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$, and ${}^{(\mathfrak{n})}\mathscr{T}$ is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],[-U_0,U_2]}$ (which is equal to $\left\{(t,u,x^2,x^3) \mid 2\tau_0 \leq {}^{(\mathfrak{n})}\tau(t,u,x^2,x^3) \leq \frac{1}{2}\tau_0,\, u \in [-U_0,U_2]\right\}$) onto $[2\tau_0,\frac{1}{2}\tau_0]\times[-U_0,U_2]\times\mathbb{T}^2$. In view of definition (A.95a) of $\mathfrak{CS}_{\text{Main}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$, we also have the following weaker result: ${}^{(\mathfrak{n})}\mathscr{T}$ is a diffeomorphism from ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],[-U_1,U_2]}$ onto $[2\tau_0,\frac{1}{2}\tau_0]\times[-U_1,U_2]\times\mathbb{T}^2$, where ${}^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0]} \subset \mathfrak{CS}_{\text{Main}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$ (i.e., for $\tau \in [2\tau_0,\frac{1}{2}\tau_0]$, the level-set portions ${}^{(\mathfrak{n})}\widetilde{\Sigma}_\tau^{[-U_1,U_2]}$ are contained in the subset $\mathfrak{CS}_{\text{Main}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$, as is claimed in the proposition).

In total, these Cauchy stability arguments yield all the conclusions of Prop. B.2, except for the following top-order energy estimates, which we will derive in the remaining steps:

[69]All of these quantities can be expressed in terms of the fluid variables, u, and their Cartesian coordinate partial derivatives.

- The L^2 estimates at the highest derivative level in (11.11a)–(11.11d), (11.12a)–(11.12c), (11.13a)–(11.13c), and (11.16a)–(11.16b).

Step 2: Estimates – without derivative loss – relative to the geometric coordinates in $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$. We first recall (A.96c): $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]} = \bigcup_{t\in[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]} \Sigma_t^{[3t-U_0,U_2]}$. In particular, $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$ is foliated by portions of the flat hypersurfaces Σ_t that are bounded by level-sets of u. Hence, we can control the solution in $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$ by constructing energies and null fluxes as in Sect. 20, but instead of using the rough time functions and rough hypersurfaces, we use the Cartesian time function t and the spacelike hypersurface portions $\Sigma_t^{[3t-U_0,u]}$ and $\underline{\mathcal{S}}^{[0,t]}$ (note that $\Sigma_t^{[3t-U_0,u]} \cup \underline{\mathcal{S}}^{[0,t]}$ is Lipschitz and piecewise smooth, which is sufficient regularity for applying the divergence theorem on $\widehat{\mathfrak{CS}}^{[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]}$ to obtain L^2 estimates for the solution). We refer to these as "flat geometric energies and null fluxes." The same arguments used in the proofs of Props. 24.1, 24.2, 24.3, and 24.4 show that analogous estimates also hold[70] for the flat geometric energies and null fluxes, where the role of the smallness parameter $\mathring{\epsilon}$ is now played by $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$. The proof is dramatically easier in the present context because the Cartesian time function t is much easier to control and because μ is uniformly positive by (B.3). In particular, the energy estimates can be derived using the standard version of Grönwall's inequality, as opposed to the very technical arguments used in e.g. Sect. 29.7.1.

There is, however, one new detail of significance that we now describe. In carrying out the above arguments, which involve using Grönwall's inequality on the sub-regions $\widehat{\mathfrak{CS}}_{[0,t],[-U_0,u]}$ defined by:

$$\widehat{\mathfrak{CS}}_{[0,t],[-U_0,u]} \overset{\text{def}}{=} \bigcup_{t'\in[0,t]} \Sigma_{t'}^{[3t-U_0,u]}, \tag{B.5}$$

we must control the top-order derivatives of Ω and S by using a Cartesian-time-function-analog of the integral identity (21.63) on $\widehat{\mathfrak{CS}}_{[0,t],[-U_0,u]}$. Since the right boundary of $\widehat{\mathfrak{CS}}_{[0,t],[-U_0,u]}$ is the $\mathbf{g}$-*spacelike* hypersurface $\underline{\mathcal{S}}^{[0,t]}$ – in contrast to the $\mathbf{g}$-*null* right boundary of the domain ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_1,\tau_2),[u_1,u_2]}$ featured in (21.63) – the new integral identity features additional $\underline{\mathcal{S}}^{[0,t]}$-integrals and tori-integrals. More precisely, for $(t,u)\in[0,T_{\text{Shock}}^{\text{PS}}-\Delta^{\text{PS}}]\times[-U_0,U_2]$, the same arguments that we used to prove (21.63) can be used to prove the following similar identity, where we have suppressed the volume and area forms to simplify the presentation, and in practice, the role of the vectorfield V is played by $\mathcal{P}^{N_{\text{top}}}\Omega$ and $\mathcal{P}^{N_{\text{top}}}S$:

$$\begin{aligned}
&\int_{\widehat{\mathfrak{CS}}_{[0,t],[-U_0,u]}} \mathscr{Q}[\boldsymbol{\partial} V, \boldsymbol{\partial} V] + \int_{\ell_{t,u}} \frac{1}{4\mu}|V|_g^2 - \int_{\ell_{t,3t-U_0}} \frac{1}{4\mu}|V|_g^2 + \int_{\ell_{t,3t-U_0}} \frac{1}{4(\mu-\frac{1}{3})}|V|_g^2 \\
&= \int_{\ell_{0,u}} \frac{1}{4\mu}|V|_g^2 - \int_{\ell_{0,-U_0}} \frac{1}{4\mu}|V|_g^2 + \int_{\ell_{0,-U_0}} \frac{1}{4(\mu-\frac{1}{3})}|V|_g^2 \\
&\quad + \int_{\Sigma_t^{[3t-U_0,u]}} \cdots + \int_{\underline{\mathcal{S}}^{[0,t]}} - \int_{\Sigma_0^{[-U_0,u]}} \cdots + \int_{\widehat{\mathfrak{CS}}_{[0,t],[-U_0,u]}} \cdots .
\end{aligned} \tag{B.6}$$

In (B.6), "$\cdots$" denotes error integrands that are similar to the ones on RHS (21.63), but are much simpler to control because they depend on the Cartesian time function (as opposed to the rough time function) and its derivatives; the error integrals of the "$\cdots$" terms can be controlled by the flat energies mentioned above, much like in the proof of Prop. 27.5. We point out that the Cauchy stability arguments from Step 1 yield that $\underline{\mathcal{S}}^{[0,t]}$ is $\mathbf{g}$-spacelike for the perturbed solution (because it is also spacelike for the background solution).

The key feature of the identity (B.6) is that the tori integrals $-\int_{\ell_{t,3t-U_0}} \frac{1}{4\mu}|V|_g^2 + \int_{\ell_{t,3t-U_0}} \frac{1}{4(\mu-\frac{1}{3})}|V|_g^2$ on LHS (B.6) sum to yield *positive definite* control of $\int_{\ell_{t,3t-U_0}} |V|_g^2$, thus allowing us to propagate the "torus regularity" corresponding to the tori integrals on the RHS of the definition (11.4) for $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}$ (note that the integrals $\int_{\ell_{0,u}} \cdots$ and $\int_{\ell_{0,-U_0}} \cdots$

[70] To derive estimates for the acoustic geometry quantities μ, L^i, χ, etc., we in particular need to control their data on Σ_0. To this end, we recall that $u|_{\Sigma_0} = -x^1$. Hence, $u|_{\Sigma_0}$ is C^∞, and it is straightforward to use the eikonal equation (3.1) and the definitions of μ, L^i, χ, etc. to "solve" for their initial data on Σ_0 in terms of the fluid variable data on Σ_0; the size of the corresponding initial data functions can then be controlled using the standard Sobolev calculus.

on RHS (B.6) are data-terms that are controlled by $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$). This coercive control allows us to prove an analog of Prop. 24.3 on the tori $\ell_{t,u}$ using only our smallness assumption on the bona fide data norm $\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$. We highlight that this coercive control stands in contrast to the *rough tori* control provided by the identity (21.63), where the integral $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u_1}} \mathfrak{P}[V,V]\, \mathrm{d}\varpi_{\widetilde{g}}$ appears on the *right-hand* side with a positive (unfavorable) sign, i.e., one needs a *new estimate* to control $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,u_1}} \mathfrak{P}[V,V]\, \mathrm{d}\varpi_{\widetilde{g}}$; see Lemma 27.3 for a proof of the needed new estimate. However, we stress that, logically speaking, the proof of Lemma 27.3 can be completed only *after* one has derived the Cauchy stability estimates of Prop. B.2, which are independent of Lemma 27.3; we explicitly pointed this out in the proof of Lemma 27.3. We also clarify that the tori integrands $\frac{1}{4\mu}|V|_g^2$ in (B.6) reflect the fact that the integrand $\mathfrak{P}[V,V]$ defined in (21.45) takes a simplified form when we use foliations by level-sets of t instead of the rough time functions. Similarly, the tori integrands $\frac{1}{4(\mu-\frac{1}{3})}|V|_g^2$ in (B.6) are analogs of the integrand $\mathfrak{P}[V,V]$ defined in (21.45), but now corresponding to the time function $3t-u$, whose U_0 level-set defines $\underline{\mathcal{S}}^{[0,5T_{\mathrm{Shock}}^{\mathrm{PS}}]}$ (see (A.95c)).

Step 3: Estimates – without derivative loss – relative to the geometric coordinates in $\mathfrak{CS}_{\mathrm{Small}}^{[0,5T_{\mathrm{Shock}}^{\mathrm{PS}}]}$. Thanks to the positivity of μ in $\mathfrak{CS}_{\mathrm{Small}}^{[0,5T_{\mathrm{Shock}}^{\mathrm{PS}}]}$ guaranteed by (B.4), we can treat this region using essentially the same arguments we used in Step 2, using foliations of $\mathfrak{CS}_{\mathrm{Small}}^{[0,5T_{\mathrm{Shock}}^{\mathrm{PS}}]}$ by portions of flat hypersurfaces Σ_t and portions of null hypersurfaces $\mathcal{P}_u$.

Recap: We have now shown that on the entire Cauchy stability region $\mathfrak{CS}^{T_{\mathrm{Shock}}^{\mathrm{PS}};\Delta^{\mathrm{PS}}}$, the fluid variables' flat geometric energies and null fluxes are bounded up to top-order by $\lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}\right)^2$, and that the same result holds for the acoustic geometry quantities (such as μ, L^i, and χ). Since $\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]} \subset \mathfrak{CS}^{T_{\mathrm{Shock}}^{\mathrm{PS}};\Delta^{\mathrm{PS}}}$, this shows in particular that the data-estimates (11.12a)–(11.12c) hold with $\mathring{\epsilon} \lesssim \mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$.

What remains to be accomplished: The previous steps have provided estimates up to top-order for the solution with respect to the geometric coordinates on portions of the hypersurfaces Σ_t of constant Cartesian time, null hypersurface portions of the form $\mathcal{P}_u^{[0,t]}$, and acoustic tori $\ell_{t,u}$. It remains for us to control the solution along suitable rough hypersurfaces $^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_1,U_2]}$, null hypersurface portions of the form $^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_1,\tau_2]}$, and $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$. Our arguments will rely on the previously derived estimates, averaging methods, and some additional energy estimates that are similar to, but much simpler than, the ones derived in the bulk of the paper.

Step 4: Estimates – without derivative loss – on a special rough hypersurface, a special null hypersurface, and special rough tori via Chebychev's inequality. Recall that our main goal is show that the perturbed solution induces data on the perturbed rough hypersurface $^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, the null hypersurface portions $\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}$, and the rough tori $^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$ that satisfy all of the assumptions of Sects. 11.2.1–11.2.3. In the present Step 4, we will combine the results of the previous steps with averaging arguments based on Chebychev's inequality to show that the desired estimates hold on nearby rough hypersurfaces and rough tori. Then, in Step 5, we will derive energy estimates, starting from the data on these nearby rough hypersurfaces and tori, showing that the estimates hold on $^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_1,U_2]}$, $\mathcal{P}_{-U_1}^{[0,\frac{4}{\mathring{\delta}_*}]}$, and $^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}$.

To proceed, we note that the standard Cauchy stability results yielded by Step 1 imply that for $\mathfrak{n} \in [0,\mathfrak{n}_0]$, the region $^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],[-U_0,U_2]}$ is contained in the region $\mathfrak{CS}^{T_{\mathrm{Shock}}^{\mathrm{PS}};\Delta^{\mathrm{PS}}}$ defined in (A.96a) (recall that $\tau_0 < 0$). Hence, by Fubini's theorem, for any non-negative function $\mathbf{F}$, we have the following estimate, where in what follows, we suppress the area and volume forms to simplify the presentation:

$$\int_{\tau'=2\tau_0}^{\frac{1}{2}\tau_0} \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_0,U_2]}} \mathbf{F}\, d\tau' = \int_{u'=-U_0}^{U_2} \int_{^{(\mathfrak{n})}\mathcal{P}_{u'}^{[2\tau_0,\frac{1}{2}\tau_0]}} \mathbf{F}\, du' = \int_{^{(\mathfrak{n})}\mathcal{M}_{[2\tau_0,\frac{1}{2}\tau_0],[-U_0,U_2]}} \mathbf{F} \leq \int_{\mathfrak{CS}^{T_{\mathrm{Shock}}^{\mathrm{PS}};\Delta^{\mathrm{PS}}}} \mathbf{F}. \tag{B.7}$$

We now let (see Def. 8.10 regarding the notation):

$$\begin{aligned}\mathbf{F} \stackrel{\text{def}}{=} & \left|\mathcal{Z}_*^{[1,N_{\text{top}}+1];1}\vec{\Psi}\right|^2 + \left|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)\right|^2 + \left|\mathcal{P}^{\leq N_{\text{top}}}(\mathcal{C},\mathcal{D})\right|^2 + \left|\boldsymbol{\partial}\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)\right|^2 \\ & + \sum_{a=1}^{3}\left|\mathcal{Z}_*^{[1,N_{\text{top}}];1}L^a\right|^2 + \left|\mathcal{P}_*^{[1,N_{\text{top}}]}\upmu\right|^2 + \left|\mathcal{P}_*^{\leq N_{\text{top}}}\mathrm{tr}_{\not g}\chi\right|^2 + \left|\not{\mathcal{L}}_{\mathcal{P}}^{\leq N_{\text{top}}}\chi\right|_{\not g}^2 .\end{aligned} \tag{B.8}$$

Note that $\mathbf{F}$ is precisely the quantity that we have controlled in $\mathfrak{CS}^{T^{\text{PS}}_{\text{Shock}};\Delta^{\text{PS}}}$ by our flat geometric energies and null fluxes and the integral identity[71] (B.6). In particular, the arguments provided by Steps 2 and 3 imply[72] the following *spacetime* integral estimate:

$$\int_{\mathfrak{CS}^{T^{\text{PS}}_{\text{Shock}};\Delta^{\text{PS}}}}\mathbf{F} \lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2 . \tag{B.9}$$

We now use (B.7) and (B.9) to deduce that $\int_{\tau'=2\tau_0}^{\frac{1}{2}\tau_0}\int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau'}^{[-U_0,U_2]}}\mathbf{F}\,d\tau' \lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2$. From this bound and Chebychev's inequality, we see that there must exist a number τ_* satisfying:

$$\tau_* \in [2\tau_0, \frac{3}{2}\tau_0] \tag{B.10}$$

such that the following estimate holds, where in the rest of the proof, we absorb factors of $\frac{1}{|\tau_0|}$ into the implicit constants:

$$\int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_*}^{[-U_0,U_2]}}\mathbf{F} \lesssim \frac{1}{|\tau_0|}\left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2 \lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2 . \tag{B.11}$$

Using Fubini's theorem again, (B.11), and the fact that $[-U_1 - \frac{1}{\mathring{\delta}_*^{\text{PS}}}, -U_1] \subset [-U_0, U_2]$, we further deduce that:

$$\int_{u'=-U_1-\frac{1}{\mathring{\delta}_*^{\text{PS}}}}^{-U_1}\int_{\widetilde{\ell}_{\tau_*,u'}}\mathbf{F}\,\mathrm{d}u' = \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_*}^{[-U_1-\frac{1}{\mathring{\delta}_*^{\text{PS}}},-U_1]}}\mathbf{F} \leq \int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_*}^{[-U_0,U_2]}}\mathbf{F} \lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2 . \tag{B.12}$$

From (B.12) and Chebychev's inequality, we find that there is a $u_* \in [-U_1 - \frac{1}{\mathring{\delta}_*^{\text{PS}}}, -U_1]$ such that the following estimate holds:

$$\int_{\widetilde{\ell}_{\tau_*,u_*}}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2 \lesssim \frac{1}{\mathring{\delta}_*^{\text{PS}}}\left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2 \lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2 . \tag{B.13}$$

From (B.13), the bound $\int_{^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_*}^{[-U_0,U_2]}}|\boldsymbol{\partial}\mathcal{P}^{\leq N_{\text{top}}}\Omega|^2 \lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2$ implied by (B.11), the fact that $[-U_1 - \frac{1}{\mathring{\delta}_*^{\text{PS}}}, U_2] \subset [-U_0, U_2]$, and fundamental theorem of calculus-type arguments similar to the ones we used to prove (20.5) – but now based on the identity (20.1b) and Grönwall's inequality with respect to u – we further deduce that:

$$\sup_{u\in[-U_1-\frac{1}{\mathring{\delta}_*^{\text{PS}}},U_2]}\int_{\widetilde{\ell}_{\tau_*,u}}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2 \lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2 . \tag{B.14}$$

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[71]More precisely, the top-order term $\left|\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}(\Omega,S)\right|^2$ on RHS (B.8), can be controlled via the integral identity (B.6), while the below-top-order terms $\left|\boldsymbol{\partial}\mathcal{P}^{\leq N_{\text{top}}-1}(\Omega,S)\right|^2$ can be controlled by the remaining terms in the definition of $\mathbf{F}$ with the help of the identities of Lemma 9.2, much like in (23.5a)–(23.5b).

[72]Actually, aside from the top-order term $|\boldsymbol{\partial}\mathcal{P}^{N_{\text{top}}}(\Omega,S)|^2$, the estimates from Steps 2 and 3 show that various energies of the terms in F on portions of the hypersurfaces Σ_t and $\mathcal{P}_u$ are bounded by $\lesssim \left(\mathring{\Delta}^{N_{\text{top}}+1}_{\Sigma_0^{[-U_0,U_2]}}\right)^2$. These hypersurface estimates imply the spacetime estimate (B.9), where the implicit constants in (B.9) depend on the size of the region $\mathfrak{CS}^{T^{\text{PS}}_{\text{Shock}};\Delta^{\text{PS}}}$ (which is compact with dimensions controlled by the background solution).

Moreover, similar arguments yield:

$$\sup_{u\in[-U_1-\frac{1}{\mathring{\delta}_*^{\text{PS}}},U_2]}\int_{\widetilde{\ell}_{\tau_*,u}}|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\Psi}|^2 \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)^2. \tag{B.15}$$

Similarly, since (B.7) and (B.9) yield $\int_{u'=-U_1-\frac{1}{\mathring{\delta}_*^{\text{PS}}}}^{-U_1}\int_{^{(\mathfrak{n})}\mathcal{P}_{u'}^{[2\tau_0,\frac{1}{2}\tau_0]}}\mathbf{F}\,\mathrm{d}u' \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)^2$, we can again use Chebychev's inequality and fundamental theorem of calculus-type arguments similar to the ones we used to prove (20.5) to deduce that there exists a $U_* > 0$ such that:[73]

$$-U_* \in [-U_1 - \frac{1}{\mathring{\delta}_*^{\text{PS}}}, -U_1 - \frac{1}{2\mathring{\delta}_*^{\text{PS}}}] \tag{B.16}$$

such that:

$$\int_{^{(\mathfrak{n})}\mathcal{P}_{-U_*}^{[2\tau_0,\frac{1}{2}\tau_0]}} F \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)^2 \tag{B.17}$$

and:

$$\sup_{\tau\in[2\tau_0,\frac{1}{2}\tau_0]}\int_{\widetilde{\ell}_{\tau,-U_*}}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2 \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)^2. \tag{B.18}$$

Step 5: The desired geometric energy estimates – without derivative loss – on the rough hypersurfaces, null hypersurfaces, and rough tori relative to the geometric coordinates in $^{(\mathfrak{n})}\mathcal{M}_{[\tau_*,\frac{1}{2}\tau_0],[-U_*,U_2]}$. In total, the arguments given in Steps 1–4 have shown that for $\mathfrak{n} \in [0,\mathfrak{n}_0]$, the bona fide initial data on $\Sigma_0^{[-U_0,U_2]}$ induce data on the rough hypersurface portion $^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_*}^{[-U_*,U_2]}$, the null hypersurface portion $^{(\mathfrak{n})}\mathcal{P}_{-U_*}^{[2\tau_0,\frac{1}{2}\tau_0]}$, the rough tori $^{(\mathfrak{n})}\widetilde{\ell}_{\tau_*,u}$ for $u \in [-U_*, U_2]$, and the rough tori $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,-U_*}$ for $\tau \in [2\tau_0, \frac{1}{2}\tau_0]$, such that on these surfaces, all of the energies and null fluxes (up to top-order) defined in Sect. 20.5 are bounded by $\lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)^2$. Starting from these "data-estimates" (including the ones on the rough tori provided by (B.14), (B.15), and (B.18)), and using the same arguments we used in the proofs of (20.57a), Props. 24.1, 24.2, 24.3, and 24.4, we can derive the same geometric energy estimates on the region $^{(\mathfrak{n})}\mathcal{M}_{[\tau_*,\frac{1}{2}\tau_0],[-U_*,U_2]}$, i.e., we can bound the geometric energies up to top-order on $^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_*,U_2]}$ for $\tau \in [\tau_*, \frac{1}{2}\tau_0]$, the geometric null fluxes up to top-order on $^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_*,\frac{1}{2}\tau_0]}$ for $u \in [-U_*, U_2]$, and the geometric rough tori energies (as in Prop. 24.3) on $^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ up to top-order for $(\tau, u) \in [\tau_*, \frac{1}{2}\tau_0] \times [-U_*, U_2]$; all of these quantities are bounded by $\lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)^2$, e.g.,

$$\sup_{(\tau,u)\in[\tau_*,\frac{1}{2}\tau_0]\times[-U_*,U_2]}\int_{\widetilde{\ell}_{\tau,u}}|\mathcal{P}^{\leq N_{\text{top}}}(\Omega,S)|^2\,\mathrm{d}\varpi_{\widetilde{g}} \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\text{top}}+1}\right)^2. \tag{B.19}$$

The analysis is in fact much simpler compared to the proofs of Props. 24.1, 24.2, 24.3, and 24.4 because in the region $^{(\mathfrak{n})}\mathcal{M}_{[\tau_*,\frac{1}{2}\tau_0],[-U_*,U_2]}$, μ is uniformly bounded from below away from 0.

Remark B.3 (The proof of (B.18) does not rely on Lemma 27.3)**.** The proof of (B.19) relies in particular on the integral identity (21.63) with $\tau_1 = \tau_*$, $\tau_2 \in [\tau_*, \frac{1}{2}\tau_0]$, $u_1 = -U_*$, and $u_2 \in [-U_*, U_2]$. The rough tori L^2 estimates (B.15) and (B.18) are needed to control the corresponding rough tori integrals $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_2,-U_1}}\cdots$, $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_*,u_2}}\cdots$, and $\int_{^{(\mathfrak{n})}\widetilde{\ell}_{\tau_*,-U_1}}\cdots$ on RHS (21.63). In particular, the rough tori estimates (B.18) provide an analog of the estimates of Lemma 27.3 that are relevant for the region under study here. We stress that our proof of (B.18) given above is independent of Lemma 27.3; this is important for the logic of the paper.

[73] Note that (B.16) implies that $U_* > U_1$. We will use this basic fact in Step 2 of our proof of Prop. 31.2.

Since $\tau_0 \in (\tau_*, \frac{1}{2}\tau_0)$, particular cases of these bounds are the ones along ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau_0}^{[-U_*,U_2]}$, which, in view of the fact that $[-U_1, U_2] \subset [-U_*, U_2]$, imply the data bounds (11.11a)–(11.11d) and (11.16a)–(11.16b) with $\mathring{\epsilon} \lesssim \mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$. Moreover, the analog of Prop. 24.3 yields:

$$\sup_{u\in[-U_*,U_2]} \int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau_0,u}} |\mathcal{P}^{\leq N_{\mathrm{top}}}(\Omega, S)|^2 \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}\right)^2, \tag{B.20}$$

which, in view of the fact that $[-U_1, U_2] \subset [-U_*, U_2]$, implies (11.13b) with $\mathring{\epsilon} \lesssim \mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$. Similarly, from the data-estimate (B.15), the same arguments we used to prove (20.57a), and the geometric energy and null-flux estimates, we conclude the rough tori L^2 estimates (11.13a) for $\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\Psi}$. Finally, we will derive the rough tori L^2 estimates (11.13c) for $\mathcal{P}^{\leq N_{\mathrm{top}}-1}(\mathcal{C}, \mathcal{D})$. To this end, we note that among the geometric energy estimates mentioned above are the following bounds: $\sup_{u\in[-U_1,U_2]} \int_{{}^{(\mathfrak{n})}\mathcal{P}_u^{[\tau_*,\frac{1}{2}\tau_0]}} |\mathcal{P}^{\leq N_{\mathrm{top}}}(\mathcal{C}, \mathcal{D})|^2 \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}\right)^2$. From this bound and arguments similar to the ones we used to prove (B.18), based on Chebychev's inequality and fundamental theorem of calculus-type estimates (cf. (20.5)), we find that for any $u \in [-U_1, U_2]$, we have $\sup_{\tau\in[\tau_*,\frac{1}{2}\tau_0]} \int_{{}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}} |\mathcal{P}^{\leq N_{\mathrm{top}}-1}(\mathcal{C}, \mathcal{D})|^2 \lesssim \left(\mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}\right)^2$. Since $\tau_0 \in (\tau_*, \frac{1}{2}\tau_0)$, this bound in particular implies (11.13c) with $\mathring{\epsilon} \lesssim \mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$.

We have therefore derived (11.11a)–(11.11d), (11.12a)–(11.12c), (11.13a)–(11.13c), and (11.16a)–(11.16b) with $\mathring{\epsilon} \lesssim \mathring{\Delta}_{\Sigma_0^{[-U_0,U_2]}}^{N_{\mathrm{top}}+1}$, thereby completing our proof sketch of Prop. B.2.

□

Appendix C. Some lessons from $1D$, scaffolded around Burgers' equation flow

Some of the subtleties in our study of the maximal classical globally hyperbolic development for the $3D$ compressible Euler equations – and some new subtleties as well – can be seen in $1D$ model problems. Hence, for illustration, in this appendix, we study shock formation in the model case of the $1D$ Burgers' equation. Importantly, we highlight some *key qualitative differences* between the global behavior solutions to Burgers' equation and solutions to the $1D$ compressible Euler equations, where the key differences ultimately stem from the presence of a "second speed of propagation" in the Euler case. Although the $1D$ Burgers' equation lacks many of the extreme technical difficulties present in multi-dimensional compressible Euler flow, nonetheless, we can use it to exhibit elementary versions of the ideas and methods found in the bulk of the paper, some of which are not readily found in the hyperbolic conservations laws or standard PDE literature. Moreover, for Burgers' equation, we also exhibit (see Sect. C.5.1) non-uniqueness of *classical* solutions, a phenomenon which has not yet been observed for globally hyperbolic classical compressible Euler solutions, i.e., solutions on domains with a Cauchy hypersurface.[74] A crucial fact is that the open sets of compressible Euler solutions studied here and in our companion work [3] do *not* suffer (at least locally[75]) from the kind of non-uniqueness for Burgers' equation that we exhibit in Sect. C.5.1. The reason is that unlike Burgers' equation, compressible Euler flow features multiple speeds of propagation, and this results in a Cauchy horizon emanating from the crease (see Fig. 1A), thereby "blocking" the kind of non-uniqueness of classical solutions to Burgers' equation that we exhibit in Sect. C.5.1. However, we caution that for "general large data" for the $3D$ compressible Euler equations, the question of whether or not maximal classical globally hyperbolic developments are always unique is not settled.

We now briefly outline this appendix. In Sects. C.1 and C.2, we study shock formation for Burgers' equation using Cartesian coordinates. While our description is sharp, the methods do not readily extend to the study of $3D$ compressible Euler flow. In Sects. C.3 and C.4, we study shock formation for Burgers' equation using a more geometric approach based on the characteristic geometry. This approach has many parallels to the methods we used in the bulk of the paper and in Appendix A. In Sect. C.5, we discuss the relationship between the two approaches, we describe various subtleties and degeneracies, and we briefly describe connections with the shock development problem.

[74] In the context of compressible Euler flow, a Cauchy hypersurface for the Lorentzian manifold $(\mathcal{M}, \mathbf{g})$, where the fluid is assumed to be a classical solution on the spacetime manifold $\mathcal{M}$ and $\mathbf{g}$ is the acoustical metric defined in (2.15a), is a hypersurface $\Sigma \subset \mathcal{M}$ such that for every $p \in \mathcal{M}$, every past-inextendible $\mathbf{g}$-causal curve through p intersects Σ. See [74, Chapter 8] for a discussion of Cauchy hypersurfaces in the context of Einstein's equations.

[75] Recall that in the bulk of the paper, we only studied compressible Euler flow in bounded subsets of spacetime.

C.1. **The Cartesian coordinate space formulation of the equation.** The Cauchy problem for Burgers' equation[76] in $1D$ is the following PDE, posed for $(t,x) \stackrel{\text{def}}{=} (x^0, x^1) \in \mathbb{R}\times\mathbb{R}$, i.e., posed in Cartesian coordinate space:

$$\partial_t \Psi + (1+\Psi)\partial_x \Psi = 0, \qquad \Psi|_{\Sigma_0} = \mathring{\Psi}. \tag{C.1}$$

In (C.1), $\partial_t \stackrel{\text{def}}{=} \partial_0$ and $\partial_x \stackrel{\text{def}}{=} \partial_1$ are the standard Cartesian coordinate partial derivative vectorfields, and throughout this appendix, much like the bulk of the paper, we define $\Sigma_{t'} \stackrel{\text{def}}{=} \{(t,x) \in \mathbb{R}\times\mathbb{R} \mid t = t'\}$. For future use, we define the Burgers' equation transport vectorfield L_{Burg} as follows:

$$L_{\text{Burg}} \stackrel{\text{def}}{=} \partial_t + (1+\Psi)\partial_x. \tag{C.2}$$

As in the bulk of the paper, L^α_{Burg} denote the components of L_{Burg} with respect to the Cartesian coordinates (t,x), i.e., $L^0_{\text{Burg}} = L_{\text{Burg}} t = 1$ and $L^1_{\text{Burg}} = L_{\text{Burg}} x = 1 + \Psi$. Note that equation (C.1) is equivalent to:

$$L_{\text{Burg}} \Psi = 0. \tag{C.3}$$

C.2. **The Cartesian coordinate space picture of the singularity.** In this section, we study shock formation for Burgers' equation in Cartesian coordinates. Our discussion here is mainly for illustration; our analysis relies on the simple form of the $1D$ Burgers' equation, and the methods do not seem to apply to the $3D$ compressible Euler equations.

C.2.1. *The characteristics and the blowup of $\partial_x \Psi$.* Given any point $(0,z) \in \Sigma_0$, we let:

$$\gamma_z(t) \stackrel{\text{def}}{=} (t, \mathfrak{x}_z(t)) \tag{C.4}$$

be the characteristic curve in the (t,x) plane associated with (C.1), where the real-valued function $\mathfrak{x}_z$ is the solution to the following ODE:

$$\frac{d}{dt}\mathfrak{x}_z(t) = L^1_{\text{Burg}}(t, \mathfrak{x}_z(t)) = 1 + \Psi(t, \mathfrak{x}_z(t)), \qquad \mathfrak{x}_z(0) = z. \tag{C.5}$$

γ_z is the future-directed characteristic curve emanating from $(0,z)$. By the chain rule, for any scalar function $f = f(t,x)$, we have the following identity, where "$\circ$" denotes composition of functions:

$$\frac{d}{dt}(f \circ \gamma_z(t)) = [L_{\text{Burg}} f] \circ \gamma_z(t). \tag{C.6}$$

Note that (C.3) and (C.6) imply that $\frac{d}{dt}(\Psi \circ \gamma_z(t)) = 0$ and thus solutions Ψ to Burgers' equation are constant along the characteristics. It follows that:

$$\Psi \circ \gamma_z(t) = \mathring{\Psi}(z), \tag{C.7}$$

and that γ_z is a straight line with slope $\frac{1}{1+\mathring{\Psi}(z)}$ in the (t,x) plane. Equivalently, we have:

$$\mathfrak{x}_z(t) = z + t\left\{1 + \mathring{\Psi}(z)\right\}. \tag{C.8}$$

Note that (C.7) implies that Ψ remains bounded, i.e., Ψ itself can never diverge to infinity along the curve $t \to \gamma_z(t)$. The situation is quite different for $\partial_x \Psi$, which can blow up along $\gamma_z(t)$. To see this, we differentiate (C.1) with ∂_x, to deduce that $L_{\text{Burg}}\left(\frac{1}{\partial_x \Psi}\right) = 1$. Equivalently, $\frac{d}{dt}\left\{\left(\frac{1}{\partial_x\Psi}\right)\circ\gamma_z(t) = 1\right\}$. Integrating in time and carrying out straightforward computations, we find that:

$$[\partial_x \Psi] \circ \gamma_z(t) = \frac{\mathring{\Psi}'(z)}{1 + t\mathring{\Psi}'(z)}, \tag{C.9}$$

where $\mathring{\Psi}'(z) \stackrel{\text{def}}{=} \frac{d}{dz}\mathring{\Psi}(z)$. It follows that along the curve $t \to \gamma_z(t)$, $\partial_x \Psi$ blows up precisely at the time t_z defined by:

$$t_z \stackrel{\text{def}}{=} -\frac{1}{\mathring{\Psi}'(z)}, \tag{C.10}$$

[76]More precisely, the standard Burgers equation is $\partial_t \Psi + \Psi \partial_x \Psi = 0$. We have made the harmless replacement $\Psi \to 1 + \Psi$ in equation (C.1) because this has the effect of shearing the characteristics in the x-direction, which makes for better pictorial comparisons with our work on compressible fluids (see, e.g., Fig. 1).

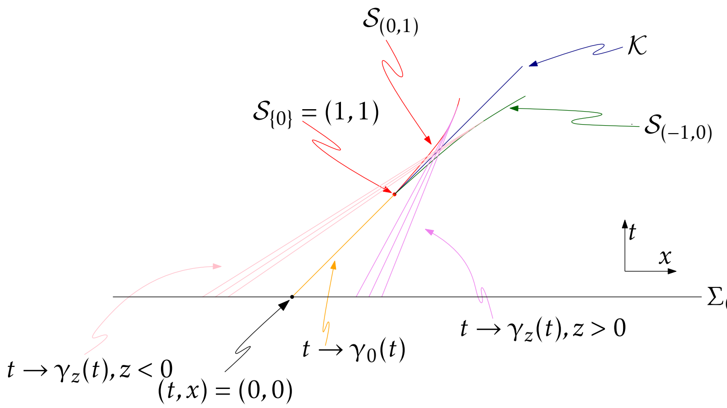


Figure 17. The singular curve and shock curve in Cartesian coordinates for $\mathring{\Psi}(x) = -x + \frac{1}{3}x^3$, not drawn to scale

and from (C.8), we see that the Cartesian coordinates of the blowup-point associated to the curve γ_z are:

$$\gamma_z(t_z) = \left(-\frac{1}{\mathring{\Psi}'(z)}, z - \frac{1 + \mathring{\Psi}(z)}{\mathring{\Psi}'(z)} \right). \tag{C.11}$$

The crudest picture of the formation of a shock is the one we have just described, i.e., $\partial_x \Psi$ blows up while Ψ remains bounded. In the ensuing discussion, we will give a much more refined description of the singularity formation.

C.2.2. *The singular curve.* We now study the singular as z varies. That is, in view of (C.11), we define the (future[77]) "singular curve" $z \to \mathscr{S}(z)$ in the (t,x)-plane as follows:

$$\mathscr{S}(z) \stackrel{\text{def}}{=} \left(-\frac{1}{\mathring{\Psi}'(z)}, z - \frac{1 + \mathring{\Psi}(z)}{\mathring{\Psi}'(z)} \right). \tag{C.12}$$

$\mathscr{S}(z)$ is the point on the image of the curve $t \to \gamma_z(t)$ where $\partial_x \Psi$ blows up as t increases from 0.

We aim to locally describe the structure of the singular curve. To begin, for subsets $J \in \mathbb{R}$, we define:

$$\mathcal{S}_J \stackrel{\text{def}}{=} \mathscr{S}(J) = \{\mathscr{S}(z) \mid z \in J\}. \tag{C.13}$$

$\mathcal{S}_J$ is the portion of the singular curve in Cartesian coordinates corresponding to $z \in J$. In Fig. 17, for the specific initial data:

$$\mathring{\Psi}(x) \stackrel{\text{def}}{=} -x + \frac{1}{3}x^3, \tag{C.14}$$

we depict[78] $\mathcal{S}_{(-1,1)} = \mathcal{S}_{(-1,0)} \cup \mathcal{S}_{\{0\}} \cup \mathcal{S}_{(0,1)}$, where the three sets on the RHS are disjoint. We will further discuss $\mathcal{S}_{(-1,1)}$ in Sect. C.5.

Next, using (C.12) and, in view of (C.2) and (C.7), identifying $\left(1, 1 + \mathring{\Psi}(z)\right)$ with the vectorfield $L_{\text{Burg}} \circ \gamma_z(t_z)$, we compute that:

$$\frac{d}{dz}\mathscr{S}(z) = \frac{\mathring{\Psi}''(z)}{[\mathring{\Psi}'(z)]^2}\left(1, 1 + \mathring{\Psi}(z)\right) = \frac{\mathring{\Psi}''(z)}{[\mathring{\Psi}'(z)]^2} L_{\text{Burg}} \circ \gamma_z(t_z). \tag{C.15}$$

In particular, (C.15) shows that along the singular curve, as long as $\mathring{\Psi}''(z) \neq 0$, the tangent vector to $\mathscr{S}$ at the corresponding point $\gamma_z(t_z)$ in Cartesian coordinate space is non-zero and *parallel* to $L_{\text{Burg}} \circ \gamma_z(t_z)$; see Fig. 17. In Sect. C.5, we will discuss the significance of these vectors being parallel. We will also discuss the shock curve, denoted by "$\mathcal{K}$" in the figure. Readers can consult [20] for similar pictures of shock formation in Burgers' equation solutions, where [20] studies formal asymptotic expansions intended to connect the behavior of solutions to Burgers' equation with small viscosity to the behavior of solutions to the inviscid problem near their shocks.

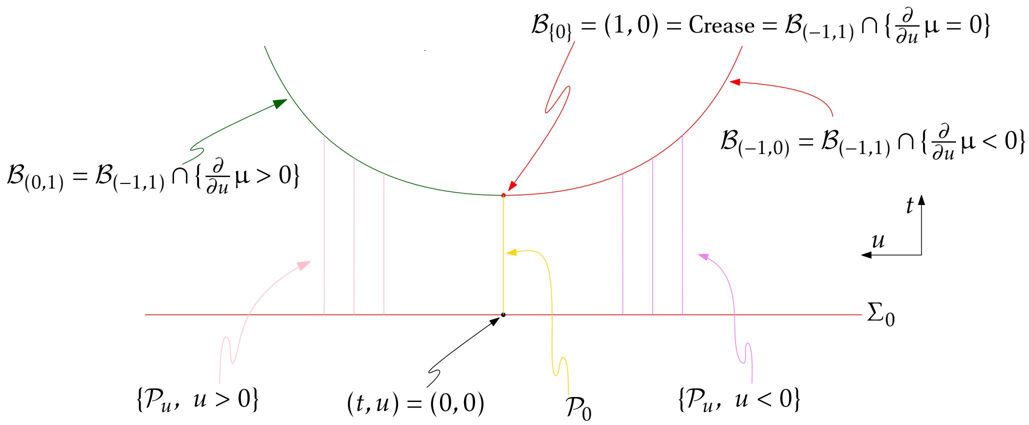


Figure 18. Portions of the curve $\mathcal{B} = \{\mu = 0\}$ for Burgers' equation in geometric coordinates for $\mathring{\Psi}(u) = u - \frac{1}{3}u^3$

C.3. **Geometric coordinates and related constructions.** In this section and Sect. C.4, we again study shock formation for Burgers' equation, but this time using alternate, more geometric methods that are even more directly tied to the characteristic geometry. Our approach here is closely related to the approach we used in the bulk of the paper and in Appendix A. In Sect. C.5 we will describe the relationship between the geometric picture of the shock formation described here and the approach from Sect. C.2, which relied on the Cartesian coordinates. Our main goal in this section is to set up the geometric framework. In Sect. C.4, we will use the framework to study the shock formation.

C.3.1. *Construction of the characteristic coordinate and the geometric coordinates.* To start, we assume that we have a solution Ψ to Burgers' equation in Cartesian coordinates, and we let L_{Burg} be the Burgers' equation transport vectorfield, as defined in (C.2). The main new ingredient is the *characteristic coordinate* u, which we define to be the solution to the following transport equation initial value problem:

$$L_{\text{Burg}} u = 0, \qquad\qquad u|_{t=0} = -x. \tag{C.16}$$

Much like in the bulk of the paper, we define (t,u) to be the *geometric coordinates*, and we define:

$$\mathcal{P}_{u'} \overset{\text{def}}{=} \Big\{(t,u) \in \mathbb{R}^2 \mid t \geq 0, u = u'\Big\}, \tag{C.17}$$

as well as the following subsets of Σ_t, where $u_1 \leq u_2$ are constants:

$$\Sigma_t^{[u_1,u_2]} \overset{\text{def}}{=} \Big\{(t,u) \in \mathbb{R}^2 \mid u_1 \leq u \leq u_2\Big\}. \tag{C.18}$$

We have chosen the initial condition $u|_{t=0} = -x$ to facilitate comparisons with the bulk of the paper, where we make the same choice (see Def. 3.1). To compare with Sect. C.1, we note that until the characteristics intersect, we can think of u as the function of (t,x) that takes on the constant value $-z$ along the characteristic curve $t \to \gamma_z(t)$. We also emphasize that in the bulk of the paper, the analog of (C.16) is the acoustic eikonal equation (3.1), which is fully nonlinear and hyperbolic and therefore much more difficult to study away from symmetry, especially from the point of view top-order energy estimates.

A key issue (here and in the bulk of the paper) is that the geometric coordinates can be diffeomorphic to the Cartesian coordinates only before a shock singularity forms; at the onset of the singularity, many interesting degeneracies emerge, and we will highlight some of them in Sect. C.5. We also note that by (C.16), the specific initial data (C.14) takes the following form in geometric coordinates:

$$\mathring{\Psi}(u) \overset{\text{def}}{=} u - \frac{1}{3}u^3. \tag{C.19}$$

[77]Using the same techniques, one could also study the "past" singular curve, i.e., the set of singular points as $t \downarrow -\infty$.

[78]More precisely, in Fig. 17, we only depict a bounded portion of $\mathcal{S}_{(-1,1)}$; for the initial data (C.14), the set $\mathcal{S}_{(-1,1)}$ is unbounded.

C.3.2. *The inverse foliation density.* Next, we define the inverse foliation density of the characteristics, denoted by μ, as follows:

$$\mu \stackrel{\text{def}}{=} -\frac{1}{\partial_x u}. \tag{C.20}$$

The vanishing of μ signifies the infinite density of the level sets of u (viewed as a function of (t,x)) and, as it turns out, the blowup of $\partial_x \Psi$; see (3.2) for the analog of μ in the context of our main results.

C.3.3. *Burgers' equation in geometric coordinates.* Using that $L_{\text{Burg}} t = 1$ (see (C.2)) and $L_{\text{Burg}} u = 0$ (see (C.16)) , we can express the vectorfield L_{Burg} relative to the geometric coordinates (t,u) as follows:

$$L_{\text{Burg}} = \frac{\partial}{\partial t} \stackrel{\text{def}}{=} \frac{\partial}{\partial t}|_u, \tag{C.21}$$

where $\frac{\partial}{\partial t}|_u$ denotes partial differentiation with respect to t at fixed u.

From (C.21), it follows that in geometric coordinates, Burgers' equation (C.1) takes the following form:

$$\frac{\partial}{\partial t}\Psi(t,u) = 0. \tag{C.22}$$

From (C.22), we find that Ψ is a function of u alone, that is, in view of the initial conditions in (C.1) and (C.16), that:

$$\Psi(t,u) = \mathring{\Psi}(u). \tag{C.23}$$

Note that on RHS (C.23), $\mathring{\Psi}(u)$ denotes the function from (C.1), but written as a function of u, where at $t=0$, $u=-x$ i.e., $\mathring{\Psi}(u) \stackrel{\text{def}}{=} \mathring{\Psi}(-x)$. We highlight that (C.22) is a *linear* PDE whose solutions do not blow up, i.e., *in geometric coordinates, initially smooth solutions to Burgers' equation remain smooth for all time*!

C.3.4. *The evolution equation for the inverse foliation density.* With the help of (C.20), we compute that $\mu\partial_x t = 0$ and $-\mu\partial_x u = 1$. It follows that:

$$\mu\partial_x = -\frac{\partial}{\partial u}, \tag{C.24}$$

where $\frac{\partial}{\partial u}$ denotes partial differentiation with respect to u at fixed t. Using (C.2), (C.16), (C.20), (C.21), and (C.23), and carrying out straightforward computations based on differentiating (C.16) with ∂_x, we compute that μ satisfies the following evolution equation in geometric coordinates:

$$\frac{\partial}{\partial t}\mu(t,u) = L_{\text{Burg}}\mu(t,u) = -\frac{\partial}{\partial u}\Psi(t,u) = -\frac{d}{du}\Psi(0,u) \stackrel{\text{def}}{=} -\frac{d}{du}\mathring{\Psi}(u). \tag{C.25}$$

Integrating (C.25) with respect to t at fixed u and using the initial condition $\mu|_{t=0} = 1$ (which follows from the initial condition in (C.16) and (C.20)), we find that:

$$\mu(t,u) = 1 - t\frac{d}{du}\mathring{\Psi}(u). \tag{C.26}$$

Next, using (C.23)–(C.26), we deduce the following identity, which is fundamental for the discussion in Sect. C.4:

$$[\partial_x\Psi](t,u) = \frac{1}{\mu(t,u)}\frac{\partial}{\partial u}\Psi(t,u) = -\frac{\frac{d}{du}\mathring{\Psi}(u)}{1 - t\frac{d}{du}\mathring{\Psi}(u)}. \tag{C.27}$$

Note that LHS (C.27) is the *Cartesian* coordinate partial derivative $\partial_x\Psi$, but written as a function of the geometric coordinates.

C.4. **The geometric coordinate space picture of the shock.** Let $I \subset \mathbb{R}$ be a subset of u-values. We start by defining:[79]

$$\mathcal{B}_I \stackrel{\text{def}}{=} \left\{(t,u) \in \mathbb{R}^2 \mid \mu(t,u) = 0,\, t \geq 0,\, u \in I\right\} = \left\{(t,u) \in \mathbb{R}^2 \mid 1 - t\frac{d}{du}\mathring{\Psi}(u) = 0,\, t \geq 0,\, u \in I\right\}, \tag{C.28}$$

where the second equality in (C.28) follows from (C.26). In what follows, we use the shorthand notation $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}_{\mathbb{R}}$. From (C.27) and (C.28), we deduce the following key result: in regions where we can *justify* the relevance of geometric

[79]The subsets $\mathcal{B}_I$ of geometric coordinate space are closely related to – but distinct from – the singular boundary portion denoted by "$\mathcal{B}^{[0,\mathfrak{n}_0]}$" in the bulk of the paper.

coordinates for reaching conclusions about the behavior of the solution in (t,x)-space, $\mathcal{B}_I$ is the subset of points within $(t,u)\in[0,\infty)\times I$ such that $[\partial_x\Psi](t,u)$ blows up.

> $\mathcal{B}$ is a Burgers'-equation-analog of the singular boundary featured in Theorem 34.1 in the context of $3D$ compressible Euler flow. However, the analogy is not perfect; unlike the $3D$ compressible Euler solutions studied in this paper and our companion work [3], Burgers' equation solutions can suffer from non-uniqueness of classical solutions and hence non-uniqueness of maximal classical developments; see Sect. C.5.1 for further discussion.

We emphasize that justifying the use of geometric coordinates in the study of Burgers' equation is a non-trivial issue in the following sense: when the characteristics cross in Cartesian coordinate space, u becomes a "multivalued" function of (t,x). Hence, when using the (t,u)-coordinates to draw conclusions about the behavior of the solution in (t,x)-space, one must be careful to limit one's attention to regions where the two coordinate systems are in bijective correspondence. In Sect. C.5.2, we will further discuss the issue of whether or not the coordinate systems (t,u) and (t,x) are in bijective correspondence. We will also describe how the change of variables map $(t,u)\to(t,x)$ *always fails* to be a diffeomorphism when μ vanishes (i.e, along $\mathcal{B}$), though *sometimes* (under appropriate assumptions), it is a homeomorphism along portions of $\mathcal{B}$. The homeomorphism property, when available, is crucial for the shock development problem, which we discuss in Sect. C.5.4; the homeomorphism property allows one to associate a *unique* (t,x)-value to the point referred to as the "crease" in Fig. 18, which is crucial for the problem.

We now observe that (C.25) and (C.28) imply that for smooth data $\mathring{\Psi}$, if $(t_*,u_*)\in\mathcal{B}_I$, then $t_*=\frac{1}{\frac{d}{du}\mathring{\Psi}(u_*)}>0$, and we have the following monotonicity:

$$L_{\mathrm{Burg}}\mu(t,u)=-\frac{d}{du}\mathring{\Psi}(u)<0,\qquad\text{for } 0\le t\le t_* \text{ and } u \text{ near } u_*. \tag{C.29}$$

A similar form of monotonicity holds in our study of $3D$ compressible Euler flow (see, e.g., (18.8a)), and it yields *dissipative terms* in the energy estimates (specifically, the spacetime integral $\mathbb{K}[f](\tau,u)$ on LHS (20.26)) that are crucial for closing the problem away from symmetry.

Next, we note that from (C.26), it follows that the Cartesian time of first blowup (i.e., the smallest positive value of t such that μ vanishes) is (where $[z]_-\stackrel{\mathrm{def}}{=}\max\{-z,0\}$):

$$T_{\mathrm{Shock}}=\frac{1}{\max_{u\in\mathbb{R}}\left[\frac{d}{du}\mathring{\Psi}(u)\right]_-}, \tag{C.30}$$

and that the subset of points within $\Sigma_{T_{\mathrm{Shock}}}\stackrel{\mathrm{def}}{=}\{(t,u)\in\mathbb{R}^2 \mid t=T_{\mathrm{Shock}}\}$ where μ vanishes is:

$$\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}=\left\{(T_{\mathrm{Shock}},u)\in\mathbb{R}^2 \mid \left[\frac{d}{du}\mathring{\Psi}(u)\right]_-=\max_{u'\in\mathbb{R}}\left[\frac{d}{du}\mathring{\Psi}(u')\right]_-\right\}. \tag{C.31}$$

We now highlight that since μ vanishes for the first time at T_{Shock}, within $\Sigma_{T_{\mathrm{Shock}}}$, μ must achieve the minimum value of 0 precisely on $\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$. Hence, since $\frac{\partial}{\partial u}\mu$ must vanish at the minima, it follows from (C.26) that:

$$(T_{\mathrm{Shock}},u)\in\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}\implies\frac{d^2}{du^2}\mathring{\Psi}(u)=0. \tag{C.32}$$

We now discuss the crucial issue of the structure of $\mathcal{B}$ near points in $\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$. To proceed, we note that since μ is positive on $\Sigma_{T_{\mathrm{Shock}}}\backslash\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$ and vanishes on $\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$, it follows that when $(T_{\mathrm{Shock}},u)\in\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$, we have $\frac{\partial^2}{\partial u^2}\mu(T_{\mathrm{Shock}},u)\ge 0$ (for initial data such that μ is a C^2 function of (t,u), i.e., when $\mathring{\Psi}$ is C^3). Within the class of solutions that are smooth with respect to the (t,u)-coordinates, the "generic behavior" at a point $(T_{\mathrm{Shock}},u_*)\in\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$ is:

$$\frac{\partial^2}{\partial u^2}\mu(T_{\mathrm{Shock}},u_*)>0. \tag{C.33}$$

Inequality (C.33) is an analog of the transversal convexity satisfied by the solutions featured in our main results; see, for example, (18.5). From (C.26) and (C.31), it follows that if $(T_{\mathrm{Shock}},u_*)\in\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$, then (C.32) and (C.33) hold at (T_{Shock},u_*) if and only if $\frac{d^2}{du^2}\mathring{\Psi}(u_*)=0$ and $\frac{d^3}{du^3}\mathring{\Psi}(u_*)>0$. Moreover, if these latter two conditions hold, then (T_{Shock},u_*) is an isolated point in $\mathcal{B}\cap\Sigma_{T_{\mathrm{Shock}}}^{\mathrm{Singular}}$. However, we stress that in our study of $3D$ compressible fluids in the bulk of the paper, transversal convexity does *not* imply that points in $\mathcal{B} \cap \Sigma_{T_{\text{Shock}}}^{\text{Singular}}$ are isolated since, roughly speaking, the Hessian of μ in the symmetry breaking directions does not have to be positive definite; see the discussion in Sect. 1.6.

Finally, to give a specific example, we note that for the specific initial data (C.19), we have $T_{\text{Shock}} = 1$, $\Sigma_{T_{\text{Shock}}}^{\text{Singular}}$ consists of the single point with geometric coordinates $(t, u) = (1, 0)$, $\frac{d^2}{du^2}\mathring{\Psi}(0) = 0$, and $\frac{d^3}{du^3}\mathring{\Psi}(0) = 2 > 0$, i.e., the conditions highlighted above hold; see Fig. 18.

C.5. **Relationship between the two approaches, subtleties and degeneracies, and the shock development problem.** In this section, we highlight various subtleties in the study of Burgers' equation solutions and connect them to our study of $3D$ compressible fluids in the bulk of the paper.

> For definiteness, throughout Sect. C.5, *we assume the specific initial data* $\mathring{\Psi} \stackrel{\text{def}}{=} -x + \frac{1}{3}x^3 = u - \frac{1}{3}u^3$, as in equations (C.14) and (C.19) and Figs. 17 and 18.

However, similar results hold (locally) for all solutions satisfying the monotonicity result (C.29) and the transversal convexity condition (C.33). That is, the basic qualitative properties of such solutions are similar to the properties of the solution with the data $\mathring{\Psi} \stackrel{\text{def}}{=} -x + \frac{1}{3}x^3 = u - \frac{1}{3}u^3$, and in particular, the properties are *stable* under perturbations[80] of the initial data. For our specific initial data, $T_{\text{Shock}} = 1$ and $\mathcal{B} \cap \Sigma_{T_{\text{Shock}}}^{\text{Singular}} = (1, 0)$. $(1, 0)$ is the lowest point on $\mathcal{B}$ in Fig. 18, and in Fig. 17, it corresponds to the lowest point on the singular curve, namely $\mathcal{S}_{\{0\}} = (1, 1)$. Much like in the bulk of the paper, in the geometric coordinates picture, we refer to $(1, 0)$ as "the crease." Note that for our specific initial data (C.19), the monotonicity (C.29) and transversal convexity condition (C.33) hold near the crease. For these data, we also have:

$$\frac{\partial}{\partial u}\mu(t, u) > 0 \text{ for } (t, u) \in \mathcal{B}_{(0,1)}, \tag{C.34}$$

$$\frac{\partial}{\partial u}\mu(t, u) = 0 \text{ at the crease } (t, u) = (1, 0), \tag{C.35}$$

$$\frac{\partial}{\partial u}\mu(t, u) < 0 \text{ for } (t, u) \in \mathcal{B}_{(-1,0)}. \tag{C.36}$$

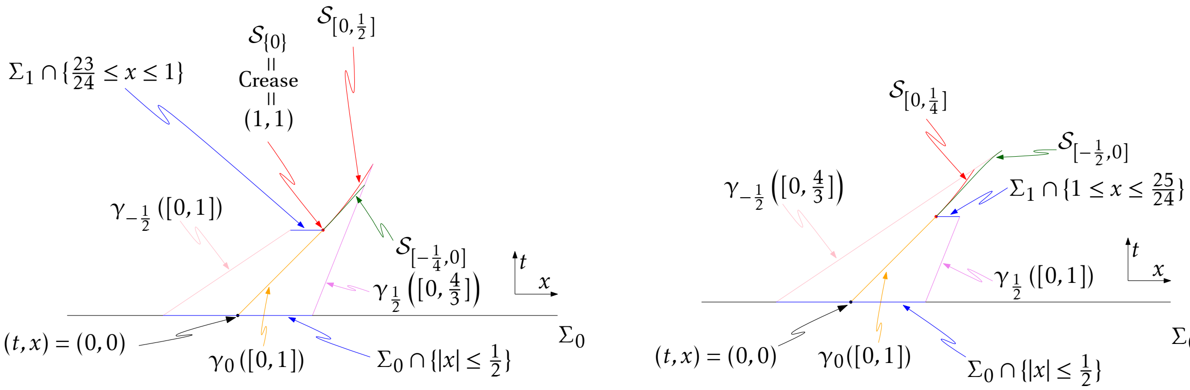
The conditions (C.34)–(C.36) are important for the ensuing discussion. In Fig. 18, we have separately labeled the three subsets of $\mathcal{B}_{(-1,1)}$ featured in (C.34)–(C.36). Note also that $\mathcal{B}_{(-1,0)}$ corresponds, in the Cartesian coordinate space picture, to the singular curve portion $\mathcal{S}_{(0,1)}$ in Fig. 17, that the crease corresponds to the cusp point $\mathscr{S}(0) = (1, 1)$ on the singular curve, and that $\mathcal{B}_{(0,1)}$ corresponds to the singular curve portion $\mathcal{S}_{(-1,0)}$; we will further discuss this in Sect. C.5.2 by studying the change of variables map from (t, u) to (t, x) coordinates.

C.5.1. *Non-uniqueness of classical solutions for Burgers' equation, though not necessarily for compressible Euler solutions.* Fig. 17 illustrates a form of non-uniqueness of classical Burgers' equation solutions in regions of Cartesian coordinate space lying to the future of the singularity. To exhibit this phenomenon in more detail, we will show (see Figs. 19–20) that there are two non-open subsets of spacetime, denoted by $\mathcal{R}_1$ and $\mathcal{R}_2$, such that the following occurs, where for convenience, we restrict our attention to portions of spacetime corresponding to characteristics that emanate from points $(0, z)$ with $|z| \leq \frac{1}{2}$:

1. For $i = 1, 2$, $\mathcal{R}_i$ contains $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$.
2. For $i = 1, 2$, there exists a classical solution Ψ_i to Burgers' equation (C.1) on $\mathcal{R}_i$ that takes on the initial condition $\Psi_i(0, x) = \mathring{\Psi}(x) = -x + \frac{1}{3}x^3$ along $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$.
3. For $i = 1, 2$, for every point $p \in \mathcal{R}_i$, there is a unique integral curve of $L_{\text{Burg}} = \partial_t + (1 + \Psi_i)\partial_x$ that passes through p, is contained in $\mathcal{R}_i$, and intersects $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$ exactly once. In this sense, we can view $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$ as a Cauchy hypersurface for $\mathcal{R}_i$.
4. $\mathcal{R}_1$ contains $\mathcal{S}_{[-\frac{1}{4},0)}$, but the solution Ψ_1 remains smooth up to $\mathcal{S}_{[-\frac{1}{4},0)}$. On the other hand, $\mathcal{S}_{[0,\frac{1}{2}]}$ belongs to the boundary of $\mathcal{R}_1$ (but not $\mathcal{R}_1$ itself), and for any $q \in \mathcal{S}_{[0,\frac{1}{2}]}$, $[\partial_x \Psi_1](p)$ blows up as q is approached by points $p \in \mathcal{R}_1$.
5. $\mathcal{R}_2$ contains $\mathcal{S}_{(0,\frac{1}{4}]}$, but the solution Ψ_2 remains smooth up to $\mathcal{S}_{(0,\frac{1}{4}]}$. On the other hand, $\mathcal{S}_{[-\frac{1}{2},0]}$ belongs to the boundary of $\mathcal{R}_2$ (but not $\mathcal{R}_2$ itself), and for any $q \in \mathcal{S}_{[-\frac{1}{2},0]}$, $[\partial_x \Psi_2](p)$ blows up as q is approached by points $p \in \mathcal{R}_2$.

[80] More precisely, our ensuing analysis will imply stability of the phenomena under discussion under C^3 perturbations of the initial data.

6. $\mathcal{R}_1 \cap \mathcal{R}_2$ is equal to the disjoint union of two non-open disconnected components, which we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$. The boundary of $\mathcal{D}_2$ contains the singular curve portions $\mathcal{S}_{[-\frac{1}{4},0]}$ and $\mathcal{S}_{[0,\frac{1}{4}]}$. Moreover, the solutions Ψ_1 and Ψ_2 *do not coincide in* $\mathcal{D}_2$ since, for example, for any $q \in \mathcal{S}_{(0,\frac{1}{4}]}$, $[\partial_x \Psi_1(p)]$ blows up as q is approached by any sequence of points $p \in \mathcal{D}_2$, while Ψ_2 is smooth in a neighborhood of q.



(A) The region $\mathcal{R}_1$ (B) The region $\mathcal{R}_2$

Figure 19. Regions $\mathcal{R}_i$ in Cartesian coordinate space on which a classical solution Ψ_i exists

We now describe the sets $\mathcal{R}_i$ and corresponding classical solutions Ψ_i such that the properties 1-6 above hold. We will use the notation from Sect. C.2.

$\underline{\mathcal{R}_1 \text{ and } \Psi_1}$. Let $\mathcal{R}_1$ be as in Fig. 19A. That is, $\mathcal{R}_1$ is the Cartesian coordinate spacetime region bounded from below by the flat hypersurface portion $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$, from above by the continuous, piecewise C^1 hypersurface portion $\left(\Sigma_1 \cap \{\frac{23}{24} \leq x \leq 1\}\right) \cup \mathcal{S}_{[0,\frac{1}{2}]}$, on the left by $\gamma_{-\frac{1}{2}}([0,1])$, and on the right by $\gamma_{\frac{1}{2}}\left([0,t_{\frac{1}{2}}]\right)$, where, as in (C.10) with the data (C.14), $t_{\frac{1}{2}} = \frac{4}{3}$ is the value of t such that the curve $t \to \gamma_{\frac{1}{2}}(t)$ intersects $\mathcal{S}_{[0,\frac{1}{2}]}$, i.e., the value of t such that $[\partial_x \Psi] \circ \gamma_{\frac{1}{2}}(t)$ blows up. We consider $\mathcal{S}_{[0,\frac{1}{2}]}$ to *not* be part of $\mathcal{R}_1$, while we consider the other boundary portions to be part of $\mathcal{R}_1$. Note that $\mathcal{R}_1$ is foliated by portions of the characteristics emanating from $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$. Hence, in view of (C.7), we can define the classical solution Ψ_1 on $\mathcal{R}_1$ such that Ψ_1 is constant along the characteristics that foliate $\mathcal{R}_1$ and such that $\Psi_1(0,x) = \mathring{\Psi}(x) = -x + \frac{1}{3}x^3$ on $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$. Ψ_1 is a classical solution on $\mathcal{R}_1$ because we have removed the singular curve portion $\mathcal{S}_{[0,\frac{1}{2}]}$, where $\partial_x \Psi_1$ blows up. Note, however, that $\partial_x \Psi_1$ is not uniformly bounded on $\mathcal{R}_1$.

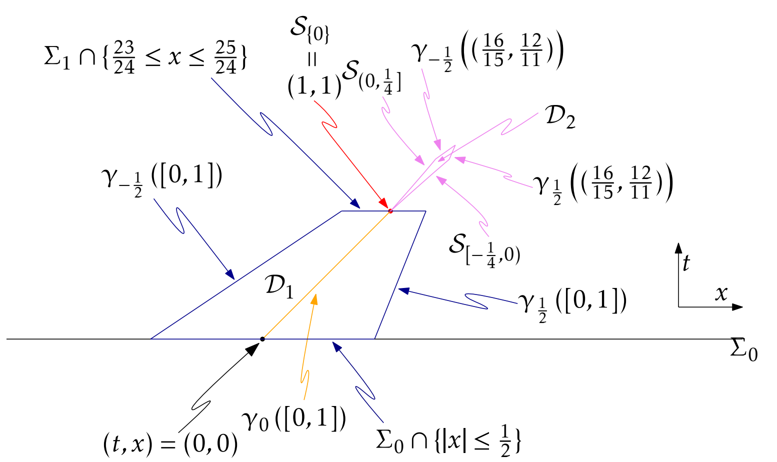


Figure 20. $\mathcal{R}_1 \cap \mathcal{R}_2 = \mathcal{D}_1 \cup \mathcal{D}_2$, illustrating non-uniqueness for $\mathring{\Psi}(x) = -x + \frac{1}{3}x^3$

$\mathcal{R}_2$ and Ψ_2. Let $\mathcal{R}_2$ be as in Fig. 19B. That is, $\mathcal{R}_2$ is the Cartesian coordinate spacetime region bounded from below by the flat hypersurface portion $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$, on the left by $\gamma_{-\frac{1}{2}}\left([0, t_{-\frac{1}{2}}]\right)$ (where, as in (C.10) with the data (C.14), $t_{-\frac{1}{2}} = \frac{4}{3}$ is the value of t such that the curve $t \to \gamma_{-\frac{1}{2}}(t)$ intersects $\mathcal{S}_{[-\frac{1}{2},0]}$), from above by $\mathcal{S}_{[-\frac{1}{2},0]} \cup \left(\Sigma_1 \cap \{1 \leq x \leq \frac{25}{24}\}\right)$, and on the right by $\gamma_{\frac{1}{2}}([0,1])$. We consider $\mathcal{S}_{[-\frac{1}{2},0]}$ to *not* be part of $\mathcal{R}_2$, while we consider the other boundary portions to be part of $\mathcal{R}_2$. Like $\mathcal{R}_1$, $\mathcal{R}_2$ is foliated by portions of the characteristics emanating from $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$, and in view of (C.7), we can define the classical solution Ψ_2 on $\mathcal{R}_2$ such that Ψ_2 is constant along the characteristics that foliate $\mathcal{R}_2$ and such that $\Psi_2(0,x) = \mathring{\Psi}(x) = -x + \frac{1}{3}x^3$ on $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$. Ψ_2 is a classical solution on $\mathcal{R}_2$ because we have removed the singular curve portion $\mathcal{S}_{[-\frac{1}{2},0]}$, where $\partial_x \Psi_2$ blows up. Note, however, that $\partial_x \Psi_2$ is not uniformly bounded on $\mathcal{R}_2$.

Description of $\mathcal{R}_1 \cap \mathcal{R}_2$. We now describe $\mathcal{R}_1 \cap \mathcal{R}_2$; see Fig. 20. $\mathcal{R}_1 \cap \mathcal{R}_2$ is a disconnected set equal to the disjoint union $\mathcal{D}_1 \cup \mathcal{D}_2$, where: **a)** $\mathcal{D}_1$ is the region bounded from below by the flat hypersurface portion $\Sigma_0 \cap \{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$, from above by the flat hypersurface portion $\left(\Sigma_1 \cap \{\frac{23}{24} \leq x \leq \frac{25}{24}\}\right)$, on the left by $\gamma_{-\frac{1}{2}}([0,1])$, and on the right by $\gamma_{\frac{1}{2}}([0,1])$, where the crease $(1,1)$ does not belong to $\mathcal{D}_1$ but the remaining boundary portions do belong to $\mathcal{D}_1$; and **b)** $\mathcal{D}_2$ is the region bounded by the curve portions $\mathcal{S}_{[0,\frac{1}{4}]}$, $\mathcal{S}_{[-\frac{1}{4},0]}$, $\gamma_{-\frac{1}{2}}\left((\frac{16}{15}, \frac{12}{11})\right)$, and $\gamma_{\frac{1}{2}}\left((\frac{16}{15}, \frac{12}{11})\right)$, where the four vertices of $\mathcal{D}_2$, starting from the crease and heading clockwise, are $(1,1)$, $(\frac{16}{15}, \frac{19}{18})$, $(\frac{12}{11}, \frac{12}{11})$, and $(\frac{16}{15}, \frac{97}{90})$. Note that the singular curve portions $\mathcal{S}_{[0,\frac{1}{4}]}$ and $\mathcal{S}_{[-\frac{1}{4},0]}$ do not belong to $\mathcal{D}_2$ but the remaining boundary portions do belong to $\mathcal{D}_2$. In obtaining the structure of $\mathcal{D}_2$, we have used the following four facts, which can be computed using the formula (C.14) for the initial data, the expression $\gamma_z(t) = \left(t, z + t(1 - z + \frac{1}{3}z^3)\right)$ (which follows from (C.4) and (C.8)), and the expression (C.12) for the singular curve: **i)** $\mathcal{S}_{[0,\frac{1}{4}]}$ and $\mathcal{S}_{[-\frac{1}{4},0]}$ intersect at the crease point $(t,x) = (1,1) = \mathscr{S}(0)$; **ii)** the curve $t \to \gamma_{-\frac{1}{2}}(t)$ intersects $\mathcal{S}_{[0,\frac{1}{2}]}$ at the point $(t,x) = (\frac{16}{15}, \frac{19}{18}) = \mathscr{S}(\frac{1}{4})$; **iii)** the curves $t \to \gamma_{-\frac{1}{2}}(t)$ and $t \to \gamma_{\frac{1}{2}}(t)$ intersect at the point $(t,x) = (\frac{12}{11}, \frac{12}{11})$, **iv)** the curve $t \to \gamma_{\frac{1}{2}}(t)$ intersects $\mathcal{S}_{[-\frac{1}{2},0]}$ at the point $(t,x) = (\frac{16}{15}, \frac{97}{90}) = \mathscr{S}(-\frac{1}{4})$.

The above example shows that in the study of Burgers' equation flow, the non-uniqueness of classical solutions past shock singularities is an unavoidable aspect of the theory. However, in the regime of $3D$ compressible Euler flow that we study in this paper, the mechanism for non-uniqueness of classical solutions that we highlighted above for Burgers' equation *does not occur*. The reason is that in the regime of compressible Euler flow that we study, there are other characteristic directions that, for classical solutions, *block the crossing of the characteristics*, even though the density of a family of characteristics can blow up (resulting in a singularity in the fluid's first derivatives). In particular, one must distinguish between the *actual crossing* of the characteristics (which, in the Burgers' equation example was tied to multi-valued solutions and lack of uniqueness) and their density becoming infinite. We will explain this phenomenon in the case of $1D$ compressible Euler flow. More precisely, in the $1D$ case, another characteristic curve, called the *Cauchy horizon*, emanates from the crease (i.e., the analog of the point $(1,1)$ in Fig. 17) and delineates a portion of the boundary of the maximal classical development. The Cauchy horizon "blocks" a family of characteristics from entering the region where they would have crossed another family and developed infinite density, thereby defeating the issue of multi-valuedness of solutions and restoring uniqueness; see Fig. 21 for a schematic depiction of the effect of a Cauchy horizon, which we denote by "$\underline{\mathcal{C}}$."

Let us further describe the emergence of the Cauchy horizon and its significance in the case of $1D$ isentropic compressible Euler flow. In this context, we can study the system using Riemann invariants $\mathcal{R}_{(+)}^{\mathrm{PS}}$ and $\mathcal{R}_{(-)}^{\mathrm{PS}}$; see the coupled system (A.4), where we imagine that the vectorfield L^{PS} in that system is an analog of L_{Burg} and that $\mathcal{R}_{(+)}^{\mathrm{PS}}$, which solves $L^{\mathrm{PS}}\mathcal{R}_{(+)}^{\mathrm{PS}} = 0$, is an analog of the Burgers' equation variable Ψ, and the vectorfield $\underline{L}^{\mathrm{PS}}$ represents a "new speed," i.e., a new characteristic direction compared to the case of Burgers' equation. Assume that one is studying shock-forming solutions such that the dynamics are dominated by L^{PS} and $\mathcal{R}_{(+)}^{\mathrm{PS}}$, i.e., the characteristics corresponding to L^{PS} develop infinite density and cause a singularity in $\partial_x \mathcal{R}_{(+)}^{\mathrm{PS}}$, while $\mathcal{R}_{(-)}^{\mathrm{PS}}$ remains C^1; with the help of the techniques introduced in Appendix A, one can show that open sets of such solutions exist. For such solutions, the blowup of $\partial_x \mathcal{R}_{(+)}^{\mathrm{PS}}$ would happen along a singular curve, denoted by "$\mathcal{B}$" in Fig. 21, which is an analog of the branch $\mathcal{S}_{[0,1)}$ from Fig. 17, which contains the crease $(1,1)$. However, unlike in Burgers equation, a Cauchy horizon, denoted by "$\underline{\mathcal{C}}$" in Fig. 21, would emanate from the crease. The Cauchy horizon is a characteristic that is generated by the *other characteristic direction* in the flow. That

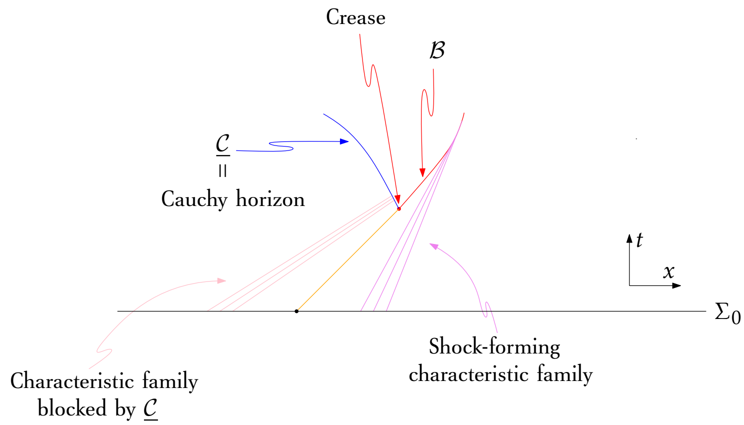


Figure 21. A schematic depiction of the effect of a Cauchy horizon in $1D$ isentropic compressible Euler solutions

is, in the context of Fig. 21, $\underline{\mathcal{C}}$ is tangent to the vectorfield $\underline{L}^{\text{PS}}$ from (A.4). See also Fig. 1A for a picture of the Cauchy horizon in the context of our main results in $3D$. It is important to appreciate that in the regime we study in this paper and our companion work [3], no singularity occurs along the Cauchy horizon, except on its past boundary (i.e., the crease in Fig. 21). Nonetheless, in compressible Euler flow, the Cauchy horizon lies in the causal future[81] of the crease, where the fluid's first partial derivatives with respect to the Cartesian coordinates blow up, and it is therefore impossible to uniquely continue the fluid solution past the Cauchy horizon to a larger globally hyperbolic region $\mathcal{M}$ such that the solution *classically* exists on $\mathcal{M}$ and such that the data-hypersurface is a Cauchy hypersurface for $\mathcal{M}$. We emphasize that for the $1D$ compressible solutions described above, the Cauchy horizon lies in the causal future of the crease precisely because it is generated by the characteristic vectorfield $\underline{L}^{\text{PS}}$.

For the $1D$ solutions described above, which are dominated by the dynamics of L^{PS} and $\mathcal{R}_{(+)}^{\text{PS}}$, the portion of the singular curve that corresponds to the branch $\mathcal{S}_{[0,1)}$ from Fig. 17 (and which is denoted by "$\mathcal{B}$" in Fig. 21) *is* part of the boundary of the maximal classical globally hyperbolic development, while the portion of the singular curve that corresponds to the branch $\mathcal{S}_{(-1,0)}$ is *not* part of the boundary. The Cauchy horizon blocks all the characteristics lying to the left of the crease in Fig. 17 from crossing any of the characteristics lying to the right of the crease in Fig. 17, thereby preventing the analog of singular curve portion $\mathcal{S}_{(-1,0)}$ in Fig. 17 from forming; hence, in Fig. 21, there is no analog of the singular curve portion $\mathcal{S}_{(-1,0)}$. In the geometric coordinates picture for $1D$ compressible Euler flow, the analog of the set $\mathcal{B}_{(0,1)}$ from see Fig. 18 corresponds to $\mathcal{S}_{(-1,0)}$. Hence, $\mathcal{B}_{(0,1)}$ does not correspond to any portion of the boundary of the maximal classical globally hyperbolic development; $\mathcal{B}_{(0,1)}$ is, in fact, the "fictitious portion" of the singular boundary that we described in Sect. 1.3. In $1D$ compressible Euler flow, the fictitious portion of the singular boundary can be identified by the condition $\frac{\partial}{\partial u}\upmu > 0$, which is satisfied along $\mathcal{S}_{(-1,0)}$; see (C.34). Moreover, in the geometric coordinates picture for $1D$ compressible fluids, one cannot even uniquely construct the solution $(\mathcal{R}_{(+)}^{\text{PS}}, \mathcal{R}_{(-)}^{\text{PS}})$ all the way up to analog of $\mathcal{S}_{(-1,0)}$ because it is cut off by the Cauchy horizon. We refer to Sects. 1.3 and 1.10.6 for further discussion of the significance of the condition $\frac{\partial}{\partial u}\upmu > 0$ (more precisely, the condition $\breve{X}\upmu > 0$ in Sects. 1.3 and 1.10.6) in compressible fluid flow and its relevance for identifying the fictitious portion of the singular boundary.

C.5.2. *The homeomorphism and diffeomorphism properties – and their breakdown – of the change of variables map* $(t,u) \to (t,x)$. Consider the change of variables map

$$\Upsilon_{\text{Burg}}(t,u) \stackrel{\text{def}}{=} (t,x), \tag{C.37}$$

where on RHS (C.37), we are viewing $x = x(t,u)$. Using (C.2), (C.21), and (C.24), we compute that the differential of Υ_{Burg} with respect to the geometric coordinates (i.e., the Jacobian matrix), which we denote by $d_{\text{geo}}\Upsilon_{\text{Burg}}$, satisfies $d_{\text{geo}}\Upsilon_{\text{Burg}} = \begin{pmatrix} 1 & 0 \\ 1+\Psi & -\upmu \end{pmatrix}$. Hence, its Jacobian determinant is $\det d_{\text{geo}}\Upsilon_{\text{Burg}} = -\upmu$. Since $\upmu$ vanishes on the set $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}_{\mathbb{R}}$ defined in (C.28), Υ_{Burg} is *not* a diffeomorphism up to $\mathcal{B}$. The breakdown in the diffeomorphism properties of Υ_{Burg}

[81] Here, we mean causal with respect to the acoustical metric $\mathbf{g}$ defined in (2.15a).

supplies another way of thinking about why the Burgers' equation solution Ψ remains smooth in (t, u) coordinates – all the way up to $\mathcal{B}$ – while forming a first-derivative singularity in (t, x) coordinates. Despite the discrepancy between the two coordinate systems, using the monotonicity (C.29), the transversal convexity condition (C.33), and the sign of $\frac{\partial}{\partial u}\mu$ on $\mathcal{B}_{(-1,0)}$ guaranteed by (C.36), one can show (see the proof of Prop. 33.1 for the main ideas) the following important property: there is a $\varsigma > 0$ (depending on the initial data) such that Υ_{Burg} is a *homeomorphism* on $\mathcal{M}^{\mathrm{Burg}}_{\mathrm{Injective}}$, where $\mathcal{M}^{\mathrm{Burg}}_{\mathrm{Injective}}$ is defined to be the compact subset of geometric coordinate space lying above the flat hypersurface portion $\Sigma_0^{[-\frac{1}{2},\varsigma]}$ (see definition (C.18)) and below the $C^{1,1}$ hypersurface $\Sigma_0^{[-\frac{1}{2},0]} \cup \mathcal{B}_{[-\varsigma,0]}$; $\mathcal{M}^{\mathrm{Burg}}_{\mathrm{Injective}}$ is a subset of the region shown in Fig. 18. In particular, $\mathcal{M}^{\mathrm{Burg}}_{\mathrm{Injective}}$ contains the singular boundary portion $\mathcal{B}_{[-\varsigma,0]}$. This injectivity is essential for justifying that the Cartesian coordinate space subset $\Upsilon_{\mathrm{Burg}}\left(\mathcal{M}^{\mathrm{Burg}}_{\mathrm{Injective}}\right)$ can be equipped with geometric coordinates and for setting up the shock development problem. In the bulk of the paper, we proved similar homeomorphism and diffeomorphism (when $\mu > 0$) properties on the region $\mathcal{M}_{\mathrm{Interesting}}$ defined in (32.1d); see Prop. 33.1.

C.5.3. *Degeneracies along* $\mathcal{B}_{[-\varsigma,0]}$. Let $\mathcal{B}_{[-\varsigma,0]}$ be the subset of $\{\mu = 0\}$ described in Sect. C.5.2. Note that $\Upsilon_{\mathrm{Burg}}\left(\mathcal{B}_{[-\varsigma,0]}\right) = \mathcal{S}_{[0,\varsigma]}$, i.e., in the Cartesian coordinate space picture, $\mathcal{S}_{[0,\varsigma]}$ is a portion of the singular curve containing the image of the crease, i.e., $\Upsilon_{\mathrm{Burg}}(1, 0) = (1, 1)$; see Fig. 17. In equation (C.15) and the discussion just below it, we showed that at every[82] point $p \in \mathcal{S}_{(0,\varsigma]}$, the tangent vector to $\mathcal{S}_{(0,\varsigma]}$ at p is *also* a tangent vector for a unique (straight-line) characteristic that emanates from $\Sigma_0^{(0,\varsigma]}$ and intersects $\mathcal{S}_{(0,\varsigma]}$ at p. The tangent vector was also shown to be parallel to $L_{\mathrm{Burg}}|_p$, i.e., to $\partial_t + (1 + \Psi|_p)\partial_x$. This exhibits two key phenomena along $\mathcal{S}_{(0,\varsigma]}$ (analogous results also hold on $\mathcal{S}_{[-\varsigma,0)}$):

I) The tangent vector to $\mathcal{S}_{(0,\varsigma]}$ everywhere points in a characteristic direction. This means that $\mathcal{S}_{(0,\varsigma]}$ is a characteristic curve. A similar phenomenon occurs in our study of $3D$ compressible Euler flow; see Prop. 33.2. In $3D$, this leads to severe degeneracies in our energy estimates since, as is well known, for quasilinear hyperbolic systems, L^2-type energies are only *semi-definite* along characteristic hypersurfaces.

II) In the Cartesian coordinate space picture, the vectorfield L_{Burg} does not have unique integral curves at points in $\mathcal{S}_{(0,\varsigma]}$. Specifically, points $p \in \mathcal{S}_{(0,\varsigma]}$ belong to $\mathcal{S}_{(0,\varsigma]}$ *and* to the straight-line characteristic mentioned above, and both curves are integral curves of L_{Burg} that share the same tangent vector $L_{\mathrm{Burg}}|_p = \partial_t + (1 + \Psi|_p)\partial_x$. This lack of uniqueness stems from the blowup of $\partial_x\Psi|_p$, which implies in particular that in Cartesian coordinates, the vectorfield L_{Burg} is not Lipschitz up to $\mathcal{S}_{(0,\varsigma]}$ (Lipschitz regularity is a standard criterion from ODE theory for uniqueness of integral curves). The phenomenon of non-uniqueness of integral curves of a characteristic vectorfield also occurs in our study of $3D$ compressible Euler flow; see Remark 33.3.

C.5.4. *Choosing a formulation, the shock development problem, and a new speed.* Consider the first singularity $\Upsilon_{\mathrm{Burg}}(1, 0) = (1, 1)$ in Cartesian coordinate space, i.e., the image of the crease in Cartesian coordinate space; see Fig. 17. Ideally, one would like to uniquely continue the solution as a weak solution past the first singularity and to fully describe the *transition* of the solution from classical to weak. To ensure the uniqueness of weak solutions, selection criteria are needed. The issue of uniqueness of weak solutions is a devilishly subtle[83] one, and we will briefly discuss it further at the end of this section. For Burgers' equation, the weak solution should be smooth on the closure of each side of a *shock curve* $\mathcal{K}$ – which has to be constructed – but will jump across $\mathcal{K}$, where the jump is vanishing at $(1, 1)$ but "starts to develop" there; see Fig. 17 for a depiction of the shock curve $\mathcal{K}$ when the initial data are as in (C.14). Our definition of a weak solution is the standard one based on integrating against test functions. More precisely, since, for smooth solutions, Burgers' equation (C.1) is equivalent to $\partial_t\Psi + \partial_x[F(\Psi)] = 0$, where:

$$F(\Psi) \stackrel{\mathrm{def}}{=} \frac{1}{2}(1 + \Psi)^2, \tag{C.38}$$

[82]The results discussed here in fact hold for every $p \in \mathcal{S}_{[0,\varsigma]}$. This cannot be seen directly from equation (C.15) because as parameterized, the tangent vector to the singular curve $z \to \mathscr{S}(z)$ vanishes at $z = 0$ (which corresponds to the crease, assuming the initial data (C.19)). However, this vanishing is only due to the parameterization; by exploiting the transversal convexity (C.33), one could reparameterize the singular curve in (C.15) by using $\sqrt{z}$ as the curve parameter for $z \in [0, \varsigma]$, which would allow us to show that the results hold on all of $\mathcal{S}_{[0,\varsigma]}$; see Prop. 33.2, in which we used a similar reparameterization to show that in the context of our main results, the singular boundary is a $C^{1,\frac{1}{2}}$ embedded submanifold-with-boundary of Cartesian coordinate space.

[83]Even in $1D$, in the large-data regime for general quasilinear hyperbolic systems of conservation laws, the question of uniqueness of weak solutions remains largely open.

the weak formulation in the half-space $(t,x) \in [0,\infty) \times \mathbb{R}$ would posit that for every smooth, compactly supported function ϕ on $[0,\infty) \times \mathbb{R}$, we have the following identity, where as before, $\mathring{\Psi}(x) = \Psi(0,x)$:

$$\int_{[0,\infty)\times\mathbb{R}} \{\Psi(t,x)\partial_t\phi(t,x) + F(\Psi(t,x))\,\partial_x\phi(t,x)\}\,dt\,dx + \int_{\mathbb{R}} \mathring{\Psi}(x)\phi(0,x)\,dx = 0. \tag{C.39}$$

The problem of constructing $\mathcal{K}$ and the corresponding piecewise smooth weak solution Ψ as well as describing the transition of the solution from classical to weak (i.e., describing the nascent, evolving jump in Ψ across $\mathcal{K}$) is called the *shock development problem*, which we discuss below in the context of Burgers' equation. It should be distinguished from, for example, the well-known *Riemann problem* in $1D$, in which the initial data are *assumed* to be piecewise constant and with a *specified* jump discontinuity across a point. In contrast, in the shock development problem, the jump is just starting to develop at the crease. We refer to Sect. 1.4 for a discussion of the relationship between the shock development problem for $3D$ compressible Euler flow and the main results from the bulk of the paper.

It is easy to see from integration by parts that for smooth data, before the solution forms a singularity, classical Burgers' equation solutions are also weak solutions, i.e., the classical and weak formulations "agree." However, at the first singularity, one must **choose** whether to keep the classical formulation or to instead impose a weak one. If one **chooses** the classical formulation, then one can at best hope to construct a maximal classical development, i.e., a "largest possible region" on which a classical solution exists (in particular, a maximal classical development does not contain any portion of the singular curve from Fig. 17), in the spirit of what we achieved in Sect. C.2. However, as we discussed in Sect. C.5.1, for Burgers' equation, one encounters non-uniqueness of classical solutions past the first singularity.

Remark C.1 (Non-uniqueness of maximal globally hyperbolic classical developments is not known for compressible Euler flow)**.** For compressible Euler flow with smooth initial data, as of present, there are no known examples of non-uniqueness of maximal globally hyperbolic classical developments (i.e., developments with a Cauchy hypersurfaces, as explained at the beginning of Appendix C). This is due to the phenomena of Cauchy horizons, as described above.

Alternatively, one could **choose** to impose a weak formulation of the flow, starting from the first singularity. It turns out (see below) that the standard weak formulation of the flow introduces a **new speed** into the problem, the speed of $\mathcal{K}$ itself. It also turns out that – like the singular curve portion $\mathcal{S}_{(0,\varsigma]}$ in Fig. 17 – $\mathcal{K}$ emanates from $(1,1)$, but it lies below $\mathcal{S}_{(0,\varsigma]}$. For this reason, the weak and classical solutions disagree in the region lying below $\mathcal{S}_{(0,\varsigma]}$ and above $\mathcal{K}$. For $3D$ compressible Euler flow, despite the irrelevance of part of the classical solution for the weak solution, it turns out that it is crucial to understand the structure of the maximal classical development to properly set up and solve the shock development problem; see Sect. 1.6 for further discussion.

As we alluded to above, a fundamental aspect of the shock development problem is the construction of the *shock curve*, denoted by $\mathcal{K}$ in Fig. 17 in the context of Burgers' equation. The evolution of $\mathcal{K}$ is coupled to that of Ψ. In Fig. 17, $\mathcal{K}$ is correctly depicted as a straight line segment, but the straightness is an artifact of the oddness of the initial data (C.14), whose corresponding solution is depicted in the figure; in general, $\mathcal{K}$ is a portion of a curve that has to be solved for. For compressible Euler flow, the shock development problem is difficult even in spherical symmetry (in part due to the multi-speed nature of the system), and the spherically symmetric problem was solved only very recently [27]; we refer to Sect. 1.9.12 for further discussion. For Burgers' equation, the shock development problem is easier to solve; see [22,43,47] for theorems that yield local existence for families of $1D$ shock development problems that include Burgers' equation as a special case. Here, we will sketch the argument of how to solve for Ψ and $\mathcal{K}$. We will use the initial data (C.14) and the corresponding Fig. 17 to guide our discussion. To proceed, we recall that for smooth solutions, Burgers' equation (C.1) is equivalent to $\partial_t\Psi + \partial_x[F(\Psi)] = 0$, where F is defined in (C.38). For clarity, we will refer to the weak (but piecewise smooth) solution as Ψ_{Weak}. Assume the following:

i) The shock curve $\mathcal{K}$ emanates from the crease (i.e., the point $(1,1)$ in the context of Fig. 17) and is parameterized in (t,x)-space by $t \to (t, \mathfrak{k}(t))$.

ii) To the left of $\mathcal{K}$ (we avoid defining Ψ_{Weak} on $\mathcal{K}$ itself), $\Psi_{\text{Weak}}(t,x) = \Psi_{\text{Left}}(t,x)$, where $\Psi_{\text{Left}}(t,x)$ is smooth up to $\mathcal{K}$ and determined by from the data by characteristics coming from the left (recall that Ψ is constant along characteristics).

iii) To the right of $\mathcal{K}$, $\Psi_{\text{Weak}}(t,x) = \Psi_{\text{Right}}(t,x)$, where $\Psi_{\text{Right}}(t,x)$ is smooth up to $\mathcal{K}$ and determined from the data by characteristics coming from the right.

Under the above assumptions, the standard Rankine–Hugoniot jump condition for weak solutions (see, for example, [33, Equation (3.1.3)]) tie the jump in Ψ_{Weak} across $\mathcal{K}$ (i.e., $\Psi_{\text{Left}} - \Psi_{\text{Right}}$) to the shock curve speed $\mathfrak{k}'(t) \overset{\text{def}}{=} \frac{d}{dt}\mathfrak{k}(t)$ and is as follows:

$$\Bbbk'(t) = \frac{F(\Psi_{\mathrm{Left}}(t,\Bbbk(t))) - F\left(\Psi_{\mathrm{Right}}(t,\Bbbk(t))\right)}{\Psi_{\mathrm{Left}}(t,\Bbbk(t)) - \Psi_{\mathrm{Right}}(t,\Bbbk(t))} = \frac{1}{2}\left\{1 + \Psi_{\mathrm{Left}}(t,\Bbbk(t))\right\} + \frac{1}{2}\left\{1 + \Psi_{\mathrm{Right}}(t,\Bbbk(t))\right\}, \qquad \Bbbk(1) = 1, \tag{C.40}$$

where the initial condition in (C.40) stems from the fact that $\mathcal{K}$ emanates from the crease point, i.e., the point $(1,1)$ in the (t,x) plane. Note that (C.40) implies that $\Bbbk'(t)$ lies between the speed $1 + \Psi_{\mathrm{Left}}(t,\Bbbk(t))$ corresponding to the Ψ_{Left}-characteristics and the speed $1 + \Psi_{\mathrm{Right}}(t,\Bbbk(t))$ corresponding to the Ψ_{Right}-characteristics. Importantly, for the initial data $\mathring{\Psi}(x) = -x + \frac{1}{3}x^3$ under consideration, $\Psi_{\mathrm{Left}}(t,\Bbbk(t))$ is larger than $\Psi_{\mathrm{Right}}(t,\Bbbk(t))$ (except at $(1,1)$, where $\Psi_{\mathrm{Left}} = \Psi_{\mathrm{Right}}$) and thus:

$$1 + \Psi_{\mathrm{Right}}(t,\Bbbk(t)) < \Bbbk'(t) < 1 + \Psi_{\mathrm{Left}}(t,\Bbbk(t)), \tag{C.41}$$

except when $t = 1$, where all three speeds in (C.41) are equal to unity. Hence, $\mathcal{K}$ is slower than the Ψ_{Left}-characteristics but faster than the Ψ_{Right}-characteristics. One can show that for the *odd* initial data (C.14), the terms $\Psi_{\mathrm{Left}}(t,\Bbbk(t))$ and $\Psi_{\mathrm{Right}}(t,\Bbbk(t))$ to the right of the second equality in (C.40) *cancel*, leading to $\Bbbk'(t) \equiv 1$; this explains why for this data, the shock curve $\mathcal{K}$ is a segment of a straight line; see Fig. 17.

Geometrically, (C.41) means that the Ψ_{Left}-characteristics impinge on $\mathcal{K}$ from the left, while the Ψ_{Right}-characteristics impinge on $\mathcal{K}$ from the right. That is, as one heads to the future, the characteristics impinge on the shock curve from both sides, as opposed to emanating from it; see Fig. 17. Inequality (C.41) (and its geometric interpretation) is an example of what is known in the hyperbolic conservation laws literature as the *Lax entropy condition*. This condition is crucial for the shock development problem and the Riemann problem. For the shock development problem, the Lax entropy condition guarantees that the Ψ_{Left}-characteristics hit $\mathcal{K}$ *before* they crash into the singular curve portion $\mathcal{S}_{(-1,0)}$, along which $\partial_x \Psi_{\mathrm{Left}}$ would have blown up (as we showed in Sect. C.2). Similarly, it guarantees that the Ψ_{Right}-characteristics hit $\mathcal{K}$ *before* they crash into the singular curve portion $\mathcal{S}_{(0,1]}$, where $\partial_x \Psi_{\mathrm{Right}}$ would have blown up. We highlight that the point $(1,1)$ (i.e., the crease in the Cartesian coordinate space picture) is of particular importance and lies on $\mathcal{K}$, $\mathcal{S}_{[-1,0]}$, and $\mathcal{S}_{[0,1]}$.

The Lax entropy condition (C.41) is a crucial inequality that is often used as an ingredient (not the only one) in ensuring uniqueness of weak solutions. If one does not impose it, then even the aforementioned Riemann problem in $1D$ can exhibit non-uniqueness of weak solutions; see, e.g., [39, Section 3.4.1]. To guarantee uniqueness of weak solutions in a suitably large function space for which global existence is know (at least for data with small BV norm), technical conditions are required. In $1D$, various criteria for uniqueness have been established in the hyperbolic conservations laws literature; see, for example, [33, Theorem (14.10.2)], which proves a uniqueness result for weak solutions to strictly hyperbolic conservation laws in $1D$ arising from BV initial data such, as long as the solution satisfies the Lax entropy condition and a “Tame Oscillation Condition,” which is an assumed quantitative constraint on the oscillation of the solution in time. We also note that for *scalar* conservation laws in arbitrary dimensions (of which Burgers’ equation is a simple example), there exists a robust theory of existence and uniqueness of global weak solutions; see, for example, [33, Theorem (6.2.2)].

Appendix D. Notation

For the reader’s convenience, in this appendix, we have gathered some of the notation and conventions used throughout the bulk of the paper.

Cartesian coordinate space and derivatives		
Symbols	Descriptions	Reference
$\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$	The ambient $(1+3)$-dimensional spacetime. The spatial domain is $\mathbb{R}\times\mathbb{T}^2$	Sect. 1.2
$\{x^\alpha\}_{\alpha=0,1,2,3}$	The Cartesian coordinates on $\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2$, relative to which the Minkowski metric m has components $m_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$. We often use the notation $t \overset{\text{def}}{=} x^0$ for Cartesian time.	Sect. 1.2
$\{\partial_\alpha\}_{\alpha=0,1,2,3}$	Cartesian coordinate partial derivative vectorfields. Note that $\{\partial_2,\partial_3\}$ can be extended to a smooth, globally defined positively oriented frame on $\mathbb{T}^2$.	Sect. 1.2
$\boldsymbol{\partial} f$	The array of Cartesian coordinate spacetime partial derivatives of a function f, i.e., $\boldsymbol{\partial} f \overset{\text{def}}{=} (\partial_t f, \partial_1, \partial_2 f, \partial_3 f)$	Sect. 1.10.3
∂f	The array of Cartesian coordinate spatial partial derivatives of f, i.e., $\partial f \overset{\text{def}}{=} (\partial_1, \partial_2 f, \partial_3 f)$.	Sect. 1.10.3
Lowercase Greek "spacetime" indices, such as α, vary over $0,1,2,3$, while Lowercase Latin "spatial" indices, such as a, vary over $1,2,3$. We use Einstein's summation convention in that repeated indices are summed over their respective ranges.		
Σ_t	The hypersurface constant Cartesian time t: $\Sigma_t \overset{\text{def}}{=} \{(x^0,x^1,x^2,x^3)\in\mathbb{R}\times\mathbb{R}\times\mathbb{T}^2 \mid x^0\equiv t\}$	Sect. 1.2
$\mathrm{curl}\, V$	The Euclidean curl of a Σ_t-tangent vectorfield V, with components $(\mathrm{curl}\, V)^k \overset{\text{def}}{=} \epsilon_{ijk}\delta^{jl}\partial_l V^k$, where ϵ_{ijk} denotes the fully antisymmetric symbol normalized by $\epsilon_{123}=1$ and δ^{jl} is the Kronecker delta.	Def. 2.3
$\mathrm{div}\, V$	The Euclidean divergence of a Σ_t-tangent vectorfield V, $\mathrm{div}\, V \overset{\text{def}}{=} \partial_a V^a$.	Def. 2.3

Fluid variables and some geometric objects		
Symbols	Descriptions	Reference
v^i,ϱ,s,p	$\{v^i\}_{i=1,2,3}$ is the Σ_t-tangent fluid velocity, ϱ is the fluid density, and s is the entropy. $p=p(\varrho,s)$ denotes the equation of state.	Sect. 1.2
$\mathbf{B}$	The material vectorfield.	(1.2)
$\bar{\varrho}$	A fixed positive constant "background density."	(2.2)
$\uprho, c$	$\uprho = \ln\left(\frac{\varrho}{\bar{\varrho}}\right)$ denotes the logarithmic density. The speed of sound is $c(\uprho,s)=\sqrt{(\bar{\varrho})^{-1}\exp(-\uprho)p_{;\uprho}}$, where $p_{;\uprho} \overset{\text{def}}{=} \frac{\partial p}{\partial \uprho}$ denotes the derivative of the equation of state with respect to the logarithmic density at fixed s.	Sect. 2.3.1
$\mathcal{R}_{(+)}$, $\mathcal{R}_{(-)}$	The almost Riemann invariants.	(2.7)
Ω^i, S^i, $\mathcal{C}^i$, $\mathcal{D}$	$\{\Omega^i\}_{i=1,2,3}$ denotes the specific vorticity, $\{S^i\}_{i=1,2,3}$ denotes the entropy gradient, and $\{\mathcal{C}^i\}_{i=1,2,3},\mathcal{D}$ denote the modified fluid variables.	Def. 2.7
$\vec{\Psi}$, $\vec{\Psi}_{(\text{Partial})}$	The array and partial array of wave-variables.	(2.11a)–(2.11b)
$\mathbf{g}$, $\mathbf{g}^{-1}$	The acoustical metric and inverse metric of spacetime.	Def. 2.10.
$G^\iota_{\alpha\beta}$, $\vec{G}_{\alpha\beta}$	$\vec{\Psi}_\iota$-derivatives of $\mathbf{g}_{\alpha\beta}$ and the array of $\vec{\Psi}_\iota$-derivatives of $\mathbf{g}_{\alpha\beta}$.	Def. 2.18a
$D\vec{\Psi}$, $\vec{G}_{V_1 V_2}\diamond D\vec{\Psi}$	Differential operators involving $\vec{\Psi}$.	Def. 2.12
$\square_{\mathbf{g}}$	The covariant wave operator of $\mathbf{g}$.	Def. 2.13
$\mathfrak{Q}^{(\mathbf{g})},\mathfrak{Q}_{\alpha\beta}$	Standard $\mathbf{g}$-null forms.	Def. 2.14

The acoustic geometry		
Symbols	Descriptions	Reference
u, μ	The eikonal function and inverse foliation density.	Def. 3.1
Σ_t, $\mathcal{P}_u$, $\ell_{t,u}$, $\Sigma_t^{[u_1,u_2]}$, $\mathcal{P}_u^{[t_1,t_2]}$	Acoustic regions and truncated acoustic regions of spacetime.	Def. 3.2
g, $g\!\!/$	First fundamental forms of Σ_t and $\ell_{t,u}$ induced by $\mathbf{g}$, respectively.	Def. 3.4
$\mathbf{D}$, $\nabla\!\!\!/$, $\mathrm{div}\!\!\!/$, $\Delta\!\!\!/$	Levi-Civita connections of g, $g\!\!/$ and associated differential operators.	Def. 3.11
$\mathbf{Riem}$, $\mathbf{Ric}$	The Riemann and Ricci curvature tensors of $\mathbf{g}$.	(6.31)–(6.32)
(t,u,x^2,x^3)	The geometric coordinate system.	Def. 3.5.
$\{\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^A}\}$	The coordinate partial derivative vectorfields in geometric coordinates.	Def. 3.5.
$\frac{\partial^{\vec{\alpha}}}{\partial(t,u,x^2,x^3)}$	Multi-index notation in geometric coordinates.	(8.1)
Uppercase Latin indices such as A vary over 2,3. We use Einstein's summation convention in that repeated indices are summed over their respective ranges.		
The contraction of a one-form ξ and $\frac{\partial}{\partial x^A}$ is denoted by $\xi_A \overset{\mathrm{def}}{=} \xi_\alpha(\frac{\partial}{\partial x^A})^\alpha$ for $A=2,3$. A similar convention is used for higher order contractions, i.e. $g\!\!/_{AB} = g\!\!/(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B})$ for $A,B=2,3$.		
$L_{(\mathrm{Geo})}$, L, X, $\breve{X}$, $Y_{(A)}$	The important acoustic vectorfields.	Def. 3.8
$L^i_{(\mathrm{Small})}$, $X^i_{(\mathrm{Small})}$, $Y^i_{(A;\mathrm{Small})}$	L^∞-small "Error parts" of the acoustic vectorfields.	(3.10), (3.13), (3.15)
$\mathcal{Z}$, $\mathcal{P}$, $\mathcal{Y}$	Sets of the commutation vectorfields, $\mathcal{P}_u$-tangent vectorfields, and $\ell_{t,u}$-tangent vectorfields, respectively.	(3.16)
$\mathcal{Z}^{N;M}$, $\mathcal{P}^N$, $\mathcal{Y}^N$, $\mathcal{Z}_*^{N;M}$, $\mathcal{P}_*^N$, $\mathcal{Z}_{**}^{N;M}$, $\mathfrak{P}^{(N)}, \mathfrak{Y}^{(N)}$	Strings of commutation vectorfields.	Def. 8.10
$\underline{L}$	A $\mathbf{g}$-null vectorfield normalized by $\mathbf{g}(\underline{L},L)=-2$	(7.1)
$\Pi, \Pi\!\!\!/, \Pi\xi, \Pi\!\!\!/\xi, \xi\!\!\!/$	Σ_t and $\ell_{t,u}$-projection tensorfields. Projections of spacetime tensors ξ onto Σ_t and $\ell_{t,u}$.	Def. 3.3
$\underline{\mathcal{L}}_Z\xi, \mathcal{L}\!\!\!/_Z\xi$,	Σ_t-projected and $\ell_{t,u}$-projected Lie derivatives.	Def. 3.12
$\mathcal{L}\!\!\!/_{\mathcal{Z}}^{N;M}\xi, \mathcal{L}\!\!\!/_{\mathcal{P}}^N\xi, \mathcal{L}\!\!\!/_{\mathcal{Y}}^N\xi$.	Strings of $\ell_{t,u}$-projected Lie derivatives.	Def. 8.10
$\mathrm{tr}_{\mathbf{g}}\xi, \mathrm{tr}_{g\!\!/}\xi$	Trace of spacetime and $\ell_{t,u}$-tangent tensors.	Def. 3.16
$\lvert\xi\rvert_{\mathbf{g}}, \lvert\xi\rvert_g, \lvert\xi\rvert_{g\!\!/}$	Pointwise norms of tensorfields.	Def. 3.17
k, χ, ζ	The second fundamental form of Σ_t, the null second fundamental form of $\ell_{t,u}$, and the $\ell_{t,u}$-tangent one form.	Def. 3.20
$\gamma, \underline{\gamma}$	The controlling quantities.	Def. 3.14
$d\!\!\!/\vec{x}$	The array of $\ell_{t,u}$-projected spatial coordinate one-forms $d\!\!\!/\vec{x} = (d\!\!\!/x^1, d\!\!\!/x^2, d\!\!\!/x^3)$	(9.2)
${}^{(Z)}\boldsymbol{\pi}$	The deformation tensor of a spacetime vectorfield Z with respect to $\mathbf{g}$.	(9.15)
$\mathring{\Delta}^{N_{\mathrm{top}}+1}_{\Sigma_0^{[u_1,u_2]}}$	Norm of the data perturbation on Σ_0.	(11.4)
$\lVert f\rVert_{H^N_{\mathrm{Cartesian}}(\Sigma_0^{[u_1,u_2]})}$, $\lVert f\rVert_{L^2_{\mathrm{Cartesian}}(\ell_{0,u})}$	Sobolev and L^2 norms on $\Sigma_0^{[u_1,u_2]}$ and $\ell_{0,u}$.	(11.5a)–(11.5b)

The rough time function and associated regions of spacetime		
Symbols	Descriptions	Reference
$\phi = \phi(u)$	The cut-off function which is identically equal to 1 when $\lvert u\rvert \leq \frac{1}{2}U_{☢}$.	(4.1)
${}^{(\mathfrak{n})}\breve{W}$	The rough transversal vectorfield ${}^{(\mathfrak{n})}\breve{W} = \breve{X} + \phi\frac{\mathfrak{n}}{L\mu}L$.	(4.2)
$\breve{\mathbb{M}}_{\mathfrak{m}}$, $\breve{\mathbb{X}}_{-\mathfrak{n}}$, $\breve{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$	Level-sets of $\mu, \breve{X}\mu$, and the μ-adapted tori.	Def. 4.2
${}^{(\mathfrak{n})}\tau$	The rough time function.	Def. 4.5
$\tau_0, -\mathfrak{m}_0$	The largest-in-magnitude value of ${}^{(\mathfrak{n})}\tau$.	(4.5)
$({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$	The adapted rough coordinates.	Def. 4.9
$\{\frac{\widetilde{\partial}}{\widetilde{\partial}{}^{(\mathfrak{n})}\tau}, \frac{\widetilde{\partial}}{\widetilde{\partial}u}, \frac{\widetilde{\partial}}{\widetilde{\partial}x^2}, \frac{\widetilde{\partial}}{\widetilde{\partial}x^3}\}$	The adapted rough coordinate partial derivative vectorfields.	Def. 4.9
$\frac{\widetilde{\partial}^{\vec{\alpha}}}{\widetilde{\partial}({}^{(\mathfrak{n})}\tau, u, x^2, x^3)}$	Multi-index notation in the adapted rough coordinates.	(8.2)
${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{I}$, ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$	The rough hypersurfaces and rough tori.	Def. 4.11
${}^{(\mathfrak{n})}\mathcal{P}_{u}^{J}$, ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$	Truncated rough foliations of spacetime.	Def. 4.11
$\breve{\mathbb{M}}_{\mathfrak{m}}^{I}$, $\breve{\mathbb{X}}_{-\mathfrak{n}}^{I}$	Truncated level-sets of μ and $\breve{X}\mu$.	Def. 4.12.
${}^{(\mathfrak{n})}\mathcal{N}_{[\tau_0, \tau_{\text{Boot}}]}$	Region of spacetime for which there is especially sharp control of μ. Specifically, the estimate (18.15) holds on it.	(18.12)

Relationships between the different coordinate systems		
Symbols	Descriptions	Reference
Υ	The map $(t, u, x^2, x^3) \mapsto (t, x^1, x^2, x^3)$.	(5.1)
${}^{(\mathfrak{n})}\mathscr{T}$	The map $(t, u, x^2, x^3) \mapsto ({}^{(\mathfrak{n})}\tau, u, x^2, x^3)$.	(5.2)
${}^{(\mathfrak{n})}\Phi$, ${}^{({}^{(\mathfrak{n})}\Phi)}\mathbf{J}$	The map $({}^{(\mathfrak{n})}\tau, u, x^2, x^3) \mapsto (\mu, \breve{X}\mu, x^2, x^3)$ and its Jacobian.	(5.4a)–(5.4b)

The rough acoustic geometry		
Symbols	Descriptions	Reference
$\widetilde{g\!\!/}, \widetilde{g}$	The first fundamental forms of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$ and ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_0, U_2]}$.	Def. 6.2
${}^{(\mathfrak{n})}\widetilde{L}$	The rough null vectorfield.	(6.3)
${}^{(\mathfrak{n})}\widetilde{\Lambda}$	The τ_0-normalized flow map of ${}^{(\mathfrak{n})}\widetilde{L}$.	Lemma 16.1
${}^{(\mathfrak{n})}U$, ${}^{(\mathfrak{n})}\breve{R}$, ${}^{(\mathfrak{n})}\hat{R}$, ${}^{(\mathfrak{n})}\widetilde{N}$, ${}^{(\mathfrak{n})}\hat{N}$	Several geometric vectorfields adapted to the rough foliations.	Def. 6.4
$\mathbf{A}_{A}^{B}$	Matrix governing the relationship between $\frac{\partial}{\partial x^A}, \frac{\widetilde{\partial}}{\widetilde{\partial}x^A}$ and $g\!\!/, \widetilde{g\!\!/}$.	(6.14)
${}^{(\mathfrak{n})}r$	Small, non-negative factor related to the size of ${}^{(\mathfrak{n})}\breve{R}$.	(6.20b).
$\widetilde{\mathfrak{Riem}}$, $\widetilde{\mathfrak{Ric}}$, $\widetilde{\mathfrak{R}}$	The Riemann and Ricci curvature tensors of $\widetilde{g\!\!/}$. The scalar and Gauss curvatures of $\widetilde{g\!\!/}$.	(6.33)–(6.36).

Norms, volume forms, and L^2 ingredients		
Symbols	Descriptions	Reference
$\|f\|_{W^{m,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{I,J})}$, $\|f\|_{C^{m,1}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{I,J})}$	L^∞-type Sobolev and Hölder norms of ${}^{(\mathfrak{n})}\mathcal{M}_{I,J}$ in the (t,u,x^2,x^3) coordinate system.	Def. 8.2
$\|f\|_{C^{m,1}_{\mathrm{rough}}(I\times J\times\mathbb{T}^2)}$	L^∞-type Hölder norms in the $({}^{(\mathfrak{n})}\tau,u,x^2,x^3)$ coordinate system.	(8.5)
$d\varpi, d\overline{\varpi}, d\underline{\varpi}, d\varpi_{\widetilde{g}}$	Volume forms induced on the rough foliations by $\mathbf{g}$ in the adapted rough coordinates used to define L^2 norms.	Def. 8.3
$\|\xi\|_{L^2({}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u})}$, $\|\xi\|_{L^2({}^{(u)}\mathcal{P}^{\mathfrak{n}}_J)}$, $\|\xi\|_{L^2({}^{(\mathfrak{n})}\widetilde{\Sigma}^I_\tau)}$, $\|\xi\|_{L^2({}^{(\mathfrak{n})}\mathcal{M}_{I,J})}$	L^2 norms relative to the volume forms $d\varpi, d\overline{\varpi}, d\underline{\varpi}, d\varpi_{\widetilde{g}}$	Def. 8.7
$d\mathrm{vol}_{\widetilde{g}}, d\mathrm{vol}_{\mathbf{g}}$	The canonical area and volume forms induced by $\widetilde{g}$ and $\mathbf{g}$, respectively.	Def. 8.5
$\mathbf{Q}[f]$	The energy-momentum tensor $\mathbf{Q}_{\alpha\beta}[f]=\mathbf{D}_\alpha\mathbf{D}_\beta f-\frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{g}^{-1}(\mathbf{D}f,\mathbf{D}f)$.	(20.18)
${}^{(Z)}\mathbf{J}[f]$	The energy current vectorfield ${}^{(Z)}\mathbf{J}^\alpha[f]=\mathbf{Q}^{\alpha\beta}[f]Z_\beta$.	(20.19)
$\breve{T}$	The multiplier vectorfield $(1+2\mu)L+2\breve{X}$.	(20.22)
$\mathbb{E}_{(\mathrm{Wave})}, \mathbb{F}_{(\mathrm{Wave})}$, $\mathbb{E}_{(\mathrm{Transport})}$, $\mathbb{F}_{(\mathrm{Transport})}$	The energies and null-fluxes for the wave and transport-variables, respectively.	Def. 20.7
$\mathbb{K}[f]$	A coercive spacetime integral used in the wave equation energy estimates.	(20.25)
${}^{(\breve{T})}\mathfrak{B}[f], {}^{(\breve{T})}\mathfrak{B}_{(1)}[f], \ldots, {}^{(\breve{T})}\mathfrak{B}_{(6)}[f]$	Bulk terms arising from $\mu\mathbf{Q}^{\alpha\beta}[f]{}^{(\breve{T})}\boldsymbol{\pi}_{\alpha\beta}[f]$ in the fundamental energy identity for wave equations.	(20.27)– (20.28f)
$\mathbb{Q}_N, \mathbb{K}_N, \mathbb{W}_N$	The N-th order wave-controlling quantities for all the wave-variables $\{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s\}$.	(20.43a)–(20.43c)
$\mathbb{Q}_N^{(\mathrm{Partial})}, \mathbb{K}_N^{(\mathrm{Partial})}$, $\mathbb{W}_N^{(\mathrm{Partial})}$	The N-th order wave-controlling quantities for all the partial wave-variables $\{\mathcal{R}_{(-)}, v^2, v^3, s\}$.	(20.44a)–(20.44c)
$\mathbb{V}_N, \mathbb{S}_N$	The N-th order specific vorticity and entropy gradient controlling quantities.	(20.45a)–(20.45b)
$\mathbb{V}_N^{(\mathrm{Rough\ Tori})}, \mathbb{S}_N^{(\mathrm{Rough\ Tori})}$	The N-th order specific vorticity and entropy gradient controlling quantities *on the rough tori*.	(20.46a)–(20.46b)
$\mathbb{C}_N, \mathbb{D}_N$	The N-th order controlling quantities for the modified fluid variables.	(20.47a)–(20.47b)
$\mathbb{C}_N^{(\mathrm{Rough\ Tori})}, \mathbb{D}_N^{(\mathrm{Rough\ Tori})}$	The N-th order controlling quantities for the modified fluid variables *on the rough tori*.	(20.48a)–(20.48b)
We use the following summation conventions: $\mathbb{Q}_{[N_1,N_2]}=\sum_{M=N_1}^{N_2}\mathbb{Q}_M$, $\mathbb{V}_{\leq N}=\sum_{M=0}^{N}\mathbb{V}_M$, and similarly for the other controlling quantities.		

Bootstrap assumptions		
Symbols	Descriptions	Reference
τ_{Boot}	The bootstrap rough-time.	(12.1)
$\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}$, $\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}$	The μ-adapted torus $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}$ is a graph over $(x^2,x^3)\in\mathbb{T}^2$: $\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}} = \{(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3) \mid (x^2,x^3)\in\mathbb{T}^2\}$.	(**BA μ − TORI STRUCTURE**)
${}^{(\mathfrak{n})}E$	The mapping defined by $(\mathfrak{m},x^2,x^3) \in (\mathfrak{m}_{\mathrm{Boot}},\mu_0]\times\mathbb{T}^2 \mapsto (\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3)$.	(12.4)
(**Quantitative improvement of bootstrap assumptions**) By this, we mean that some quantity Q was assumed to satisfy $A_1\le Q\le A_2$ in the bootstrap assumptions (where A_1,A_2 are real numbers), and we derive the improved bound $B_1\le Q\le B_2$, where $A_1<B_1\le B_2<A_2$.		
(**From soft to quantitative improvement of bootstrap assumptions**) By this, we mean that in the bootstrap assumptions, we assumed that some function Q belonged to some function space and had a finite norm in that space, and our improvement is a quantitative estimate for the norm of Q.		
(**Extension to the closure improvement of bootstrap assumptions**) By this, we mean that our bootstrap assumptions involved an assumption on the "open-at-the-top" domain ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}),[-U_0,U_2]}$, and we derive an improved result showing that the assumption holds on the closed domain ${}^{(\mathfrak{n})}\mathcal{M}_{[\tau_0,\tau_{\mathrm{Boot}}],[-U_0,U_2]}$.		

Embeddings and flow maps		
Symbols	Descriptions	Reference
${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}$	A subset of $\mathbb{R}\times\mathbb{T}^2$ which is diffeomorphic to $\check{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}})}$.	(14.4)
${}^{(\mathfrak{n})}H, {}^{(\mathfrak{n})}h$	The function ${}^{(\mathfrak{n})}H(t,x^2,x^3)=(t,{}^{(\mathfrak{n})}h(t,x^2,x^3),x^2,x^3)$ is an embedding from ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}\to{}^{(\mathfrak{n})}\mathcal{M}_{[\tau,\tau_{\mathrm{Boot}}),[-\frac{3}{4}U_1,\frac{3}{4}U_1]}$ whose image is $\check{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}})}$	Lemma 14.1
${}^{(\mathfrak{n})}\iota_{\Delta u}$	The flow map of $\check{W}$.	(14.9)
${}^{(\mathfrak{n})}F, {}^{(\mathfrak{n})}\mathscr{F}$	${}^{(\mathfrak{n})}F(\Delta u,t,x^2,x^3)={}^{(\mathfrak{n})}\iota_{\Delta u}\circ{}^{(\mathfrak{n})}H(t,x^2,x^3)$, and ${}^{(\mathfrak{n})}\mathscr{F}$ is its domain.	Lemma 14.2.
${}^{(\mathfrak{n})}M$	The 4×4 matrix-valued function on ${}^{(\mathfrak{n})}\mathscr{H}_{(\mathfrak{m}_{\mathrm{Boot}},\mathfrak{m}_0]}$ whose first column is $(0,1,0,0)^\top$ and whose last three columns are the Jacobian $d_{(t,x^2,x^3)}{}^{(\mathfrak{n})}H$.	(14.23)
$\mathscr{A}$	An "ambient" spacetime function of class $W^{3,\infty}_{\mathrm{geo}}({}^{(\mathfrak{n})}\mathcal{M}_{(\tau_0,\tau_{\mathrm{Boot}}),(-U_0,U_2)})$, satisfying the constraint $({}^{(\mathfrak{n})}\check{W}\mathscr{A})\vert_{\check{\mathbb{X}}_{-\mathfrak{n}}^{[\tau_0,\tau_{\mathrm{Boot}})}}=0$	Lemma 15.1
$\mathrm{t}_{\tau,\mathfrak{n}}$	Describes the Cartesian time function t as a function of (u,x^2,x^3) on the rough hypersurfaces. That is, ${}^{(\mathfrak{n})}\widetilde{\Sigma}_{\tau}^{[-U_0,U_2]}=\left\{(t,u,x^2,x^3)\mid t=\mathrm{t}_{\tau,\mathfrak{n}}(u,x^2,x^3),\,(u,x^2,x^3)\in[-U_0,U_2]\times\mathbb{T}^2\right\}$.	(15.28)
${}^{(\mathfrak{n})}\mathrm{I}$	The map defined by ${}^{(\mathfrak{n})}\mathrm{I}(x^2,x^3)=\Upsilon\circ\left(\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),\mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3),x^2,x^3\right)$, which is a diffeomorphism from $\mathbb{T}^2$ onto $\Upsilon\left(\check{\mathbf{T}}_{\mathfrak{m},-\mathfrak{n}}\right)$.	(18.21)

The elliptic–hyperbolic integral identities		
Symbols	Descriptions	Reference
$\overline{\Pi}$	The projection tensorfield $\overline{\Pi}_{\beta}^{\ \alpha} = \delta_{\beta}^{\ \alpha} + \frac{1}{2}\underline{L}^{\alpha} L_{\beta}$ onto $\mathcal{P}_u$.	(21.1)
$\mathbf{h}, \mathbf{h}^{-1}$	Spacetime Riemannian metric and inverse metrics.	(21.3a)–(21.3b)
$\overline{e}, \overline{E}$	Positive semi-definite $\binom{0}{2}, \binom{2}{0}$ tensorfields which restrict to Riemannian metric and inverse metrics on $\mathcal{P}_u$.	(21.4)–(21.5)
$\lvert ZV \rvert_g$	The g-norm of the Σ_t-tangent vectorfield ZV^a, where Z is an arbitrary spacetime vectorfield and V^a is a Σ_t-tangent vectorfield.	Def. 21.5
$\lvert\xi\rvert_{\mathbf{h}}$	The $\mathbf{h}$-norm of a $\binom{m}{n}$ spacetime tensorfield.	Def. 21.6
$\mathscr{Q}[\boldsymbol{\partial} V, \boldsymbol{\partial} V]$	The coercive elliptic-hyperbolic quadratic form.	(21.23)
$\mathscr{J}[V, \boldsymbol{\partial} V]$	The characteristic current, also referred to as the elliptic-hyperbolic current.	(21.29)
$\mathscr{W}$	An arbitrary weight function.	Lemma 21.12
$\mathfrak{J}_{(\text{Antisymmetric})}[\boldsymbol{\partial} V, \boldsymbol{\partial} V]$, $\mathfrak{J}_{(\text{Div})}[\boldsymbol{\partial} V, \boldsymbol{\partial} V]$	Error terms arising in the covariant divergence identity for $\mathscr{J}$ that are *quadratic* in $\boldsymbol{\partial} V^a$.	(21.31a)–(21.31b)
$\mathfrak{J}_{(\boldsymbol{\partial}\mathscr{W})}[V, \boldsymbol{\partial} V]$, $\mathfrak{J}_{(\text{Absorb-1})}[V, \boldsymbol{\partial} V]$, $\mathfrak{J}_{(\text{Absorb-2})}[V, \boldsymbol{\partial} V]$, $\mathfrak{J}_{(\text{Null Geometry})}[V, \boldsymbol{\partial} V]$	Error terms arising in the covariant divergence identity for $\mathscr{J}$ that are *linear* in $\boldsymbol{\partial} V^a$.	(21.31c)–(21.31g)
$\mathfrak{P}[V, V]$	Term arising in the boundary terms for the characteristic current identity which is a *perfect* ${}^{(\mathfrak{n})}\breve{R}$ derivative. It is positive definite in Σ_t-tangent vectorfields in the sense that $\mathfrak{P}[V, V] \approx \frac{1}{\mu - \phi \frac{\mathfrak{n}}{L\mu}} \lvert V \rvert_g^2$.	(21.45)
$\mathfrak{E}_{(\text{Principal})}[V, \boldsymbol{\partial} V]$	Principal order error terms arising in the boundary terms for the characteristic current identity.	(21.46)
$\mathfrak{E}_{(\text{Lower-order})}[V, V]$	Lower order error terms arising in the boundary terms for the characteristic current identity.	(21.47)
$\mathfrak{M}[V, \boldsymbol{\partial} V]$	Spacetime bulk error term arising in the elliptic-hyperbolic integral identity.	(21.64)

Error terms in the geometric wave equations		
Symbols	Descriptions	Reference
$\text{Harmless}^{[1,N]}_{(\text{Wave})}$	N-th order harmless wave error terms.	(22.1)
$\vec{\mathfrak{G}}, \mathfrak{G}_\iota$	For $\vec{\Psi} = (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4) = (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$, $\vec{\mathfrak{G}} = (\mathfrak{G}_0, \cdots, \mathfrak{G}_4)$ is the vector array of the inhomogeneous terms in the covariant wave equations $\mu \square_{\mathbf{g}} \Psi_\iota = \mathfrak{G}_\iota$	Prop. 22.3

Avoiding derivative loss when controlling the acoustic geometry		
Symbols	Descriptions	Reference
$\mathcal{A}$	Error term arising in the key identity verified by $\mu\mathbf{Ric}_{LL}$. The identity requires the use of the geometric wave equations (2.22a)–(2.22d).	(19.2)
$\mathfrak{B}$	Error term arising from key identity verified by $\mathbf{Ric}_{LL}$. The identity does not require that the geometric wave equations (2.22a)–(2.22b) are satisfied.	(19.4)
${}^{(\mathcal{P}^N)}\mathscr{X}$	Fully modified version of $\mu\mathcal{P}^N \mathrm{tr}_{\not g}\chi$ that satisfies the favorable transport equation (19.10).	(19.6a)
$\mathfrak{X}$	Inhomogeneous term in the modified quantity ${}^{(\mathcal{P}^N)}\mathscr{X} = \mu\mathcal{P}^N \mathrm{tr}_{\not g}\chi + \mathcal{P}^N\mathfrak{X}$.	(19.6b)
${}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}}$	Partially modified version of $\mathcal{P}^N \mathrm{tr}_{\not g}\chi$ that satisfies the favorable transport equation (19.11).	(19.7a)
${}^{(\mathcal{P}^N)}\widetilde{\mathfrak{X}}, \widetilde{\mathfrak{X}}$	N-th and 0-th order inhomogeneous terms in the partially modified quantity ${}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}}$.	(19.7b)–(19.8)
${}^{(\mathcal{P}^{N-1})}\mathfrak{B}$	Error terms arising in the transport equation for ${}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}}$.	(19.12)
$\lvert\xi\rvert_{\widetilde{\not g}}$	Norms of ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tensors with respect to $\widetilde{\not g}$.	Def. 6.9
$\mathrm{tr}_{\widetilde{\not g}}\widetilde{\xi}$	The $\widetilde{\not g}$-trace of a type $\binom{0}{2}$ tensorfield on ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$: $\mathrm{tr}_{\widetilde{\not g}}\widetilde{\xi} = (\widetilde{\not g}^{-1})^{\alpha\beta}\xi_{\alpha\beta}$.	(6.25)
$\widetilde{\not d}\varphi$	The rough differential of a scalar function.	Def. 6.12
$\widetilde{\mathrm{div}\!\!\!/}\,\xi$	The rough divergence of a ${}^{(\mathfrak{n})}\widetilde{\ell}_{\tau,u}$-tangent tensorfield ξ.	Def. 6.13
$\{e_A\}_{A=2,3}$, $\{f_A\}_{A=2,3}$	The frames obtained from applying Gram–Schmidt to $\{\frac{\partial}{\partial x^A}\}_{A=2,3}$ and $\{\frac{\widetilde{\partial}}{\partial x^A}\}_{A=2,3}$ with $\not g$, $\widetilde{\not g}$ respectively.	Def. 28.2
$\mathscr{O}_{AB}$, λ_A	Change of frame coefficients between $\{e_A\}_{A=2,3}$ and $\{f_A\}_{A=2,3}$.	Lemma 28.4

Developments of the data, the singular boundary and crease, and a new time function		
Symbols	Descriptions	Reference
$\mathcal{M}_{\text{Left}}$	The portion of the development sandwiched between ${}^{(U_2)}\mathcal{P}^{\mathfrak{n}}_{[\tau_0,0]}$ and $\breve{\mathbb{X}}_0^{[\tau_0,0]}$.	(32.1a)
$\mathcal{M}_{\text{Right}}$	The portion of the development sandwiched between $\breve{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$ and ${}^{(-U_0)}\mathcal{P}^{\mathfrak{n}}_{[\tau_0,0]}$.	(32.1c)
$\mathcal{M}_{\text{Singular}}$	The portion of the development sandwiched between $\breve{\mathbb{X}}_0^{[\tau_0,0]}$ and $\breve{\mathbb{X}}_{-\mathfrak{n}_0}^{[\tau_0,0]}$.	(32.1b)
$\mathcal{M}_{\text{Interesting}}$	The union of $\mathcal{M}_{\text{Left}}, \mathcal{M}_{\text{Singular}}, \mathcal{M}_{\text{Right}}$.	(32.1d)
$\mathcal{B}^{[0,\mathfrak{n}_0]}$	The singular boundary portion of the development, given by $\mathcal{B}^{[0,\mathfrak{n}_0]} = \bigcup_{\mathfrak{n}\in[0,\mathfrak{n}_0]} \breve{\mathbf{T}}_{0,-\mathfrak{n}}$.	(32.3)
$\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]}$	The crease, given by $\partial_{-}\mathcal{B}^{[0,\mathfrak{n}_0]} \stackrel{\text{def}}{=} \breve{\mathbf{T}}_{0,0}$.	(32.4)
$\mathfrak{L}$	The map given by $\mathfrak{L}(\mathfrak{m},\mathfrak{n},x^2,x^3) = (\mathfrak{T}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3), \mathfrak{U}_{\mathfrak{m},-\mathfrak{n}}(x^2,x^3), x^2, x^3)$, which is a $C^{1,1}$ diffeomorphism from $[0,\mathfrak{m}_0]\times[0,\mathfrak{n}_0]\times\mathbb{T}^2$ onto $\mathcal{M}_{\text{Singular}}$.	(32.7)
$\mathscr{D}_{\mathfrak{m}}^{[0,\mathfrak{n}_0]}$	The $\mathfrak{m}$-level-sets of μ in the singular development as a graph.	(32.16)
$\breve{T}_{\mathfrak{m}}$	The graph of the Cartesian t as a function of (u,x^2,x^3) along the $\mathfrak{m}$-level-sets of μ in the singular development.	(32.18)
${}^{(\text{Interesting})}\tau$	The time function whose level-sets foliate $\mathcal{M}_{\text{Interesting}}$.	(32.34)
$({}^{(\text{Interesting})}\tau, u, x^2, x^3)$	The interesting coordinate system.	Def. 32.9
${}^{(\text{Interesting})}\mathscr{T}$	The map $(t,u,x^2,x^3)\mapsto({}^{(\text{Interesting})}\tau,u,x^2,x^3)$.	(32.36)
${}^{(\text{Interesting})}\mathfrak{t}_{\tau}$	The function on $[-U_0,U_2]\times\mathbb{T}^2$ whose graph is the Cartesian t on $\mathcal{M}_{\text{Interesting}}$.	(32.37)
$\breve{H}$	The vectorfield that is the coordinate partial derivative with respect to u in the interesting coordinate system.	(32.38a)
$\breve{G}$	The vectorfield that is the coordinate partial derivative with respect to ${}^{(\text{Interesting})}\tau$ in the interesting coordinate system.	(32.38b)

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